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BOUNDARY TEMPERATURE RECONSTRUCTION IN AN INVERSE HEAT CONDUCTION PROBLEM USING BOUNDARY INTEGRAL EQUATION METHOD

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Abstract. In this paper, we consider an inverse boundary value problem for two-dimensional heat equation in an annular domain. This problem consists of determining the temperature on the interior boundary curve from the Cauchy data (boundary temperature and heat flux) on the exterior boundary curve. To this end, the boundary integral equation method is used. Since the resulting system of linear algebraic equations is ill-posed, the Tikhonov first-order regularization procedure is employed to obtain a stable solution. Determination of regularization parameter is based on L-curve technique. Some numerical examples for the feasibility of the proposed method are presented.
Keywords: Inverse boundary problem, heat equation, boundary integral equation method, regularization.
MSC(2010): Primary: 65N21; Secondary: 65N38.

1. Introduction

In many scientific and engineering applications it is very important to estimate some heat characteristics in places where their measurement is impossible. Calculation of unknown heat parameters on the basis of temperature measurements within or on accessible boundary parts of a body is the matter of inverse heat conduction problems (IHCPs) [6, 11].

The inverse problems for heat conduction equation can be classified into some classes with respect to the unknown parameters [22]. An inverse problem is called an inverse boundary problem, if it is required to determine the unknown functions in the boundary conditions [22]. The IHCP which we are interested in is to determine the temperature on an inaccessible boundary part of a body from measurements of the temperature and heat flux on the rest part

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of the boundary; see, for example, [1, 16, 28]. This situation can be modeled as Cauchy problem for the heat equation. These kind of inverse problems are known as sideways heat equation too [5].

It is well-known that this IHCP is an ill-posed problem in the sense as described by Hadamard [15], i.e. small errors in the input data (e.g. measured temperatures) may lead to errors of catastrophic magnitude in the computed solution.

Various methods for solving this class of inverse heat conduction problems have been proposed; see, e.g. the boundary element method [21, 25], the fundamental solution method [19], the Laplace transform and the finite-difference method [6] and the iterative methods [2,3,29]. Due to the advantages of boundary integral equation method (BIEM), e.g. reduction of the dimension by one and reduction of an unbounded exterior domain to a bounded boundary, this method has been considered in the study of inverse problems by many researchers [21].

The aim of this paper is to use the BIEM and the Tikhonov regularization technique to solve an IHCP in two spatial dimensions. In this study, we seek the solution of the heat equation in the form of a single-layer potential and investigate whether this method works for our IHCP [20]. Our methodology leads to a system of integral equations. For the numerical solution of the integral equations, we proceed analogous to [7, 9]. Our approach is somewhat similar to the one used by X. Z. Jia and Y. B. Wang [21] for one-dimensional heat equation and the methods proposed by R. Chapko et al. [4, 9, 10] for Laplace equation.

This paper is organized as follows. In Section 2, the mathematical model of our inverse problem is presented. In Section 3, we will describe the application of BIEM for solving the inverse problem, namely to transform the inverse problem into a system of boundary integral equations and then discretisize the resulting system of integral equations. In Section 4, we use the Tikhonov regularization procedure to obtain a stable solution to the proposed inverse problem. Some numerical experiments are presented in Section 5 to demonstrate the efficiency of the proposed method. Finally, Section 6 is devoted to some comments.

2. Problem formulation

The mathematical formulation of our IHCP can be described as follows. Assume that D_1 and D_2 are two bounded simply-connected domains in $\mathbb{R}^2$ such that $\overline{D_1} \subset D_2$. The boundaries of D_1 and D_2 are assumed to be of class C^2 and we denote them by Γ_1 and Γ_2, respectively. Further let $D := D_2 \setminus \overline{D_1}$, $T > 0$ and $I = (0, T)$. Figure 2 demonstrates a geometrical sketch of the problem.

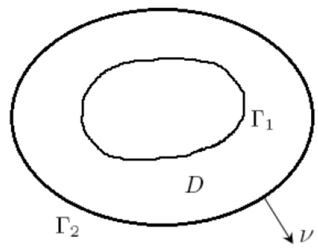


FIGURE 1. An example of domain D and its boundary parts.

We consider the following heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0, \qquad \text{in } D \times (0, T), \tag{2.1a}$$

subject to the homogeneous initial condition

$$u(., 0) = 0, \qquad \text{in } D, \tag{2.1b}$$

and the boundary conditions

$$u = \psi, \qquad \text{on } \Gamma_1 \times [0, T], \tag{2.1c}$$

$$\frac{\partial u}{\partial \nu} = \phi, \qquad \text{on } \Gamma_2 \times [0, T]. \tag{2.1d}$$

Here ψ and ϕ are given functions on Γ_1 and Γ_2, respectively and ν is the outward unit normal to Γ_2. Our inverse problem is to reconstruct the surface temperature ψ on the interior boundary Γ_1 in the problem (2.1) from a knowledge of the temperature on the exterior boundary Γ_2 as

$$u|_{\Gamma_2 \times [0,T]} = \varphi. \tag{2.2}$$

For simplicity, the initial data in this IHCP is assumed to be zero. The general case can be reduced to this problem [3].

It is well-known that the Cauchy problem given by equations (2.1a), (2.1b), (2.1d) and (2.2) is ill-posed and one can show that the solution of this problem is unique [3, 30]. Furthermore we assume that data are given such that there exists a solution to this Cauchy problem.

We introduce an operator $A : L^2(\Gamma_1 \times (0, T)) \to L^2(\Gamma_2 \times (0, T))$ as

$$A(\psi) := u|_{\Gamma_2 \times [0,T]} \qquad \text{for } \psi \in L^2(\Gamma_1 \times (0, T)),$$

where u is the solution of the mixed boundary value problem (2.1). According to this operator, our inverse problem consists of finding the function ψ from the following equation

$$A(\psi) = \varphi. \tag{2.3}$$

In the next section, a boundary integral formulation of our problem is introduced and investigated numerically.

3. **Boundary integral formulation**

In this section, we describe the application of BIEM for solving the equation

$$A(\psi) = \varphi,$$

for the function ψ. Here φ is a given (measured) temperature on the boundary curve Γ_2. Solving this equation is equivalent to determining the unknown function ψ in the initial-boundary value problem (2.1) from a knowledge of the temperature φ on Γ_2.

The solution of the equation (2.1a) is assumed to possess the form of a single-layer potential

$$u(x,t) = \int_0^t \int_{\partial D} G(x,y;t,\tau)\ q(y,\tau)\ \mathrm{d}s(y)\ \mathrm{d}\tau, \qquad (x,t) \in D \times I, \tag{3.1}$$

where $q \in C(\partial D \times (0,T))$ shows the density and G is the fundamental solution to the heat equation defined as

$$G(x,y;t,\tau) = \frac{1}{4\pi(t-\tau)} \exp\left\{ -\frac{|x-y|^2}{4(t-\tau)} \right\}, \qquad t > \tau.$$

The single-layer heat potential (3.1) satisfies the heat equation (2.1a) and the homogeneous initial condition (2.1b). As $x_0 \in D$ tends non-tangentially to $x \in \partial D$, this heat potential satisfies the jump relations [14]

(3.2a)

$$\lim_{x_0 \to x} u(x_0,t) = u(x,t),$$

(3.2b)

$$\lim_{x_0 \to x} \frac{\partial u}{\partial \nu(x)}(x_0,t) = \frac{1}{2} q(x,t) + \int_0^t \int_{\partial D} \frac{\partial G}{\partial \nu(x)}(x,y;t,\tau)\ q(y,\tau)\ \mathrm{d}s(y)\ \mathrm{d}\tau.$$

By imposing the boundary conditions (2.1c), (2.1d) and (2.2) on the potential (3.1) and using the jump relations (3.2), we obtain a system of integral equations as follows

(3.3a)
$$\frac{1}{2}q(x,t)+\int_0^t\int_{\partial D}\frac{\partial G}{\partial\nu(x)}(x,y;t,\tau)\ q(y,\tau)\ \mathrm{d}s(y)\ \mathrm{d}\tau=\phi(x,t),\quad (x,t)\in\Gamma_2\times I,$$
(3.3b)
$$\int_0^t\int_{\partial D}G(x,y;t,\tau)\ q(y,\tau)\ \mathrm{d}s(y)\ \mathrm{d}\tau=\psi(x,t),\qquad (x,t)\in\Gamma_1\times I,$$
(3.3c)
$$\int_0^t\int_{\partial D}G(x,y;t,\tau)\ q(y,\tau)\ \mathrm{d}s(y)\ \mathrm{d}\tau=\varphi(x,t),\qquad (x,t)\in\Gamma_2\times I.$$

We assume that the boundary curves have the smooth, regular and 2π-periodic parametric representation as

$$\Gamma_{\mathrm{k}}=\{x_{\mathrm{k}}(s)=(x_{\mathrm{k},1}(s),x_{\mathrm{k},2}(s))\ :\ 0\le s\le 2\pi\},\qquad \mathrm{k}=1,2, \tag{3.4}$$

where $x_{\mathrm{k}}:\mathbb{R}\to\mathbb{R}^2$ are of class C^2 such that the Jacobian $|x_{\mathrm{k}}'(s)|$ is strictly positive for all s. Using these parameterizations, we can transform the system of equations (3.3) into the following parametric form

$$\mu_2(s,t)+\int_0^t\int_0^{2\pi}\sum_{\mathrm{k}=1}^{2}H_{\mathrm{k}}(s,\sigma;t,\tau)\ \mu_{\mathrm{k}}(\sigma,\tau)\ \mathrm{d}\sigma\ \mathrm{d}\tau=g_1(s,t), \tag{3.5a}$$

$$\int_0^t\int_0^{2\pi}\sum_{\mathrm{k}=1}^{2}K_{1\mathrm{k}}(s,\sigma;t,\tau)|x_{\mathrm{k}}'(\sigma)|\ \mu_{\mathrm{k}}(\sigma,\tau)\ \mathrm{d}\sigma\ \mathrm{d}\tau=f(s,t), \tag{3.5b}$$

$$\int_0^t\int_0^{2\pi}\sum_{\mathrm{k}=1}^{2}K_{2\mathrm{k}}(s,\sigma;t,\tau)|x_{\mathrm{k}}'(\sigma)|\ \mu_{\mathrm{k}}(\sigma,\tau)\ \mathrm{d}\sigma\ \mathrm{d}\tau=g_2(s,t), \tag{3.5c}$$

for $(s,t)\in[0,2\pi]\times I$. Here we set $\mu_{\mathrm{k}}(s,t):=q(x_{\mathrm{k}}(s),t)$ for $\mathrm{k}=1,2$, $f(s,t):=\psi(x_1(s),t)$, $g_1(s,t):=2\phi(x_2(s),t)$ and $g_2(s,t):=\varphi(x_2(s),t)$. In addition, we assume that the kernels are given by

$$H_{\mathrm{k}}(s,\sigma;t,\tau)=\frac{\nu(x_2(s)).[x_{\mathrm{k}}(\sigma)-x_2(s)]|x_{\mathrm{k}}'(\sigma)|}{4\pi(t-\tau)^2}\exp\left\{-\frac{|x_2(s)-x_{\mathrm{k}}(\sigma)|^2}{4(t-\tau)}\right\},$$

for $\mathrm{k}=1,2$ and $t>\tau$, and

$$K_{\mathrm{qk}}(s,\sigma;t,\tau)=G(x_{\mathrm{q}}(s),x_{\mathrm{k}}(\sigma);t,\tau),$$

for $\mathrm{q},\mathrm{k}=1,2$ and $s\neq\sigma$.

For the numerical solution of the system of integral equations (3.5), we proceed analogous to the algorithm described in [7, 9]. For this purpose, we choose equidistant meshes on $[0,T]$ and $[0,2\pi]$ by setting

$$t_{\mathrm{n}} = \mathrm{n}h_t, \qquad \mathrm{n}=0,\dots,N,\ h_t=\frac{T}{N},$$
$$s_{\mathrm{j}} := \frac{\mathrm{j}\pi}{M}, \qquad \mathrm{j}=0,\dots,2M-1,\ M\in\mathbb{N}.$$

Then by applying a collocation method with respect to the time and space variables, we obtain the sequence of the linear systems for approximate values $\tilde{\mu}_{\mathrm{k,n;j}} \approx \mu_{\mathrm{k}}(s_{\mathrm{j}},t_{\mathrm{n}})$ and $\tilde{f}_{\mathrm{n,j}} \approx f(s_{\mathrm{j}},t_{\mathrm{n}})$ as follows

$$\tilde{\mu}_{2,\mathrm{n;i}} + \sum_{\mathrm{j}=0}^{2M-1}\sum_{\mathrm{k}=1}^{2} b_{\mathrm{k,ij}}\ \tilde{\mu}_{\mathrm{k,n;j}} = \tilde{F}_{2,\mathrm{n;i}}, \tag{3.6a}$$

$$2\tilde{f}_{\mathrm{n,i}} - \sum_{\mathrm{j}=0}^{2M-1}\sum_{\mathrm{k}=1}^{2} a_{1\mathrm{k;ij}}\ \tilde{\mu}_{\mathrm{k,n;j}} = \tilde{F}_{1,\mathrm{n;i}}, \tag{3.6b}$$

$$\sum_{\mathrm{j}=0}^{2M-1}\sum_{\mathrm{k}=1}^{2} a_{2\mathrm{k;ij}}\ \tilde{\mu}_{\mathrm{k,n;j}} = \tilde{G}_{\mathrm{n,i}}, \tag{3.6c}$$

for $\mathrm{i}=0,\dots,2M-1$, $\mathrm{n}=1,\dots,N$, with the matrix elements

$$b_{\mathrm{k,ij}} := \frac{\pi}{M} H_{\mathrm{k}}^{(0)}(s_{\mathrm{i}},s_{\mathrm{j}}),$$
$$a_{\mathrm{qk;ij}} := \begin{cases} \left(-R_{|\mathrm{i}-\mathrm{j}|} + \dfrac{1}{2M} K_{\mathrm{qq}}^{(0,1)}(s_{\mathrm{i}},s_{\mathrm{j}})\right)|x_{\mathrm{k}}'(s_{\mathrm{j}})|; & \mathrm{q}=\mathrm{k},\\ \dfrac{|x_{\mathrm{k}}'(s_{\mathrm{j}})|}{2M} K_{\mathrm{qk}}^{(0)}(s_{\mathrm{i}},s_{\mathrm{j}}); & \mathrm{q}\neq\mathrm{k}, \end{cases}$$

for $\mathrm{q,k}=1,2$, and the right-hand sides

$$\tilde{F}_{2,\mathrm{n;i}} := g_1(s_{\mathrm{i}},t_{\mathrm{n}}) - \frac{\pi}{M}\sum_{\mathrm{m}=1}^{\mathrm{n}-1}\sum_{\mathrm{k}=1}^{2}\sum_{\mathrm{j}=0}^{2M-1} \tilde{\mu}_{\mathrm{k,m;j}}\ H_{\mathrm{k}}^{(\mathrm{n}-\mathrm{m})}(s_{\mathrm{i}},s_{\mathrm{j}}),$$

$$\tilde{F}_{1,\mathrm{n;i}} := \frac{1}{2M}\sum_{\mathrm{m}=1}^{\mathrm{n}-1}\sum_{\mathrm{k}=1}^{2}\sum_{\mathrm{j}=0}^{2M-1} \tilde{\mu}_{\mathrm{k,m;j}}\ K_{1\mathrm{k}}^{(\mathrm{n}-\mathrm{m})}(s_{\mathrm{i}},s_{\mathrm{j}})\ |x_{\mathrm{k}}'(s_{\mathrm{j}})|,$$

$$\tilde{G}_{\mathrm{n,i}} := 2g_2(s_{\mathrm{i}},t_{\mathrm{n}}) - \frac{1}{2M}\sum_{\mathrm{m}=1}^{\mathrm{n}-1}\sum_{\mathrm{k}=1}^{2}\sum_{\mathrm{j}=0}^{2M-1} \tilde{\mu}_{\mathrm{k,m;j}}\ K_{2\mathrm{k}}^{(\mathrm{n}-\mathrm{m})}(s_{\mathrm{i}},s_{\mathrm{j}})\ |x_{\mathrm{k}}'(s_{\mathrm{j}})|.$$

For more details about the method used for numerical solution of the system of integral equations (3.5), including discretization process and an error and convergence analysis see [7–9, 23].

4. Method of solution

The application of BIEM for our proposed inverse problem results in the systems of algebraic equations (3.6) by $3(2M)$ linear equations with $3(2M)$

unknowns. These systems of equations need to be solved recursively for $\mathrm{n} = 1,\dots,N$. By using the equations (3.6a) and (3.6b), the discretized densities $\tilde{\mu}_{\mathrm{k,n;j}}$ for $\mathrm{k} = 1,2,\ \mathrm{j} = 0,\dots,2M-1$ can be expressed as a function of the discretized temperatures $\tilde{f}_{\mathrm{n,i}}$ for $\mathrm{i} = 0,\dots,2M-1$ and the densities $\tilde{\mu}_{\mathrm{k,m;j}}$ for $\mathrm{m} = 1,\dots,\mathrm{n}-1$. Then, the aforementioned elimination in equations (3.6a) and (3.6b) leads to the linear systems for approximate values $\tilde{\mu}_{\mathrm{k,n;j}}$ as follows

$$\tilde{\mu}_{2,\mathrm{n;i}} + \sum_{\mathrm{j}=0}^{2M-1}\sum_{\mathrm{k}=1}^{2} b_{\mathrm{k,ij}}\ \tilde{\mu}_{\mathrm{k,n;j}} = \tilde{F}_{2,\mathrm{n;i}}, \tag{4.1a}$$

$$\sum_{\mathrm{j}=0}^{2M-1}\sum_{\mathrm{k}=1}^{2} a_{1\mathrm{k;ij}}\ \tilde{\mu}_{\mathrm{k,n;j}} = 2\tilde{f}_{\mathrm{n,i}} - \tilde{F}_{1,\mathrm{n;i}}, \tag{4.1b}$$

for $\mathrm{i} = 0,\dots,2M-1,\ \mathrm{n} = 1,\dots,N$. We assume that $\mathbf{Q}$ is the coefficient matrix in the linear system (4.1) and $\mathbf{P} = (p_{\mathrm{rc}})_{4M\times 4M}$ is its inverse. Substituting the resulting expressions for the solution of the system of equations (4.1) into equation (3.6c) yields a sequence of linear systems of $2M$ linear equations with $2M$ unknowns. This systems can be written in the generic form as

$$\mathbf{A}\tilde{\mathbf{f}}_{\mathrm{n}} = \mathbf{b}_{\mathrm{n}}, \qquad \mathrm{n} = 1,\dots,N, \tag{4.2}$$

where $\mathbf{A} = (A_{\mathrm{ij}})$ for $\mathrm{i,j} = 0,\dots,2M-1$ is a matrix with the elements

$$A_{\mathrm{ij}} = 2\sum_{\mathrm{k}=0}^{2M-1}\left(a_{21;\mathrm{ik}}\ p_{\mathrm{k},(\mathrm{j}+2M)} + a_{22;\mathrm{ik}}\ p_{(\mathrm{k}+2M),(\mathrm{j}+2M)}\right),$$

$\tilde{\mathbf{f}}_{\mathrm{n}} = (\tilde{f}_{\mathrm{n},0},\dots,\tilde{f}_{\mathrm{n},2M-1})^{\mathrm{T}}$ is the unknown vector and $\mathbf{b}_{\mathrm{n}} = (b_{\mathrm{n,i}})$ for $\mathrm{i} = 0,\dots,2M-1$ is a known vector that its elements are dependent on the densities $\tilde{\mu}_{\mathrm{k,m;j}}$ for $\mathrm{k} = 1,2,\ \mathrm{j} = 0,\dots 2M-1,\ \mathrm{m} = 1,\dots \mathrm{n}-1$. For this reason, for every time step, the linear systems (4.2) and (4.1) respectively, need to be solved.

Due to the large value of the condition number of the matrix $\mathbf{A}$, using the usual analytical and iterative methods to solve the system of linear algebraic equations (4.2) may produce a highly unstable solution. To stabilize the results, the first-order Tikhonov regularization method is employed. The Tikhonov method is a famous approach for regularizing ill-posed problems, wherein a regularization parameter needs to be determined. To remedy the sensitivity to noise, the regularization procedures is often used. Intensive studies on the applications, stability and convergence of the Tiknohov regularization method in terms of the noise level can be found in the literature, see, e.g. [12, 13, 17, 18, 24, 27, 31]. Furthermore the determination of optimal values for the regularization parameter plays a vital role in the regularization theory. After the pioneering work of Tikhonov and Arsenin [31], a number of methods have been established to determine a suitable regularization parameter such as the L-curve method proposed by Hansen et. al. [17, 18], the discrepancy principles [12], and the iterative technique [24]. Here in order to regularize the obtained linear system, we replace (4.2) by the following least-squares problem to minimize the penalized residual

$$R(\tilde{\mathbf{f}}_{\mathrm{n}}) := \min \left\{ \|\mathbf{A}\tilde{\mathbf{f}}_{\mathrm{n}} - \mathbf{b}_{\mathrm{n}}\|_2^2 + \alpha \|\mathbf{L}\tilde{\mathbf{f}}_{\mathrm{n}}\|_2^2 \right\},$$

where $\alpha > 0$ denotes the regularization parameter and $\mathbf{L}$ is the roughening matrix where defined as

$$\mathbf{L} = \begin{pmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & \dots & & & \\ & & -1 & 1 & & \\ & & & -1 & 1 \end{pmatrix}.$$

Remark 4.1. In Section 5 the values of u on Γ_2, where u is the solution of the initial-boundary value problem (2.1), need to be computed. To this end, we seek the solution of (2.1a) in the form of a single-layer potential and match the boundary conditions. With discretization of the resulting integral equations, we obtain the sequence of linear systems (4.1) where $(\tilde{\mu}_{1,\mathrm{n};0}, \dots, \tilde{\mu}_{1,\mathrm{n};2M-1}, \tilde{\mu}_{2,\mathrm{n};0}, \dots, \tilde{\mu}_{2,\mathrm{n};2M-1})$ is the vector of the unknown discretisized densities on Γ_1 and Γ_2.

From the jump relation (3.2a) and using the solution of the linear systems (4.1), we conclude the approximate values

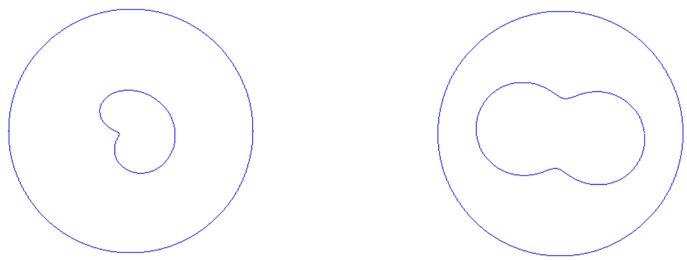
$$u(x_2(s_{\mathrm{i}}), t_{\mathrm{n}}) = \frac{1}{4M} \sum_{\mathrm{m}=1}^{\mathrm{n}} \sum_{\mathrm{j}=0}^{2M-1} \sum_{\mathrm{k}=1}^{2} \tilde{K}_{\mathrm{k,ij}}^{(\mathrm{n-m})}\ \tilde{\mu}_{\mathrm{k,m;j}},$$

where $\tilde{K}_{\mathrm{k,ij}}^{(0)} := 2Ma_{2\mathrm{k;ij}}$, $\mathrm{k} = 1,2$ with $a_{2\mathrm{k;ij}}$ such as defined in (3.6c) and $\tilde{K}_{\mathrm{k,ij}}^{(\mathrm{p})} := K_{2\mathrm{k}}^{(\mathrm{p})}(s_{\mathrm{i}}, s_{\mathrm{j}})|x'_{\mathrm{k}}(s_{\mathrm{j}})|$ for $\mathrm{p} = 1, \dots, N-1$ and $\mathrm{k} = 1, 2$.

5. Numerical experiments and discussions

In this section, some numerical results are presented to show how our proposed numerical procedure works. To this end, some test problems are investigated. In these problems, we consider a mixed initial-boundary value problem in the form (2.1) and try to reconstruct the temperature on the interior boundary curve from temperature measurements on the exterior boundary Γ_2.

To solve the linear systems (3.6) with the procedure introduced in Section 4, first we solve the linear system (4.2) to calculate the approximate values of temperature $\tilde{f}_{\mathrm{n,i}}$ on Γ_1. To calculate the temperature values on Γ_1 at time $t = t_{\mathrm{n}}$, the temperature and heat flux values on Γ_2 at time $t = t_{\mathrm{n}}$ and the values $\tilde{\mu}_{\mathrm{k,m;j}}$ for $\mathrm{m} = 1, \dots, \mathrm{n}-1$ are needed. After calculating the values of temperature on Γ_1 at time $t = t_{\mathrm{n}}$, i.e. $\tilde{f}_{\mathrm{n,i}}$, the linear system (4.1) is solved to compute the approximate values $\tilde{\mu}_{\mathrm{k,n;j}}$. Therefore, for $\mathrm{n} = 1, \dots, N$ the linear systems (4.2) and (4.1) respectively, need to be solved.



(a) (b)

FIGURE 2. Solution domains in the numerical examples

In each of the following examples, the absolute errors

$$E_{\text{a}} := \|\psi_{\text{approx}} - \psi_{\text{exact}}\|_{L^2(\Gamma_1)},$$

and the relative errors

$$E_{\text{r}} := \frac{\|\psi_{\text{approx}} - \psi_{\text{exact}}\|_{L^2(\Gamma_1)}}{\|\psi_{\text{exact}}\|_{L^2(\Gamma_1)}},$$

of the temperatures at different time steps are presented. In addition, the regularization parameters α at different time steps are presented. These regularization parameters are chosen by using the L-curve function from the Regularization Tools

In the following examples, we use $[0, T] = [0, 1]$ as time interval and the discretization parameters are chosen as $M = 32$, $N = 10$.

Example 5.1. In the first test problem, the exterior boundary curve Γ_2 is given by a circle of radius 1.5 and center at the origin as

$$\Gamma_2 = \{x_2(s) = (1.5\cos s, 1.5\sin s) \ : \ 0 \leq s \leq 2\pi\},$$

and the boundary curve Γ_1 is considered as bean-shaped with the parametric representation of the form

$$x_1(s) = r(s)(\cos s, -\sin s), \qquad 0 \leq s \leq 2\pi, \tag{5.1}$$

with the radial function (see Figure 2(a))

$$r(s) = \frac{1.0 + 0.9\cos s + 0.1\sin 2s}{2.0 + 1.5\cos s}.$$

In addition, the boundary conditions assumed as

$$\psi(x, t) = 0, \qquad (x, t) \in \Gamma_1 \times [0, T],$$
$$\phi(x, t) = t^2 \exp(-4(t + |x|^2) + 2), \qquad (x, t) \in \Gamma_2 \times [0, T].$$

We solve this direct initial-boundary value problem to calculate the value of the trace of the solution u on the exterior boundary Γ_2, i.e. $\varphi = u|_{\Gamma_2}$. Our inverse problem is to reconstruct the temperature ψ on the boundary Γ_1 from the Cauchy data

$$\begin{cases} u = \varphi(x,t), & (x,t) \in \Gamma_2 \times [0,T], \\ \dfrac{\partial u}{\partial \nu} = \phi(x,t), & (x,t) \in \Gamma_2 \times [0,T]. \end{cases} \tag{5.2}$$

We apply the proposed procedure to solve this inverse problem to determine the temperature on Γ_1. The reconstructed temperature on Γ_1 is plotted in Figure 3 at four different time steps and in Figure 4 at two positions $\pi/4$ and $3\pi/4$ on Γ_1.

From Figure 3 and Table 1 it can be observed that the numerical results are in good agreement with exact data. In Figure 3 one may see that the accuracy of the numerical results in the interval $[135^0, 180^0]$ on the interior boundary Γ_1, compared with the other parts of the boundary, is lower. This behavior may result from this fact that at this interval the distance between the interior boundary where the unknowns are located and the exterior boundary where measurements are taken is maximum.

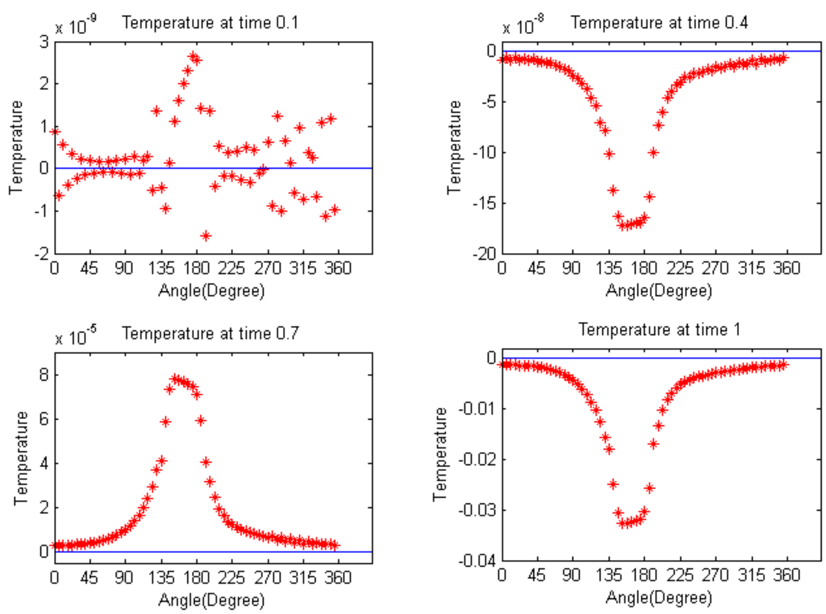


FIGURE 3. The numerical results for the boundary temperature on Γ_1 at four different times for Example 5.1. "$*$"shows the approximate temperature and "—" shows the exact temperature data.

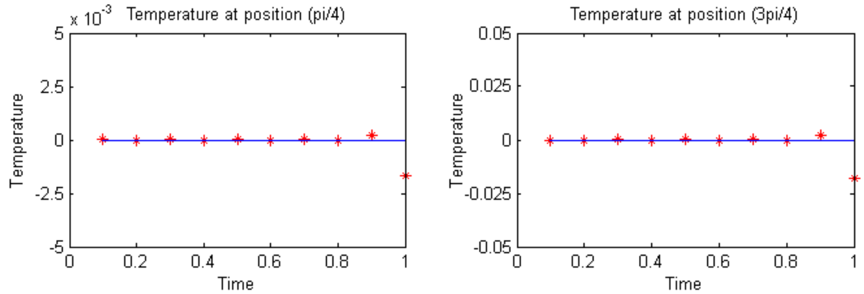


FIGURE 4. The numerical results for the boundary temperature at two positions on Γ_1 for Example 5.1. "$*$" shows the approximate temperature and "—" shows the exact temperature data.

TABLE 1. The absolute errors $E_{\rm a}$ at different times for Example 5.1.

Time	α	Absolute errors $E_{\rm a}$
0.1	E−13	0.2251E−8
0.2	E−11	0.3120E−8
0.3	3E−12	0.2280E−7
0.4	4E−13	0.1721E−6
0.5	5E−14	0.1316E−5
0.6	3E−14	0.9851E−5
0.7	4E−15	0.7519E−4
0.8	4E−15	0.5640E−3
0.9	4E−15	0.4249E−2
1	4E−15	0.3198E−1

Example 5.2. We consider the case when the boundary curve Γ_2 is the same as in the Example 5.1 and the interior boundary Γ_1 is a peanut-shaped curve given in the form (5.1) with with the radial function (see Figure 2(b))

$$r(s) = \sqrt{\cos^2 s + 0.26 \sin^2(s + 0.5)}.$$

The boundary conditions are

$$\psi(x,t) = -t^2 \exp(-t) \cos(|x|), \qquad (x,t) \in \Gamma_1 \times [0,T],$$
$$\phi(x,t) = 0, \qquad (x,t) \in \Gamma_2 \times [0,T].$$

First we solve this direct mixed boundary value problem to obtain the value of $\varphi = u|_{\Gamma_2}$ and consider an inverse problem with the Cauchy data as follows

$$\begin{cases} u = \varphi(x,t), & (x,t) \in \Gamma_2 \times [0,T], \\ \dfrac{\partial u}{\partial \nu} = 0, & (x,t) \in \Gamma_2 \times [0,T]. \end{cases}$$

The comparison between the exact temperature and the numerical results for the boundary temperature is illustrated in Figure 5 at four different time steps and in Figure 6 at two different positions on Γ_1.

Figure 6 shows that the numerical results of the proposed method have good accuracy even at higher time steps. The errors presented in Table 2 can confirm our observations. As can be seen from Table 2, the absolute and the relative errors of the temperatures, respectively, remain below 2E$-$3 and 3E$-$3.

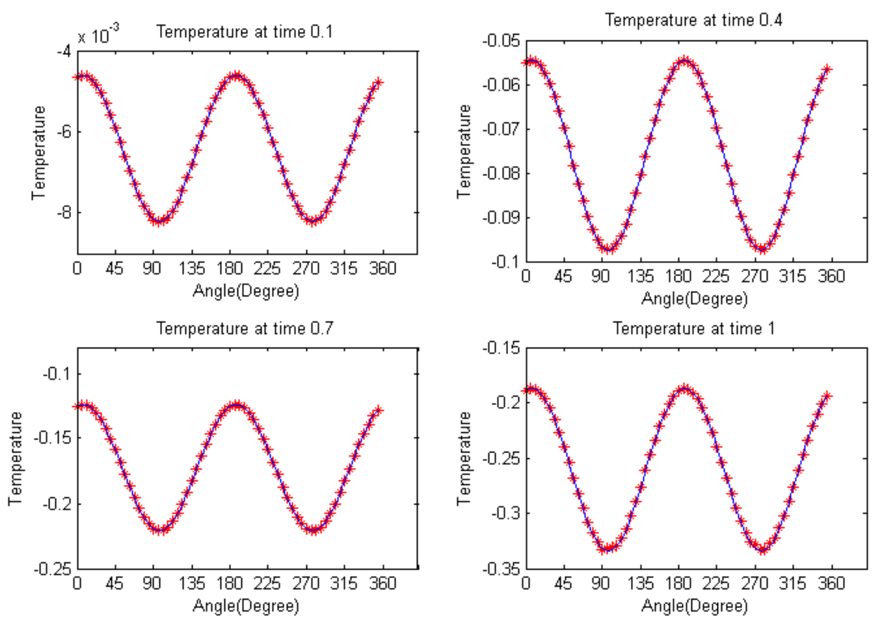


FIGURE 5. The numerical results for the boundary temperature on Γ_1 at four different times for Example 5.2. "$*$" shows the approximate temperature and "—" shows the exact temperature data.

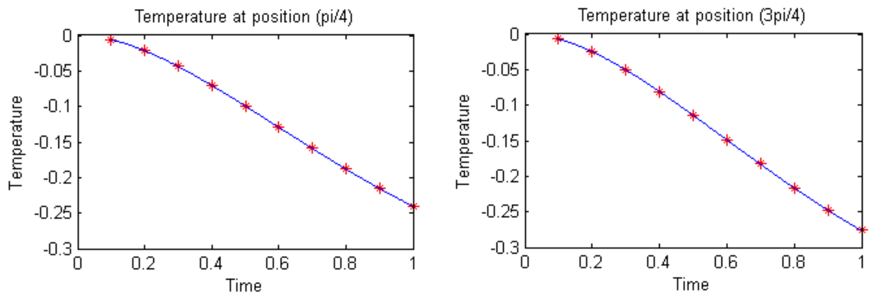


FIGURE 6. The numerical results for the boundary temperature at two positions on Γ_1 for Example 5.2. "$*$" shows the approximate temperature and "—" shows the exact temperature data.

TABLE 2. The absolute errors E_{a} and the relative errors E_{r} at different times for Example 5.2.

Time	α	Absolute errors E_{a}	Relative errors E_{r}
0.1	3E−14	0.4797E−5	0.2942E−3
0.2	3E−14	0.2624E−4	0.4447E−3
0.3	3E−14	0.6006E−4	0.4998E−3
0.4	3E−14	0.4868E−4	0.2518E−3
0.5	3E−14	0.1552E−3	0.5680E−3
0.6	3E−14	0.2516E−3	0.7067E−3
0.7	3E−14	0.2314E−3	0.5276E−3
0.8	3E−14	0.5511E−3	0.1063E−2
0.9	E−13	0.5977E−3	0.1007E−2
1	3E−14	0.1925E−2	0.2904E−2

Example 5.3. In this example, the boundary curves Γ_1 and Γ_2 are the same as in the Example 5.2 and the boundary conditions ψ and ϕ are given by

$$\psi(x,t) = -t^2 \exp(-t)\cos(|x|), \qquad (x,t) \in \Gamma_1 \times [0,T],$$
$$\phi(x,t) = 4t^2 \exp(-4t+2), \qquad (x,t) \in \Gamma_2 \times [0,T].$$

After calculating the value $\varphi = u|_{\Gamma_2}$, where u is the solution of the above direct problem, we solve an inverse problem with the following Cauchy data

$$\begin{cases} u = \varphi(x,t), & (x,t) \in \Gamma_2 \times [0,T], \\ \dfrac{\partial u}{\partial \nu} = 4t^2 \exp(-4t+2), & (x,t) \in \Gamma_2 \times [0,T], \end{cases}$$

to retrieve the boundary temperature ψ.

The spatial distribution of the temperature data on Γ_1, obtained using the proposed method in this paper are displayed in Figure 7 at four different time values. Also, Figure 8 displays the time plot for the boundary temperature at four different spatial positions on the interior curve Γ_1. From these figures, one can observe that up to the final time of interest $T = 1$, the numerical results obtained from the our proposed method provide a good estimate of the exact temperature data.

In comparison with the results obtained in Example 5.2, it can be seen that the numerical results when $\phi(x,t) = 0$ on the exterior boundary Γ_2 become slightly better than the results with nonhomogeneous Neumann boundary condition $\phi(x,t)$ on Γ_2.

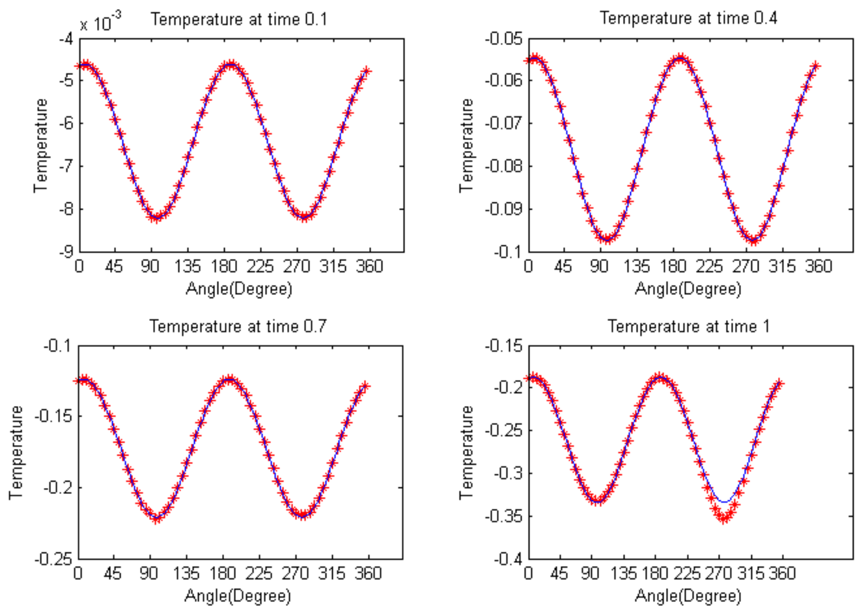


FIGURE 7. The numerical results for the boundary temperature on Γ_1 at four different times for Example 5.3. "$*$" shows the approximate temperature and "—" shows the exact temperature data.

6. Conclusion

In this study, determination of the unknown boundary temperature on an inaccessible boundary part of a 2D domain in an inverse heat conduction problem

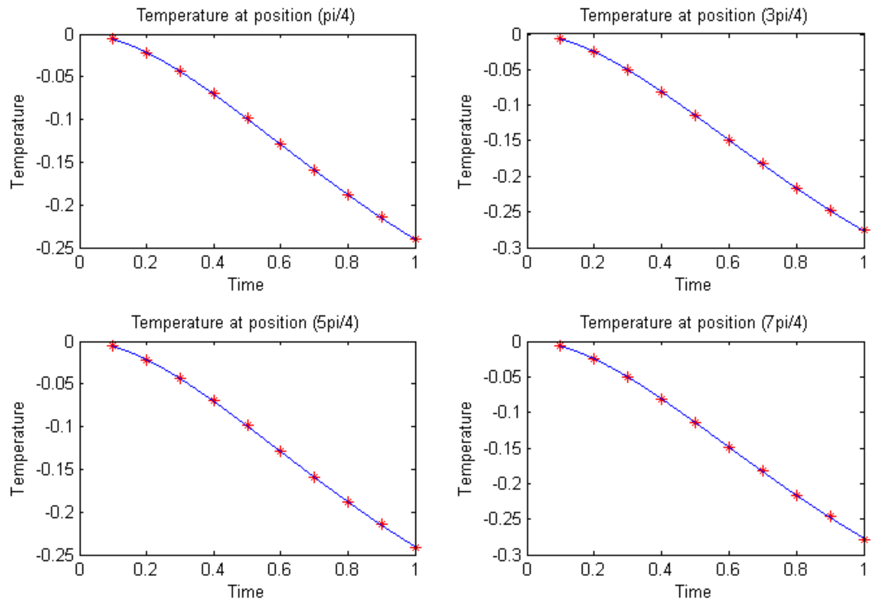


FIGURE 8. The numerical results for the boundary temperature at four positions on Γ_1 for Example 5.3. "$*$" shows the approximate temperature and "—" shows the exact temperature data.

TABLE 3. The absolute errors E_{a} and the relative errors E_{r} at different times for Example 5.3.

Time	α	Absolute errors E_{a}	Relative errors E_{r}
0.1	9E−14	0.2535E−4	0.1555E−2
0.2	3E−14	0.9896E−4	0.1677E−2
0.3	3E−14	0.5041E−3	0.4195E−2
0.4	2E−13	0.1509E−3	0.7807E−3
0.5	3E−14	0.8164E−3	0.2988E−2
0.6	4E−14	0.1710E−2	0.4802E−2
0.7	3E−14	0.9826E−3	0.2241E−2
0.8	2E−13	0.2146E−2	0.4142E−2
0.9	E−13	0.5461E−2	0.9201E−2
1	E−13	0.1442E−1	0.2176E−1

through measurements of the temperature on the rest part of the boundary has been considered. For solving this inverse problem, the boundary integral equation method is used accompanied by the Tikhonov first-order regularization procedure. As mentioned, the boundary integral equation method is used to solve a Cauchy problem in [21] for one-dimensional heat equation and [4, 9, 10] for Laplace equation. In these papers, the solution of the equation is represented as a single-layer potential with unknown density and to find this density, the representation of the solution is matched up with the given Cauchy data. Once the discrete densities have been obtained, they can be used to find an approximation to the solution and its normal derivative on the inaccessible part of the boundary of the solution domain. The implementation of the method in this paper is slightly different. In this paper, after representing the solution as a single-layer potential and matching it with the given Cauchy data, the trace on the interior boundary of the solution have been taken. This approach has resulted in a system of three boundary integral equations. With discretization of these integral equations and doing some simple calculations on the resulting discrete system, we have obtained two systems of linear equations for the discrete approximate values of the temperature on the inner boundary and the densities which in each time step, respectively, have been solved. The accuracy of the numerical results obtained from the proposed method in this paper show the feasibility of this method.

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References

[1] O. M. Alifanov, E. A. Artyukhin and S. V. Rumyantsev, Extreme Methods for Solving Ill-Posed Problems with Applications to Inverse Heat Transfer Problems, Begell House, New York, 1995.

[2] S. Andrieux, T. Baranger and A. Ben Abda, Solving Cauchy problems by minimizing an energy-like functional, *Inverse Problems* **22** (2006), no. 1, 115–133.

[3] G. Bastay, V. A. Kozlov and B. O. Turesson, Iterative methods for an inverse heat conduction problem, *J. Inverse Ill-Posed Probl.* **9** (2001), no. 4, 375–388.

[4] C. Babenko, R. Chapko and B. T. Johansson, On the numerical solution of the Laplace equation with complete and incomplete Cauchy data using integral equations, *CMES: Comput. Model. Eng. Sci.* **101** (2014), no. 5, 299–317.

[5] A.S. Carasso, Determining surface temperatures from interior observations, *SIAM J. Appl. Math.* 42 (1982), no. 3, 558–574.

[6] H. T. Chen, S. Y. Lin and L. C. Fang, Estimation of surface temperature in two-dimensional inverse heat conduction problems, *Int. J. Heat Mass Transfer* **44** (2001) 1455–1463.

[7] R. Chapko, R. Kress and J. R. Yoon, On the numerical solution of an inverse boundary value problem for the heat equation, *Inverse Problems* **14** (1998), no. 4, 853–867.

[8] R. Chapko and R. Kress, On a quadrature method for a logarithmic integral equation of the first kind, Contributions in numerical mathematics, 127–140, World Sci. Ser. Appl. Anal., 2, World Sci. Publ., River Edge, NJ, 1993.

[9] R. Chapko, B. T. Johansson and Y. Savka, On the use of an integral equation approach for the numerical solution of a Cauchy problem for Laplace equation in a doubly connected planar domain, *Inverse Probl. Sci. Eng.* **22** (2014), no. 1, 130–149.

[10] R. Chapko, B. T. Johansson, On the numerical solution of a Cauchy problem for the Laplace equation via a direct integral equation approach, *Inverse Probl. Imaging* **6** (2012), no. 1, 25–38.

[11] L. B. Drenchev and J. Sobczak, Inverse Heat Conduction Problems and Application to Estimate of Heat Parameters in 2-D Experiments, in 2nd Int. Conf. High Temperature Capillarity, Cracow, Poland, 29 June-2 July 1997, Foundry Research Institute, Krakow (Poland), 1998, pp. 355-361.

[12] H. W. Engl, Discrepancy principles for Tikhonov regularization of ill-posed problems leading to optimal convergence rates, *J. Optim. Theory Appl.* **52** (1987), no. 2, 209–215.

[13] H. Fenga and D. Jiangb, Convergence rates of Tikhonov regularization for parameter identification in a Maxwell system, *Appl. Anal.* **94** (2015), no. 2, 361–375.

[14] R. B. Guenther and J. W. Lee, Partial Differential Equations of Mathematical Physics and Integral Equations, Dover Publications, Inc., Mineola, 1996.

[15] J. Hadamard, Lectures on Cauchy's problem in linear partial differential equations, Dover Publications, New York, 1953.

[16] D. N. Hào, Methods for inverse heat conduction problems Habilitationsschrift, University of Siegen, Siegen, 1996, Methoden und Verfahren der Mathematischen Physik, 43, Peter Lang, Frankfurt am Main, 1998.

[17] P. C. Hansen, Analysis of discrete ill-posed problems by means of the L-curve, *SIAM Rev.* **34** (1992), no. 4, 561–580.

[18] P. C. Hansen and D. P. O'Leary, The use of the L-curve in the regularization of discrete ill-posed problems, *SIAM J. Sci. Comput.* **14** (1993), no. 6, 1487–1503.

[19] Y. C. Hon and T. Wei, A fundamental solution method for inverse heat conduction problem, *Eng. Anal. Bound. Elem.* **28** (2004) 489–495.

[20] G. C. Hsiao and J. Saranen, Boundary integral solution of the two-dimensional heat equation, *Math. Methods Appl. Sci.* **16** (1993), no. 2, 87–114.

[21] X. Z. Jia and Y. B. Wang, A Boundary Integral Method for Solving Inverse Heat Conduction Problem, *J. Inverse Ill-Posed Probl.* **14** (2006), no. 4, 375–384.

[22] S. I. Kabanikhin, Definitions and examples of inverse and ill-posed problems, *J. Inverse Ill-Posed Probl.* **16** (2008), no. 4, 317–357.

[23] R. Kress and I. H. Sloan, On the numerical solution of a logarithmic integral equation of the first kind for the Helmholtz equation, *Numer. Math.* **66** (1993), no. 2, 199–214.

[24] K. Kunisch and J. Zou, Iterative choices of regularization parameters in linear inverse problems, *Inverse Problems* **14** (1998), no. 5, 1247–1264.

[25] D. Lesnic, L. Elliott and D. B. Ingham, Application of the boundary element method to inverse heat conduction problems, *Int. J. Heat Mass Transfer* **39** (1996) 1503–1517.

[26] N. N. Lebedev, Special Functions and Their Applications, Revised English edition, Translated and edited by Richard A. Silverman, Prentice Hall Inc., Englewood Cliffs, 1965.

[27] C. S. Liu, A dynamical Tikhonov regularization for solving ill-posed linear algebraic systems, *Acta Appl. Math.* **123** (2013) 285–307.

[28] D. A. Murio, The mollification method and the numerical solution of ill-posed problems, John Wiley & Sons, Inc., New York, 1993.

[29] H. J. Reinhardt, D. N. Hào, J. Frohne and F. T. Suttmeier, Numerical solution of inverse heat conduction problems in two spatial dimensions, *J. Inverse Ill-Posed Probl.* **15** (2007), no. 2, 181–198.

[30] J. C. Saut and B. Scheurer, Remarques sur un théorème de prolongement unique de Mizohata, *C. R. Acad. Sci. Paris Sér. I Math.* **296** (1983), no. 6, 307–310.

[31] A. N. Tikhonov and V. Y. Arsenin, Solutions of ill-posed problems, Translated from the Russian, Preface by translation editor Fritz John, Scripta Series in Mathematics, V. H. Winston & Sons, Washington, D.C., John Wiley & Sons, New York-Toronto, Ont.-London, 1977.

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