ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 42 (2016), No. 5, pp. 1057–1076

Title:

On a generalization of condition (PWP)

Author(s):

X. Liang and Y. Luo

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 5, pp. 1057–1076
Online ISSN: 1735-8515

ON A GENERALIZATION OF CONDITION (PWP)

X. LIANG* AND Y. LUO

(Communicated by Ali Reza Ashrafi)

Abstract. There is a flatness property of acts over monoids called Condition (PWP) which, so far, has received much attention. In this paper, we introduce Condition GP-(P), which is a generalization of Condition (PWP). Firstly, some characterizations of monoids by Condition GP-(P) of their (cyclic, Rees factor) acts are given, and many known results are generalized. Moreover, some possible conditions on monoids that describe when their diagonal acts satisfy Condition GP-(P) are found. Finally, using some new types of epimorphisms, an alternative description of Condition GP-(P) (resp., Condition (PWP)) is obtained, and directed colimits of these new epimorphisms are investigated.
Keywords: S-act, Condition (PWP), condition GP-(P), generally left annihilating right ideal, quasi G-2-pure epimorphism.
MSC(2010): Primary: 20M30; Secondary: 20M50.

1. **Introduction and preliminaries**

Throughout this paper, S always stands for a monoid and $\mathbb{N}$ the set of natural numbers. A non-empty set A is called a *right S-act*, usually denoted by A_S, if there exists a mapping $A \times S \to A$, $(a, s) \mapsto as$, such that $(as)t = a(st)$ and $a1 = a$ for all $a \in A$ and $s, t \in S$. Left S-acts $_SA$ are defined dually. Every right (left) ideal I of S is in a natural way a right (resp. left) S-act. Let A_S and B_S be two right S-acts. A mapping $f : A_S \to B_S$ is called an *S-morphism* if $f(as) = f(a)s$ for all $a \in A$ and $s \in S$. Analogously, S-morphisms of left S-acts are defined.

In 1970, Kilp [7] initiated a study of flatness of acts. A right S-act A_S is called *flat* if the functor $A_S \otimes -$ preserves all monomorphisms. In 1983, Kilp [9] further investigated the (principal) weak version of flatness under the name of (principal) weak flatness. A right S-act A_S is called (*principally*) *weakly flat* if this functor $A_S \otimes -$ preserves all embeddings of (principal) left ideals into S.

Article electronically published on October 31, 2016.
Received: 16 January 2015, Accepted: 23 June 2015.
*Corresponding author.

In 1987, Normark [13] studied Condition (P), which was first considered by Stenström [17]. A right S-act A_S is said to satisfy *Condition* (P) if for all $a, a' \in A$, $s, t \in S$, $as = a't$ implies that there exist $a'' \in A$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vt$. As was shown in [13] that Condition (P) strictly implies flatness. Until 2001, Laan [11] defined Conditions (WP) and (PWP), which are the weak form and principal weak form of Condition (P), respectively. A right S-act A_S satisfies Condition (WP) if and only if, for all elements $s, t \in S$, all S-morphisms $f : {}_S(Ss \cup St) \to_S S$, and all $a, a' \in A$, if $af(s) = a'f(t)$ then there exist $a'' \in A$, $u, v \in S$, $s', t' \in \{s, t\}$ such that $f(us') = f(vt')$, $a \otimes s = a'' \otimes us'$ and $a' \otimes t = a'' \otimes vt'$ in $A_S \otimes_S (Ss \cup St)$. A right S-act A_S satisfies Condition (PWP) if and only if, for all $a, a' \in A$, $s \in S$, $as = a's$ implies that there exist $a'' \in A$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vs$. Also, we know from [11] that Condition (WP) strictly implies weak flatness, and Condition (PWP) strictly implies principal weak flatness.

In 2012, Qiao et al. [15] defined GP-flatness of acts, which is a generalization of principal weak flatness. Moreover, using this property, some new classes of monoids are characterized, such as generally regular monoids, generally left almost regular monoids and so on. A right S-act A_S is called *GP-flat*, if for any $s \in S$, $a, a' \in A$, $a \otimes s = a' \otimes s$ in $A_S \otimes_S S$ implies that there exists $n \in \mathbb{N}$ such that $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes_S Ss^n$. According to the above statements, the relations of these properties are as follows:

$$\begin{array}{ccccccc} \text{Condition (P)} & \Rightarrow & \text{Condition (WP)} & \Rightarrow & \text{Condition (PWP)} & & \\ \Downarrow & & \Downarrow & & \Downarrow & & \\ \text{flatness} & \Rightarrow & \text{w. flatness} & \Rightarrow & \text{p. w. flatness} & \Rightarrow & \text{GP-flatness} \end{array}$$

Motivated by the work of [15], in this paper, we naturally investigate a generalization of Condition (PWP). Moreover, we prove that this generalization can imply GP-flatness.

In Section 2, we first introduce Condition GP-(P), and characterize monoids by this new property of their acts. Furthermore, many unknown results for Condition (PWP) are obtained, such as Theorems 2.10, 2.11 and so on. From [14, Theorem 2.4], we know that all torsion free right S-acts satisfy Condition (PWP) if and only if S is a right cancellative monoid. But the situation for Condition GP-(P) is slightly different.

In Section 3, we define a new ideal of a monoid S, and present an equivalent description of Rees factor acts satisfying Condition GP-(P). Moreover, we determine the relationship between Condition GP-(P) and GP-flatness.

In [4], Bulman-Fleming and Gilmour initiated the study of flatness properties of diagonal acts, such as freeness, strongly flatness, Condition (P) and so on. In Section 4, similar to the techniques of [4], we continue to investigate the diagonal acts satisfying Condition GP-(P).

In [17], Stenström proved that a right S-act A_S is strongly flat if and only if every epimorphism $\psi : B_S \to A_S$ is pure. Recently, Bailey and Renshaw [1] defined a generalization of pure epimorphisms called n-pure epimorphisms ($n \in \mathbb{N}$), and gave a necessary characterization of Condition (P) by 2-pure epimorphisms (see [1, Proposition 3.12]). However, the situation for Condition (PWP) is presently unknown. In Section 5, we give some new generalizations of 2-pure epimorphisms, and obtain an equivalent description of Condition GP-(P) (resp., Condition (PWP)). Moreover, we study directed colimits of these new epimorphisms.

For more details about the notions used in this paper, we refer the reader to [6, 10], and for an account of flatness properties of acts the reader is referred to [3, 5, 11].

2. Acts satisfying Condition GP-(P)

In this section we give a characterization of monoids by Condition GP-(P) of right acts.

Definition 2.1. We say a right S-act A_S satisfies *Condition GP-(P)* if for any $a, a' \in A$ and $s \in S$, $as = a's$ implies that there exist $n \in \mathbb{N}$, $a'' \in A$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us^n = vs^n$.

It is immediate from the above definition that, if $n = 1$, then Condition GP-(P) is in fact Condition (PWP). So Condition GP-(P) is a general form of Condition (PWP), but we will show that Condition GP-(P) does not imply Condition (PWP) (at the end of Section 3).

We first present an equivalent description of Condition GP-(P) by pullback diagram.

Proposition 2.2. *A right S-act A_S satisfies Condition GP-(P) if and only if, for all $a, a' \in A$, $x, y, s \in S$, and all S-morphisms $f : {}_SSs \to {}_SS$, $af(xs) = a'f(ys)$ implies that there exist $n \in \mathbb{N}$, $a'' \in A$, $u, v \in S$ such that $f(us^n) = f(vs^n)$, $a \otimes xs^n = a'' \otimes us^n$ and $a' \otimes ys^n = a'' \otimes vs^n$ in $A_S \otimes_S Ss^n$.*

Proof. **Necessity**. This follows immediately from Definition 2.1.

Sufficiency. Let $at = a't$ for $a, a' \in A$, $t \in S$. We consider the right translation (by t) $\rho_t : {}_SS \to {}_SS$, i.e., $\rho_t(z) = zt$ for every $z \in S$. Then $at = a't$ means that $a\rho_t(1) = a'\rho_t(1)$. By our assumption, there exist $n \in \mathbb{N}$, $a'' \in A$, $u, v \in S$ such that $\rho_t(u1^n) = \rho_t(v1^n)$, $a \otimes 1^n = a'' \otimes u1^n$ and $a' \otimes 1^n = a'' \otimes v1^n$ in $A_S \otimes_S S$. Using the definition of ρ_t and the standard isomorphism between $A_S \otimes_S S$ and A_S, it follows that $ut = vt$, $a = a''u$ and $a' = a''v$. This shows that A_S satisfies Condition GP-(P). □

The following proposition shows that right S-acts satisfying Condition GP-(P) are closed under directed colimits. For more information about directed colimit of families of right S-acts, the reader is referred to [1].

Proposition 2.3. *Let S be a monoid. Then every directed colimit of a direct system of right S-acts that satisfy Condition GP-(P), satisfies Condition GP-(P).*

Proof. Let $(A_i, \phi_{i,j})$ be a direct system of right S-acts satisfying Condition GP-(P) over a directed index set I with directed colimit (A_S, α_i). Suppose that $as = a's$ in A_S for $a, a' \in A$, $s \in S$. Then there exist $i, j \in I$, $a_i \in A_i$, $a_j \in A_j$ with $a = \alpha_i(a_i)$, $a' = \alpha_j(a_j)$. Since I is directed, by [1, Lemma 2.1], there exists $k \geq i, j$ with $\phi_{i,k}(a_i)s = \phi_{j,k}(a_j)s$ in A_k. Since A_k satisfies Condition GP-(P), there exist $n \in \mathbb{N}$, $a'' \in A_k$ and $u, v \in S$ such that $\phi_{i,k}(a_i) = a''u$, $\phi_{j,k}(a_j) = a''v$ and $us^n = vs^n$. We can calculate that $a = \alpha_i(a_i) = \alpha_k\phi_{i,k}(a_i) = \alpha_k(a'')u$. In a similar way, $a' = \alpha_k(a'')v$, and so A_S satisfies Condition GP-(P). □

Notice from the proof of Proposition 2.3 that we can also show that right S-acts satisfying Condition (PWP) are closed under directed colimits.

From [14, Lemma 2.1], we know that the S-act $A(I)$ does not satisfy Condition (PWP). Even for Condition GP-(P), the result is still valid.

Lemma 2.4. *Let I be a proper right ideal of a monoid S. Then $A(I)$ fails to satisfy Condition GP-(P).*

Proof. The proof is similar to that of [14, Lemma 2.1]. □

Golchin et al. [5] investigated Condition (P') lying strictly between Condition (P) and Condition (PWP). A right S-act A_S is said to satisfy *Condition* (P') if for all $a, a' \in A$, $s, t, z \in S$, $as = a't$ and $sz = tz$ imply that there exist $a'' \in A$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us = vt$. Also, according to [3, Proposition 9], all right S-acts are weakly pullback flat if and only if S is a group. Further, using Lemma 2.4, the following proposition is an evident result for Condition GP-(P).

Proposition 2.5. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All right S-acts are weakly pullback flat;*
(2) *All right S-acts satisfy Condition (P);*
(3) *All right S-acts satisfy Condition (P');*
(4) *All right S-acts satisfy Condition (WP);*
(5) *All right S-acts satisfy Condition (PWP);*
(6) *All right S-acts satisfy Condition GP-(P);*
(7) *S is a group.*

As we know that Condition (PWP) implies principal weak flatness. Parallel to this fact, we have the following result.

Proposition 2.6. *If a right S-act A_S satisfies Condition GP-(P), then A_S is GP-flat.*

Proof. Suppose that A_S satisfies Condition GP-(P) and let $a \otimes s = a' \otimes s$ in $A_S \otimes_S S$ for $a, a' \in A$ and $s \in S$. Using the standard isomorphism between $A_S \otimes_S S$ and A_S, it follows $as = a's$. By assumption, there exist $n \in \mathbb{N}$, $a'' \in A$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $us^n = vs^n$. So we can compute that

$$a \otimes s^n = a''u \otimes s^n = a'' \otimes us^n = a'' \otimes vs^n = a''v \otimes s^n = a' \otimes s^n$$

in the tensor product $A_S \otimes_S Ss^n$. Hence A_S is GP-flat. □

Notice that the implication in the above proposition is strict, we will give an example to show this fact.

Now it is natural to consider monoids over which all GP-flat right acts satisfy Condition GP-(P). In what follows we need the following notions and results.

Recall that a monoid S is called a *left PP monoid* if every principal left ideal of S is projective (as a left S-act). According to Kilp [8], a monoid S is left PP if and only if for every $s \in S$ there exists an idempotent $e \in S$ such that $es = s$ and for all $u, v \in S$, $us = vs$ implies $ue = ve$. By Liu and Yang [12], a monoid S is called a *left PSF monoid* if every principal left ideal of S is strongly flat (as a left S-act); Or, equivalently, a monoid S is left PSF if and only if $su = tu$ for $s, t, u \in S$ there exists $r \in S$ such that $ru = u$ and $sr = tr$. It is clear that every left PP monoid is left PSF.

Lemma 2.7. ([2]) *Let I be a proper right ideal of a monoid S. Then the following statements are equivalent:*

(1) *$A(I)$ is flat;*

(2) *I is left stabilizing (that is, for every $j \in I$, $j \in Ij$).*

Lemma 2.8. ([12]) *The following statements on a monoid S are equivalent:*

(1) *Every proper right ideal I of S is not left stabilizing (that is, there exists $j \in I - Ij$);*

(2) *For every infinite sequence $(x_0, x_1, x_2, \cdots)$ with $x_i = x_{i+1}x_i$, $x_i \in S$, $i = 0, 1, \cdots$, there exists a positive integer n such that $x_n = x_{n+1} = \cdots = 1$.*

Lemma 2.9. *Let S be a left PSF monoid. A right S-act A_S is GP-flat if and only if, for every $a, a' \in A$ and $s \in S$, $as = a's$ implies that there exist $n \in \mathbb{N}$, $u \in S$ such that $au = a'u$ and $us^n = s^n$.*

Proof. **Necessity.** Suppose that A_S is a GP-flat right S-act. If $as = a's$ for $a, a' \in A$, $s \in S$, then we have $a \otimes s = a' \otimes s$ in $A_S \otimes_S S$. In view of [15, Lemma 2.2], there exist $m, n \in \mathbb{N}$, $s_1, t_1, \cdots, s_m, t_m \in S$ and $a_1, \cdots, a_m \in A$ such that

$$\begin{aligned} a &= a_1 s_1 & & \\ a_1 t_1 &= a_2 s_2 & s_1 s^n &= t_1 s^n \\ a_2 t_2 &= a_3 s_3 & s_2 s^n &= t_2 s^n \\ &\vdots & &\vdots \\ a_m t_m &= a' & s_m s^n &= t_m s^n. \end{aligned}$$

Since S is left PSF, from $s_1s^n = t_1s^n$ we obtain $u_1 \in S$ such that $u_1s^n = s^n$ and $s_1u_1 = t_1u_1$. So we deduce $s_2u_1s^n = s_2s^n = t_2s^n = t_2u_1s^n$, and this implies that there exists $u_2 \in S$ such that $u_2s^n = s^n$ and $s_2u_1u_2 = t_2u_1u_2$. Now letting $u' = u_1u_2 \in S$, we have $u's^n = s^n$, $s_1u' = t_1u'$ and $s_2u' = t_2u'$. By induction we can find $u \in S$ such that

$$us^n = s^n, \quad s_iu = t_iu, \quad i = 1, 2, \cdots, m.$$

Hence we can deduce that

$$au = a_1s_1u = a_1t_1u = a_2s_2u = \cdots = a_ms_mu = a_mt_mu = a'u.$$

Sufficiency. Let $a \otimes s = a' \otimes s$ in $A_S \otimes_S S$ for $a, a' \in A$, $s \in S$. This means $as = a's$ in A_S, and so by assumption, there exist $n \in \mathbb{N}$, $u \in S$ such that $au = a'u$ and $us^n = s^n$. Now we can compute that

$$a \otimes s^n = a \otimes us^n = au \otimes s^n = a'u \otimes s^n = a' \otimes us^n = a' \otimes s^n$$

in $A_S \otimes_S Ss^n$. This completes the proof. □

Now we can establish the following result.

Theorem 2.10. *Let S be a left PSF monoid. Then the following statements are equivalent:*

(1) *All principally weakly flat right S-acts satisfy Condition GP-(P);*
(2) *All principally weakly flat right S-acts satisfy Condition (PWP);*
(3) *All GP-flat right S-acts satisfy Condition GP-(P);*
(4) *All GP-flat right S-acts satisfy Condition (PWP);*
(5) *Every proper right ideal I of S is not left stabilizing.*

Proof. The implications (4) ⇒ (3) ⇒ (1) and (4) ⇒ (2) ⇒ (1) are obvious.

(1) ⇒ (5). This follows from Lemmas 2.4 and 2.7.

(5) ⇒ (4). Suppose that A_S is a GP-flat right S-act. Let $as = a's$ for $a, a' \in A$ and $s \in S$. In view of Lemma 2.9, there exist $n \in \mathbb{N}$, $u \in S$ such that $au = a'u$ and $us^n = s^n$. Since S is left PSF, from $us^n = s^n$ we get $x_1 \in S$ such that $x_1s^n = s^n$ and $ux_1 = x_1$. Again, since S is left PSF, from $ux_1 = x_1$ we obtain $x_2 \in S$ such that $x_2x_1 = x_1$ and $ux_2 = x_2$. By continuing this process, letting $x_0 = s^n$ we can find an infinite sequence $(x_0, x_1, \cdots)$, such that

$$x_{i+1}x_i = x_i, \quad ux_i = x_i, \quad i = 0, 1, \cdots.$$

By Lemma 2.8, there exists a positive integer m such that $x_m = x_{m+1} = \cdots = 1$. Thus, we have $u = 1$, and so $a = a'$. This shows that A_S satisfies Condition (PWP). □

For the situation of idempotent monoids, we have the following

Theorem 2.11. *Let S be an idempotent monoid. Then the following statements are equivalent:*

(1) *All flat right S-acts satisfy Condition GP-(P);*

(2) *All weakly flat right S-acts satisfy Condition GP-(P);*
(3) *All principally weakly flat right S-acts satisfy Condition GP-(P);*
(4) *All GP-flat right S-acts satisfy Condition GP-(P);*
(5) $S = \{1\}$.

Proof. The implications (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are obvious.

(1) $\Rightarrow$ (5). Assume $S \neq \{1\}$. If $e \in S\backslash\{1\}$, then $eS \neq S$. Since $I = eS$ is a left stabilizing proper right ideal of S, from Lemma 2.7, it follows that $A(I)$ is flat. By assumption, $A(I)$ satisfies Condition GP-(P), this contradicts Lemma 2.4. Thus, we have $S = \{1\}$.

(5) $\Rightarrow$ (4). It is clear. □

From [14, Theorem 2.4], it follows that all torsion free right S-acts satisfy Condition (PWP) if and only if S is a right cancellative monoid. But, the situation for Condition GP-(P) is slightly different.

Theorem 2.12. *Let S be a left PP monoid. Then the following statements are equivalent:*

(1) *All flat right S-acts satisfy Condition GP-(P);*
(2) *All weakly flat right S-acts satisfy Condition GP-(P);*
(3) *All principally weakly flat right S-acts satisfy Condition GP-(P);*
(4) *All GP-flat right S-acts satisfy Condition GP-(P);*
(5) *All torsion free right S-acts satisfy Condition GP-(P);*
(6) *S is a right cancellative monoid.*

Proof. The implications (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are obvious.

(1) $\Rightarrow$ (6). If S is not right cancellative, then $I = \{s \in S | s$ is not right cancellable$\}$ is a proper right ideal of S. Next we show that $A(I)$ is flat. For every $i \in I$, since S is a left PP monoid, there exists $e \in E(S)$ such that $ei = i$ and for every $u, v \in S$, $ui = vi$ implies $ue = ve$. If $e \notin I$, then e is right cancellable. So for every $u, v \in S$, $ui = vi$ implies $u = v$. Therefore, i is right cancellable, which is a contradiction. Thus $e \in I$ and $ei = i$, from Lemma 2.7 it follows that $A(I)$ is flat. By assumption, $A(I)$ satisfies Condition GP-(P), this contradicts Lemma 2.4. Therefore, S is right cancellative.

(6) $\Rightarrow$ (5). If S is right cancellative, then by [14, Theorem 2.4], all torsion free right S-acts satisfy Condition (PWP), and so all torsion free right S-acts satisfy Condition GP-(P). □

Note that in this theorem, if we replace "a left PP monoid" by "a regular monoid", then "a right cancellative monoid" will be replaced by "a group". And if we replace "a left PP monoid" by "a left PSF monoid", "a left almost regular monoid" or "there exists a regular left S-act", then we have the same result as for the left PP monoid.

From [5, Theorem 2.6 (Theorem 2.7)], it follows that all right S-acts satisfying Condition (P') are free (projective generators) if and only if $S = \{1\}$. By analogy with Condition (P'), we give the following result.

Proposition 2.13. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All right S-acts satisfying Condition GP-(P) are free (projective generators);*
(2) *All finitely generated right S-acts satisfying Condition GP-(P) are free (projective generators);*
(3) *All cyclic right S-acts satisfying Condition GP-(P) are free (projective generators);*
(4) *All monocyclic right S-acts satisfying Condition GP-(P) are free (projective generators);*
(5) $S = \{1\}$.

Proof. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious.

(4) ⇒ (5). Suppose that all monocyclic right S-acts satisfying Condition GP-(P) are free (projective generators). Then all monocyclic right S-acts satisfying Condition (P) are free (projective generators), and so by [10, Theorem 4.12.8], $S = \{1\}$.

(5) ⇒ (1). It is obvious. □

As we know monoids over which all right S-acts satisfying Condition (PWP) are free (projective generators) are unknown since now. Here, from Proposition 2.13 we immediately get the following result as a corollary.

Corollary 2.14. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All right S-acts satisfying Condition (PWP) are free (projective generators);*
(2) *All finitely generated right S-acts satisfying Condition (PWP) are free (projective generators);*
(3) *All cyclic right S-acts satisfying Condition (PWP) are free (projective generators);*
(4) *All monocyclic right S-acts satisfying Condition (PWP) are free (projective generators);*
(5) $S = \{1\}$.

3. Cyclic (Rees factor) acts satisfying Condition GP-(P)

In this section we give a classification of monoids by Condition GP-(P) of their cyclic (Rees factor) acts.

We first give a description for cyclic acts satisfying Condition GP-(P).

Proposition 3.1. *Let ρ be a right congruence on a monoid S. The cyclic right S-act S/ρ satisfies Condition GP-(P) if and only if*

$$(\forall x, y, s \in S)[(xs)\rho(ys) \Longrightarrow (\exists n \in \mathbb{N})(\exists u, v \in S)(x\rho u \wedge y\rho v \wedge us^n = vs^n)].$$

Proof. It is a routine matter. □

Recall from [15] that a monoid S is called *generally regular*, if for every $s \in S$, there exist $n \in \mathbb{N}$ and $x \in S$ such that $s^n = sxs^n$.

Theorem 3.2. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All cyclic right S-acts satisfy Condition GP-(P);*
(2) *S is generally regular, and S satisfies the following condition: for every $x, y, s \in S$, there exist $n \in \mathbb{N}$ and $u, v \in S$ such that $x\rho(xs, ys)u$, $y\rho(xs, ys)v$ and $us^n = vs^n$.*

Proof. (1) ⇒ (2). Suppose that all cyclic right S-acts satisfy Condition GP-(P). Then all cyclic right S-acts are GP-flat, and so by [15, Theorem 3.7], S is generally regular. Now, let $x, y, s \in S$. By assumption, $S/\rho(xs, ys)$ satisfies Condition GP-(P). Since $(xs)\rho(xs, ys)(ys)$, from Proposition 3.1 we obtain $n \in \mathbb{N}$ and $u, v \in S$ such that $x\rho(xs, ys)u$, $y\rho(xs, ys)v$ and $us^n = vs^n$, as required.

(2) ⇒ (1). Let ρ be a right congruence on S and let $(xs)\rho(ys)$ for $x, y, s \in S$. By assumption, there exist $n \in \mathbb{N}$ and $u, v \in S$ such that $x\rho(xs, ys)u$, $y\rho(xs, ys)v$ and $us^n = vs^n$. Since $(xs)\rho(ys)$, we have $\rho(xs, ys) \subseteq \rho$, and so $x\rho u$ and $y\rho v$. From Proposition 3.1, it follows that S/ρ satisfies Condition GP-(P). □

Recall that a proper right ideal K of a monoid S is called *left annihilating*, if

$$(\forall s \in S)(\forall x, y \in S\backslash K)(xs, ys \in K \Longrightarrow xs = ys).$$

In order to depict Rees factor acts satisfying Condition GP-(P), we need to introduce the following

Definition 3.3. We say that a proper right ideal K of a monoid S is *generally left annihilating*, if

$$(\forall s \in S)(\forall x, y \in S\backslash K)[xs, ys \in K \Longrightarrow (\exists\, n \in \mathbb{N})(xs^n = ys^n)].$$

It is obvious that, every left annihilating proper right ideal of a monoid S is generally left annihilating. But, that the converse is not true follows from the next example.

Example 3.4. Let $S = \{1, e, f, 0\}$ denote the monoid with the Cayley table

	1	e	f	0
1	1	e	f	0
e	e	0	0	0
f	f	0	f	0
0	0	0	0	0

and let $K_S = \{e, 0\}$. Then K is a proper right ideal of S. Since $1, f \in S \backslash K$, $1e, fe \in K$, but $1e \neq fe$, so K is not left annihilating. However, we can verify that K is generally left annihilating.

We will use the following result.

Lemma 3.5. ([15]) *Let K be a proper right ideal of a monoid S. Then the following conditions are equivalent:*

(1) *S/K is GP-flat;*

(2) *For every $s \in S$, if there exists $r \in S \backslash K$ such that $rs \in K$, then there exist a natural number $n \in \mathbb{N}$, and $j \in K$ such that $rs^n = js^n$.*

For convenience, we say a proper right ideal K of a monoid S is *generally left stabilizing*, if K satisfies the condition (2) of Lemma 3.5.

For a Rees factor act to satisfy Condition GP-(P) we can now give the following description.

Theorem 3.6. *Let K be a proper right ideal of a monoid S. Then the right Rees factor act S/K satisfies Condition GP-(P) if and only if K is generally left stabilizing and generally left annihilating.*

Proof. **Necessity.** If S/K satisfies Condition GP-(P), then S/K is GP-flat, and so by Lemma 3.5, K is generally left stabilizing. So assume that $xs, ys \in K$ for $x, y \in S \backslash K$, $s \in S$. Then we have $(xs)\rho_K(ys)$. In view of Proposition 3.1, there exist $n \in \mathbb{N}$ and $u, v \in S$ such that $us^n = vs^n$, $x\rho_K u$ and $y\rho_K v$. Now $x, y \notin K$ yields $x = u$ and $y = v$ by the definition of ρ_K. Thus, we have $xs^n = ys^n$, and this shows that K is generally left annihilating.

Sufficiency. Let K be a generally left stabilizing and generally left annihilating, proper right ideal of S. We use Proposition 3.1 to check that S/K satisfies Condition GP-(P). Let $(xs)\rho_K(ys)$ for $x, y, s \in S$. If $xs = ys$, then we can take $u = x$ and $v = y$. So we may assume that $xs, ys \in K$. We have the following four cases to consider.

Case 1: $x, y \in K$. Then we can take $u = v = x$;

Case 2: $x \in K$, $y \notin K$. Since K is generally left stabilizing, we can find for $ys \in K$ a natural number $n \in \mathbb{N}$ and an element $k \in K$ such that $ys^n = ks^n$. It remains to take $u = k$ and $v = y$;

Case 3: This is analogous to the previous case;

Case 4: $x, y \notin K$. Since K is generally left annihilating, there exists $n \in \mathbb{N}$ such that $xs^n = ys^n$, so we take $u = x$ and $v = y$.

As stated above, S/K satisfies Condition GP-(P). □

For the one-element act, we can easily prove

Corollary 3.7. *For any monoid S, the one-element act Θ_S always satisfies Condition GP-(P).*

Our next task is to describe monoids for which all Rees factor acts satisfy Condition GP-(P).

Theorem 3.8. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All right Rees factor acts of S satisfy Condition GP-(P);*
(2) *S is a generally regular monoid, and also for all $x, y, s \in S$, $x, y \notin xsS \cup ysS$ implies that $xs^n = ys^n$ for some $n \in \mathbb{N}$.*

Proof. (1) ⇒ (2). Suppose that all right Rees factor acts of S satisfy Condition GP-(P). Then all right Rees factor acts of S are GP-flat, and so by [15, Theorem 3.7], S is a generally regular monoid. Now let $x, y \notin xsS \cup ysS$ for $x, y, s \in S$ and let $K_S = xsS \cup ysS$. By assumption, S/K satisfies Condition GP-(P), and so by Theorem 3.6, K is generally left annihilating. Since $xs, ys \in K$, we can find a natural number n such that $xs^n = ys^n$, exactly as needed.

(2) ⇒ (1). Suppose that K is a right ideal of S. If K is proper, since S is generally regular, it is easy to verify that K is generally left stabilizing. Now let $xs, ys \in K$ for $x, y \in S \backslash K$, $s \in S$. Then $xsS \cup ysS \subseteq K$, and so $x, y \notin xsS \cup ysS$. By assumption, there exists $n \in \mathbb{N}$ such that $xs^n = ys^n$, and this shows that K is generally left annihilating. Thus, using Theorem 3.6, S/K satisfies Condition GP-(P). But if $K = S$, then by Corollary 3.7, $S/K \cong \Theta_S$ satisfies Condition GP-(P), and the proof is complete. □

As we will see that, for Rees factor acts Condition GP-(P) does not imply Condition (PWP). So it is natural to consider the following

Theorem 3.9. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All right Rees factor acts of S satisfying Condition GP-(P) satisfy Condition (PWP);*
(2) *Every generally left stabilizing and generally left annihilating proper right ideal of S is left stabilizing and left annihilating.*

Proof. (1) ⇒ (2). If K is a generally left stabilizing and generally left annihilating proper right ideal of S, then by Theorem 3.6, S/K satisfies Condition GP-(P). By assumption, S/K satisfies Condition (PWP). Thus, from [11, Theorem 10], it follows that K is left stabilizing and left annihilating.

(2) ⇒ (1). Suppose that K is a right ideal of S and S/K satisfies Condition GP-(P). If K is proper, then by Theorem 3.6, K is generally left stabilizing and generally left annihilating. By assumption, K is left stabilizing and left

annihilating, and so from [11, Theorem 10] it follows that S/K satisfies Condition (PWP). But if $K = S$, then by [11, Corollary 11], $S/K \cong \Theta_S$ satisfies Condition (PWP), and so we are done. □

For monoids over which every Rees factor act satisfying Condition GP-(P) have other flatness property (where here, the property is stronger in general than Condition GP-(P)), their characterizations are similar in nature to the above theorem, and so here will be omitted.

As we will also see for Rees factor acts GP-flatness does not imply Condition GP-(P), so we naturally consider the following

Theorem 3.10. *Let S be a monoid. Then the following statements are equivalent:*

(1) *All GP-flat right Rees factor acts of S satisfy Condition GP-(P);*
(2) *Every generally left stabilizing proper right ideal of S is generally left annihilating.*

Proof. (1) ⇒ (2). Suppose that K is a generally left stabilizing proper right ideal of S. In view of Lemma 3.5, S/K is GP-flat. By assumption, S/K satisfies Condition GP-(P), and so by Theorem 3.6, K is generally left annihilating.

(2) ⇒ (1). Suppose that K is a right ideal of S and S/K is GP-flat. If K is proper, then by Lemma 3.5, K is generally left stabilizing. By assumption, K is also generally left annihilating, and so using Theorem 3.6, S/K satisfies Condition GP-(P). But if $K = S$, then from Corollary 3.7, it follows that $S/K \cong \Theta_S$ satisfies Condition GP-(P). This completes the proof. □

At the end of this section, we present two examples to show that Condition GP-(P) lies strictly between Condition (PWP) and GP-flatness. Meanwhile, these two examples can also reveal that Condition GP-(P) and principal weak flatness (resp., weak flatness, flatness) are independent notions.

Example 3.11. ([11, Example 1]) [GP-flatness $\not\Rightarrow$ Condition GP-(P)] Let $S = \{1, e, f, 0\}$ be a semilattice with $ef = 0$ and $K = eS$. It is clear that K is a proper right ideal of S. It is shown in [11] that S/K is flat. Hence S/K is GP-flat. On the other hand, since $1, f \in S\backslash K$, $1e, fe \in K$, there is no natural number n such that $1e^n = fe^n$. Hence K is not generally left annihilating, and so by Theorem 3.6, S/K does not satisfy Condition GP-(P).

Example 3.12. [Condition GP-(P) $\not\Rightarrow$ Condition (PWP)] Let $S = K \cup \{I\}$ with

$$K = \left\{ \left(\begin{array}{ccc} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{array} \right) \middle| a, b, c \in \mathbb{Z} \right\}, \quad I = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right).$$

Then S is a monoid and K is a generally left stabilizing and generally left annihilating proper right ideal. In view of Theorem 3.6, S/K satisfies Condition

GP-(P). On the other hand, since $l = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in K$, there is no $k \in K$ such that $kl = l$. Hence K is not left stabilizing. From [11, Proposition 8], it follows that S/K is not principally weakly flat, so S/K is not (weakly) flat, and also fails to satisfy Condition (PWP).

4. Diagonal acts satisfying Condition GP-(P)

In this section, our goal is to give some possible conditions on a monoid that describe when its diagonal act satisfies Condition GP-(P).

If S is a monoid, then the right S-act $S \times S$, equipped with componentwise S-action, that is $(s,t)u = (su, tu)$ for $s, t, u \in S$, is called the *diagonal right act of* S. This act will be denoted by $D(S)$. If S is a semigroup without identity, then the right S^1-act $(S \times S)_{S^1}$ is called *deleted diagonal right act of* S, usually denoted by $D_d(S)$. For more information about the diagonal acts the reader is referred to [4, 16].

Let S be a monoid. For the diagonal right act $D(S)$, we define

$$L(s^n, s^n) := \{(u,v) \in D(S) |\ us^n = vs^n\}$$

for any $n \in \mathbb{N}$. Clearly, $L(s^n, s^n)$ is a left S-act.

We first provide an alternative description for Condition GP-(P) when it comes to diagonal acts.

Theorem 4.1. *Let S be a monoid. Then the following statements are equivalent:*

(1) *$D(S)$ satisfies Condition GP-(P);*
(2) *For every $s \in S$, $L(s,s)$ is either empty or for each 2 elements (u,v), $(u',v') \in L(s,s)$, there exists $(p,q) \in L(s^n, s^n)$ for some $n \in \mathbb{N}$ such that $(u,v), (u',v') \in S(p,q)$.*

Proof. (1) $\Rightarrow$ (2). Suppose that $D(S)$ satisfies Condition GP-(P). Let (u,v), $(u',v') \in L(s,s)$ for any $s \in S$. Then we have $us = vs$ and $u's = v's$, from which it follows that $(u,u')s = (v,v')s$. Since $D(S)$ satisfies Condition GP-(P), there exist $n \in \mathbb{N}$, $(w,w') \in D(S)$ and $p, q \in S$ such that $ps^n = qs^n$, $(u,u') = (w,w')p$ and $(v,v') = (w,w')q$. This translates into $(p,q) \in L(s^n, s^n)$ and $(u,v), (u',v') \in S(p,q)$, as required.

(2) $\Rightarrow$ (1). Let $(a,b)s = (a',b')s$ for $(a,b), (a',b') \in D(S)$, $s \in S$. Then we see $as = a's$ and $bs = b's$, these imply that $(a,a'), (b,b') \in L(s,s) \neq \emptyset$. By assumption, we get $(p,q) \in L(s^n, s^n)$ for some $n \in \mathbb{N}$ such that $(a,a'), (b,b') \in S(p,q)$, that is, there exist $w, w' \in S$ such that $(a,a') = w(p,q)$ and $(b,b') = w'(p,q)$. It follows that $ps^n = qs^n$, $(a,b) = (w,w')p$ and $(a',b') = (w,w')q$. Hence $D(S)$ satisfies Condition GP-(P). $\square$

As an extension of Theorem 4.1, the following result is obtained.

Theorem 4.2. *Let S be a monoid. Then the following statements are equivalent:*

(1) $(S^I)_S$ *satisfies Condition GP-(P) for every nonempty set I;*
(2) *For every $s \in S$, $L(s,s)$ is either empty or there exists $(p,q) \in L(s^n,s^n)$ for some $n \in \mathbb{N}$ such that $L(s,s) \subseteq S(p,q)$.*

Proof. (1) $\Rightarrow$ (2). Assume $s \in S$ and $L(s,s) \neq \emptyset$. Write $L(s,s) = \{(u_i,v_i) | \ i \in I\}$. Let u, v be the elements of S^I whose ith components are u_i, v_i, respectively. Then we see $us = vs$ in S^I. Since S^I satisfies Condition GP-(P), there exist $n \in \mathbb{N}$, $w \in S^I$ and $p, q \in S$ such that $u = wp$, $v = wq$ and $ps^n = qs^n$. So $(p,q) \in L(s^n,s^n)$. Now let the ith component of w be w_i $(i \in I)$. Then for each $i \in I$ we have $u_i = w_ip$ and $v_i = w_iq$. From this it follows that $(u_i,v_i) = w_i(p,q)$ for all $i \in I$, and so $L(s,s) \subseteq S(p,q)$.

(2) $\Rightarrow$ (1). Suppose that $us = vs$ for $u, v \in S^I$ and $s \in S$. Let the ith components of u and v be u_i and v_i $(i \in I)$, respectively. Then we have $u_is = v_is$ for all $i \in I$, and so $(u_i,v_i) \in L(s,s) \neq \emptyset$. By assumption, there exists $(p,q) \in L(s^n,s^n)$ for some $n \in \mathbb{N}$ such that $L(s,s) \subseteq S(p,q)$. So for each $i \in I$ there exists $w_i \in S$ such that $(u_i,v_i) = w_i(p,q)$. Setting $w = (w_i)_{i \in I}$, we have $ps^n = qs^n$, $u = wp$ and $v = wq$, exactly as needed. □

It is well-known that, for any monoid S the diagonal act $D(S)$ is always torsion free. From Theorem 2.12, it follows that if S is a right cancellative monoid then $D(S)$ satisfies Condition GP-(P). So it is natural to consider the following

Proposition 4.3. *Let S be a monoid. Then the following statements are equivalent:*

(1) $D(S)$ *satisfies Condition (P') and $|E(S)| = 1$;*
(2) $D(S)$ *satisfies Condition (PWP) and $|E(S)| = 1$;*
(3) $D(S)$ *satisfies Condition GP-(P) and $|E(S)| = 1$;*
(4) S *is right cancellative.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4) Suppose that $D(S)$ satisfies Condition GP-(P) and $|E(S)| = 1$. Let $sz = tz$, for $s, t, z \in S$. Then we have $(1,s)z = (1,t)z$, and by assumption, we obtain $n \in \mathbb{N}$, $(c,d) \in D(S)$ and $u, v \in S$ such that $uz^n = vz^n$, $(1,s) = (c,d)u$ and $(1,t) = (c,d)v$. Further, $(1,s) = (c,d)u$ and $(1,t) = (c,d)v$ imply that $1 = cu = cv$, and so we can deduce $uc, vc \in E(S)$. From the condition $|E(S)| = 1$, it follows $uc = vc$. Then we may compute that $u = ucu = vcu = v$. Thus, $s = du = dv = t$, and we are done.

(4) $\Rightarrow$ (1). Assume S is a right cancellative monoid. It is easy to see that $|E(S)| = 1$. Since $D(S)$ is torsion free, by [5, Theorem 2.2], $D(S)$ satisfies Condition (P'). □

The next result gives a nice characterization of inverse monoids whose diagonal acts satisfies Condition GP-(P).

Proposition 4.4. *Let S be an inverse monoid. Then the following statements are equivalent:*
(1) *$D(S)$ satisfies Condition (P');*
(2) *$D(S)$ satisfies Condition (PWP);*
(3) *$D(S)$ satisfies Condition GP-(P);*
(4) *S is a group.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Note that S is an inverse and a right cancellative monoid if and only if S is a group. So we only need to prove that S is a right cancellative monoid. Assume $sz = tz$ for $s, t, z \in S$. Then we have $(s, 1)z = (t, 1)z$. Applying Condition GP-(P) to this equality, obtaining $n \in \mathbb{N}$, $x, y, u, v \in S$ such that $(s, 1) = (x, y)u$, $(t, 1) = (x, y)v$ and $uz^n = vz^n$. It then follows that $yu = 1 = yv$, and so $yuy = y$ and $uyu = u$, this shows u is an inverse of y. Similarly, we get v is also an inverse of y. Since S is an inverse monoid, we have $u = v$. Thus, we can deduce that $s = xu = xv = t$, proving that S is right cancellative.

(4) $\Rightarrow$ (1). It is clear. □

For a commutative monoid S, the following result gives a useful description of when $D(S)$ satisfies Condition GP-(P).

Proposition 4.5. *Let S be a commutative monoid. Then the following statements are equivalent:*
(1) *$D(S)$ satisfies Condition (P');*
(2) *$D(S)$ satisfies Condition (PWP);*
(3) *$D(S)$ satisfies Condition GP-(P);*
(4) *S is cancellative.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4) Suppose that $D(S)$ satisfies Condition GP-(P). Let $sz = tz$, for $s, t, z \in S$. From the equality $(1, s)z = (1, t)z$, it follows that there exist $n \in \mathbb{N}$, $(c, d) \in D(S)$ and $u, v \in S$ such that $(1, s) = (c, d)u$, $(1, t) = (c, d)v$ and $uz^n = vz^n$. From $(1, s) = (c, d)u$ and $(1, t) = (c, d)v$ we see $1 = cu = cv$. Since S is commutative, we can calculate that $u = cuu = cvu = cuv = v$. Therefore, $s = du = dv = t$, we obtain the desired result.

(4) $\Rightarrow$ (1). It follows directly from [5, Theorem 6.1]. □

Proposition 4.6. *Let S be a semigroup. Then $D(S^1)$ satisfies Condition GP-(P) if and only if S is right cancellative.*

Proof. In view of Proposition 4.3, it suffices to show $|E(S^1)| = 1$. Suppose, by way of contradiction, that there exists $e \in S$ such that e is idempotent. Then we have $(e,1)e = (1,e)e$. If $D(S^1)$ satisfies Condition GP-(P), then we could find $n \in \mathbb{N}$, $(s,t) \in D(S^1)$ and $u, v \in S$ such that $ue^n = ve^n$, $(e,1) = (s,t)u$ and $(1,e) = (s,t)v$. So we see $tu = 1$ and then because 1 is isolated, $t = u = 1$. But if $t = 1$, then from the equality $(1,e) = (s,t)v$ we must get that $v = e$, which would make it impossible for $sv = 1$. Thus, we have $|E(S^1)| = 1$. $\square$

Observing the proof of Proposition 4.6, we know that if a monoid S with an isolated identity 1 has any non-trivial idempotent, then $D(S)$ will fail to satisfy Condition GP-(P). But, completely 0-simple semigroups surely contain a non-trivial idempotent namely 0. This means that, all completely 0-simple semigroups with an identity adjoined fail to have diagonal acts satisfying Condition GP-(P). In such situation, it is somewhat natural to consider deleted diagonal acts of completely (0-)simple semigroups, represented here as regular Rees matrix semigroups (with zero).

A left group is a semigroup of the form $L \times G$, where L is a left zero semigroup (that is, $st = s$ for all $s, t \in L$) and G is a group. It is not hard to check that if $S = \mathcal{M}(G; I, \Lambda; P)$ is a completely simple semigroup and $|\Lambda| = 1$ then S is in fact a left group.

Proposition 4.7. *Let $S = \mathcal{M}(G; I, \Lambda; P)$ be a completely simple semigroup that is not a group. Then $D_d(S)$ satisfies Condition GP-(P) if and only if S is a left group.*

Proof. **Necessity.** Suppose that $D_d(S)$ satisfies Condition GP-(P), and let $i \in I$, $\lambda, \mu \in \Lambda$. If e denotes the identity element of G, then

$$((i,e,\lambda),(i,e,\mu))(i,e,\mu) = ((i, P_{\lambda i}P_{\mu i}^{-1}, \mu), (i, P_{\mu i}P_{\lambda i}^{-1}, \lambda))(i,e,\mu)$$

and so, by Condition GP-(P), there exist a natural number n and elements $a \in S \times S$, $u, v \in S^1$ such that

$$\begin{aligned} ((i,e,\lambda),(i,e,\mu)) &= au, && (1)\\ ((i, P_{\lambda i}P_{\mu i}^{-1}, \mu), (i, P_{\mu i}P_{\lambda i}^{-1}, \lambda)) &= av, \text{ and} && (2)\\ u(i,e,\mu)^n &= v(i,e,\mu)^n. \end{aligned}$$

From the equality (1) or (2), we can easily deduce that $\lambda = \mu$. Hence, $|\Lambda| = 1$ and S is a left group.

Sufficiency. If S is a left group, then by [4, Proposition 25] $D_d(S)$ satisfies Condition (P), and hence $D_d(S)$ satisfies Condition GP-(P). $\square$

As mentioned above, for any completely 0-simple semigroup with an identity adjoined, its diagonal acts fail to satisfy Condition GP-(P). But even for any completely 0-simple semigroup, its deleted diagonal acts is also not true.

Proposition 4.8. *Let $S = \mathcal{M}_0(G; I, \Lambda; P)$ be a completely 0-simple semigroup. Then $D_d(S)$ does not satisfy Condition GP-(P).*

Proof. Apply the technique used in [4, Proposition 26(2)]. $\square$

From Sections 2 and 3, we know that Condition GP-(P) implies GP-flatness but not conversely, even for diagonal acts the two properties are distinct. We shall give an example of a monoid over which its diagonal act is GP-flat but does not satisfy Condition GP-(P). Let S denote the monoid $\{0, x, 1 | \ x^2 = 0\}$. It is clear to check that S is a generally regular monoid. From [15, Theorem 3.4], it follows that the diagonal act $D(S)$ is GP-flat. On the other hand, since S is commutative but not cancellative, using Proposition 4.5, $D(S)$ fails to satisfy Condition GP-(P).

5. **Purity and epimorphisms**

In this section, we give a new generalization of 2-pure epimorphisms, and obtain an equivalent characterization of Condition GP-(P). Moreover, we study directed colimits of this new epimorphism.

We say that an epimorphism $\psi : B_S \to A_S$ is *n-pure* [1], if for every family $a_1, \cdots, a_n \in A$ and relations

$$a_{\alpha_i} s_i = a_{\beta_i} t_i \quad (i = 1, \cdots, m),$$

there exist $b_1, \cdots, b_n \in B$ such that $\psi(b_r) = a_r$ for all $1 \le r \le n$ and $b_{\alpha_i} s_i = b_{\beta_i} t_i$ for all $1 \le i \le m$. In particular, when $n = 2$, we say ψ is *2-pure*.

Definition 5.1. Let $\psi : B_S \to A_S$ be an epimorphism. We say ψ is *quasi G-2-pure* if for any $a_1, a_2 \in A$, $s \in S$, $a_1 s = a_2 s$ implies that there exist $n \in \mathbb{N}$ and $b_1, b_2 \in B$ such that $\psi(b_1) = a_1, \psi(b_2) = a_2$ and $b_1 s^n = b_2 s^n$. In the case $n = 1$, we call the ψ *quasi 2-pure*.

Clearly, every 2-pure epimorphism is quasi G-2-pure, but the converse is not true. Indeed, let S be the monoid $(\mathbb{N}; +)$. We consider the one-element act $\Theta_S = \{\theta\}$ over S and the epimorphism $\psi : S_S \to \Theta_S$. Note that $\theta \cdot 0 = \theta \cdot 0$ and $\theta \cdot 0 = \theta \cdot 1$, but there cannot exist $m, n \in S$ such that $m + 0 = n + 0$ and $m + 0 = n + 1$. Hence ψ is not 2-pure. But, we can verify that ψ is quasi G-2-pure.

Proposition 5.2. *For any right S-act A_S, the following statements are equivalent:*

(1) *A_S satisfies Condition GP-(P);*
(2) *Every epimorphism $B_S \to A_S$ is quasi G-2-pure;*
(3) *There exists a quasi G-2-pure epimorphism $B_S \to A_S$ with B_S satisfies Condition GP-(P).*

Proof. (1) $\Rightarrow$ (2). Suppose that $\psi : B_S \to A_S$ is an epimorphism and suppose that $a_1, a_2 \in A$, $s \in S$ are such that $a_1 s = a_2 s$ in A_S. Since A_S satisfies Condition GP-(P), there exist $n \in \mathbb{N}$, $u, v \in S$ and $a \in A$ such that $a_1 = au$, $a_2 = av$ and $us^n = vs^n$. Applying surjectivity of ψ, there exists $b \in B$ with $\psi(b) = a$. Then we have $bus^n = bvs^n$ in B_S, $a_1 = au = \psi(bu)$ and

$a_2 = av = \psi(bv)$. Taking $b_1 = bu$, $b_2 = bv$, we see that we have reached the desired conclusion.

(2) $\Rightarrow$ (3). It is clear.

(3) $\Rightarrow$ (1). Let A_S be a right S-act. By hypothesis there exists a quasi G-2-pure epimorphism $\psi : B_S \to A_S$ with B_S satisfies Condition GP-(P). Now suppose that $a_1, a_2 \in A$, $s \in S$ are such that $a_1 s = a_2 s$ in A_S. Then there exist a natural number $n \in \mathbb{N}$ and elements b_1, $b_2 \in B$ such that $b_1 s^n = b_2 s^n$ in B, $\psi(b_1) = a_1$ and $\psi(b_2) = a_2$. Also, since B_S satisfies Condition GP-(P), from equality $b_1 s^n = b_2 s^n$, we obtain $m \in \mathbb{N}$, $b \in B$ and $u, v \in S$ such that $b_1 = bu$, $b_2 = bv$ and $u(s^n)^m = v(s^n)^m$. Consequently, $a_1 = \psi(b)u$ and $a_2 = \psi(b)v$ in A_S, and $us^{nm} = vs^{nm}$ for $nm \in \mathbb{N}$, as required. □

Using an argument similar to that of Proposition 5.2, we have the following result.

Proposition 5.3. *For any right S-act A_S, the following statements are equivalent:*

(1) *A_S satisfies Condition (PWP);*
(2) *Every epimorphism $B_S \to A_S$ is quasi 2-pure;*
(3) *There exists a quasi 2-pure epimorphism $B_S \to A_S$ with B_S satisfies Condition (PWP).*

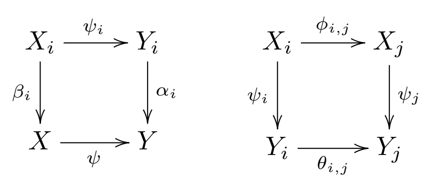
Applying Propositions 5.2 and 5.3, the following two conclusions hold.

Corollary 5.4. *Let S be a monoid and let $\psi : B_S \to A_S$ be an epimorphism, where B_S satisfies Condition GP-(P). Then A_S satisfies Condition GP-(P) if and only if ψ is quasi G-2-pure.*

Corollary 5.5. *Let S be a monoid and let $\psi : B_S \to A_S$ be an epimorphism, where B_S satisfies Condition (PWP). Then A_S satisfies Condition (PWP) if and only if ψ is quasi 2-pure.*

At last, we discuss directed colimits of quasi G-2-pure epimorphisms of right S-acts.

By [1], suppose that $(X_i, \phi_{i,j})$ and $(Y_i, \theta_{i,j})$ are direct systems of right S-acts and S-morphisms. Suppose that for each $i \in I$ there exists an S-morphism $\psi_i : X_i \to Y_i$ and suppose that (X, β_i) and (Y, α_i), the directed colimits of these systems are such that the diagrams



commute for all $i \leq j \in I$. Then we shall refer to ψ as the *directed colimit of the ψ_i*.

Proposition 5.6. *Let S be a monoid. Directed colimits of quasi G-2-pure epimorphisms of right S-acts are quasi G-2-pure.*

Proof. Suppose that $(X_i, \phi_{i,j})$ and $(Y_i, \theta_{i,j})$ are direct systems of right S-acts and S-morphisms. Suppose that for each $i \in I$ there exists a quasi G-2-pure S-morphism $\psi_i : X_i \to Y_i$, and suppose that (X, β_i) and (Y, α_i), the directed colimits of these systems are such that the diagrams

$$\begin{array}{ccc} X_i & \xrightarrow{\psi_i} & Y_i \\ {\scriptstyle\beta_i}\downarrow & & \downarrow{\scriptstyle\alpha_i} \\ X & \xrightarrow[\psi]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} X_i & \xrightarrow{\phi_{i,j}} & X_j \\ {\scriptstyle\psi_i}\downarrow & & \downarrow{\scriptstyle\psi_j} \\ Y_i & \xrightarrow[\theta_{i,j}]{} & Y_j \end{array}$$

commute for all $i \le j \in I$.

Assume that $y_1 s = y_2 s$ for $y_1, y_2 \in Y$, $s \in S$. Then there exist $i, j \in I$, $y_i \in Y_i$, $y_j \in Y_j$ with $\alpha_i(y_i) = y_1$ and $\alpha_j(y_j) = y_2$. So we deduce $\alpha_i(y_i s) = \alpha_i(y_i)s = \alpha_j(y_j)s = \alpha_j(y_j s)$. Since I is directed, there exists some $k \ge i, j$ such that $\theta_{i,k}(y_i)s = \theta_{j,k}(y_j)s$. Since ψ_k is quasi G-2-pure, there exist $n \in \mathbb{N}$, $x_1, x_2 \in X_k$ such that $\psi_k(x_1) = \theta_{i,k}(y_i)$, $\psi_k(x_2) = \theta_{j,k}(y_j)$ and $x_1 s^n = x_2 s^n$. Then, we can calculate that

$$y_1 = \alpha_i(y_i) = \alpha_k \theta_{i,k}(y_i) = \alpha_k \psi_k(x_1) = \psi(\beta_k(x_1)),$$
$$y_2 = \alpha_j(y_j) = \alpha_k \theta_{j,k}(y_j) = \alpha_k \psi_k(x_2) = \psi(\beta_k(x_2))$$

and $\beta_k(x_1)s^n = \beta_k(x_2)s^n$. Hence ψ is quasi G-2-pure. □

Corollary 5.7. *Let S be a monoid. Directed colimits of quasi 2-pure epimorphisms of right S-acts are quasi 2-pure.*

Acknowledgements

The authors are thankful to the anonymous referee for his expert and valuable comments to improve this article. This research was partially supported by the National Natural Science Foundation of China (Grant No. 11371177, 11201201).

References

[1] A. Bailey and J. Renshaw, Covers of acts over monoids and pure epimorphisms, *Proc. Edinb. Math. Soc. (2)* **57** (2014), no. 3, 589–617.

[2] S. Bulman-Fleming, Flat and strongly flat S-systems, *Comm. Algebra* **20** (1992), no. 9, 2553–2567.

[3] S. Bulman-Fleming, M. Kilp and V. Laan, Pullbacks and flatness properties of acts II, *Comm. Algebra* **29** (2001), no. 2, 851–878.

[4] S. Bulman-Fleming and A. Gilmour, Flatness properties of diagonal acts over monoids, *Semigroup Forum* **79** (2009), no. 2, 298–314.

[5] A. Golchin and H. Mohammadzadeh, On condition (P'), *Semigroup Forum* **86** (2013), no. 2, 413–430.

[6] J. M. Howie, Fundamentals of Semigroup Theory, London Mathematical Society Monographs, Oxford University Press, New York, 1995.
[7] M. Kilp, On flat acts, (Russian) *Tartu Riikl. Ül. Toimetised Vih.* **253** (1970) 66–72.
[8] M. Kilp, Commutative monoids all of whose principal ideals are projective, *Semigroup Forum* **6** (1973), no. 4, 334–339.
[9] M. Kilp, Characterization of monoids by properties of their left Rees factors, (Russian) *Tartu Riikl. Ül. Toimetised Vih.* **640** (1983) 29–37 .
[10] M. Kilp, U. Knauer and A. V. Mikhalev, Monoids, Acts and Categories, Walter de Gruyter & Co., Berlin, 2000.
[11] V. Laan, Pullbacks and flatness properties of acts I, *Comm. Algebra* **29** (2001), no. 2, 829–850.
[12] Z. K. Liu and Y. B. Yang, Monoids over which every flat right act satisfies condition (P), *Comm. Algebra* **22** (1994), no. 8, 2861–2875.
[13] P. Normak, On equalizer-flat and pullback-flat acts, *Semigroup Forum* **36** (1987), no. 3, 293–313.
[14] H. S. Qiao, Some new characterizations of right cancellative monoids by condition (PWP), *Semigroup Forum* **71** (2005), no. 1, 134–139.
[15] H. S. Qiao, On a generalization of principal weak flatness property, *Semigroup Forum* **85** (2012), no. 1, 147–159.
[16] M. Sedaghatjoo, V. Laan and M. Ershad, Principal weak flatness and regularity of diagonal acts, *Comm. Algebra* **40** (2012), no. 11, 4019–4030.
[17] B. Stenström, Flatness and localization over monoids, *Math. Nachr.* **48** (1971) 315–334.

(Xingliang Liang) Department of Mathematics, Lanzhou University, Lanzhou, Gansu 730000, P.R. China.
E-mail address: lxl_119@126.com

(Yanfeng Luo) Department of Mathematics, Lanzhou University, Lanzhou, Gansu 730000, P.R. China.
E-mail address: luoyf@lzu.edu.cn