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APPROXIMATE SOLUTION OF DUAL INTEGRAL EQUATIONS

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Abstract. We study dual integral equations which appear in formulation of the potential distribution of an electrified plate with mixed boundary conditions. These equations will be converted to a system of singular integral equations with Cauchy type kernels. Using Chebyshev polynomials, we propose a method to approximate the solution of Cauchy type singular integral equation which will be used to approximate the solution of the main dual integral equations. Numerical results demonstrate effectiveness of this method.
Keywords: Dual integral equation, Cauchy type integral equation, Fourier transform.
MSC(2010): Primary: 45F10; Secondary: 45F10, 45E05.

1. Introduction

Many problems in different branches of mathematical physics and in particular boundary value problems with mixed boundary conditions (See [1] and [5]) can be formulated as dual integral equations (DIEs). Sneddon[9] has solved certain DIEs which appear in solving mixed boundary value problems by Hankel transforms. Nasim et al. [8] studied some DIEs with the kernels involving Bessel, Hankel and trigonometric functions. Manam [6] proposed a quick method of solution for DIEs involving trigonometric kernels.

Solving DIEs can be reduced to solving a Cauchy type singular integral equation (SIE) or a system of such equations. The SIEs have been studied by many researchers (see for example Ioakimids [4] and Chakrabarti et al. [2, 3]).

In the next section we illustrate formulation of a mixed boundary value problem, which will result in the DIEs

$$\int_{-\infty}^{\infty} A(\lambda)\, e^{i\lambda y}\, d\lambda = \sqrt{2\pi} g(y)\ , \qquad |y| < a \tag{1.1}$$

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and

$$\int_{-\infty}^{\infty} |\lambda| \, A(\lambda)\, e^{i\lambda y}\, d\lambda = 0 \,, \qquad |y| > a \tag{1.2}$$

in terms of the unknown function $A(\lambda)$. Decomposing the kernels of these DIEs will result a system of DIEs with trigonometric kernels which will be converted to a system of Cauchy type SIEs. Section 3 demonstrates the proposed method based on orthogonality of Chebyshev polynomials on the interval $[-1, 1]$ and use of Fourier-Chebyshev series. In the last section illustrative examples are given to show efficiency of our proposed method.

2. Modeling

We consider the potential distribution $u(x, y)$ in the half-plane $x > 0$ (because of symmetry), due to the presence of a plate of width $2a$ (see Fig.1) placed along the y axis with center at the origin and extending in the z direction (see [5], p.124). The plate is kept at a potential $u(0, y) = g(y), -a < y < a$ and where the rest of the yz plane is assumed to be insulated. The potential

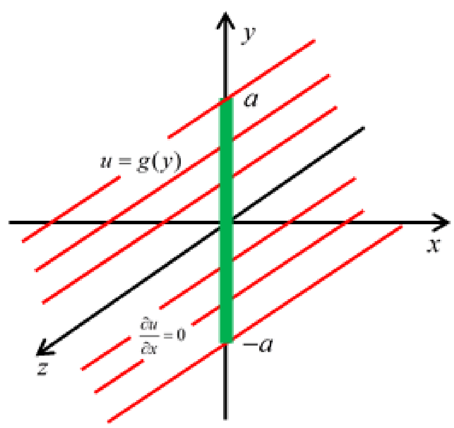

FIGURE 1. The electrified plate

distribution in free space here is assumed to be independent of z. Thus $u(x, y)$ is formulated as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,, \qquad x > 0\,, \qquad -\infty < y < \infty \tag{2.1}$$

with the following boundary conditions

$$u(0, y) = g(y)\,, \qquad -a < y < a \tag{2.2}$$

and

$$\frac{\partial u(0,y)}{\partial x} = 0\,, \qquad |y| > a. \tag{2.3}$$

Using the Fourier exponential transform, we let

$$U(x,\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,y) e^{-i\lambda y} dy, \tag{2.4}$$

and use the Fourier-transform to the Laplace equation (2.1) to obtain

$$\frac{d^2 U(x,\lambda)}{dx^2} - \lambda^2 U(x,\lambda) = 0. \tag{2.5}$$

General solution of (2.5) is as

$$U(x,\lambda) = A(\lambda) e^{-|\lambda| x} + B(\lambda) e^{|\lambda| x}. \tag{2.6}$$

Here, as $x \to \infty$ the term involving the positive exponential factor $e^{|\lambda|x}$ will blow up, therefore in (2.6), one should let $B(\lambda) = 0$, consequently

$$U(x,\lambda) = A(\lambda) e^{-|\lambda| x}. \tag{2.7}$$

The mixed conditions (2.2)-(2.3) at $x = 0$, are not suitable for the substitution in (2.4), since they are given as function for $|y| < a$ and as derivative of a function for $|y| > a$, respectively. Therefore we can't find $A(\lambda)$ in (2.7), hence we proceed to find $u(x,y)$ as the inverse Fourier transform of (2.7),

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(\lambda) e^{-|\lambda| x} e^{i\lambda y} d\lambda. \tag{2.8}$$

Now, we can apply the mixed boundary conditions (2.2)-(2.3) on $u(x,y)$ above to obtain the DIEs (1.1) and (1.2) in $A(\lambda)$.

Throughout this paper, we set $a = 1$ (this can be considered by rescaling the parameters). To find a real type solution in (1.1)-(1.2), we consider $\sqrt{2\pi}\, g(y) = g_1(y) + i g_2(y)$, where g_1 and g_2 are real functions on $[0,1]$. Equating the real and imaginary parts in (1.1)-(1.2) yields the following system of dual integral equations with trigonometric kernels

$$\int_0^{\infty} E(\lambda)\, \cos \lambda y \, d\lambda = g_1(y)\ , \qquad 0 < y < 1 \tag{2.9}$$

$$\int_0^{\infty} \lambda\, E(\lambda)\, \cos \lambda y \, d\lambda = 0\ , \qquad y > 1 \tag{2.10}$$

and

$$\int_0^{\infty} O(\lambda)\, \sin \lambda y \, d\lambda = g_2(y)\ , \qquad 0 < y < 1 \tag{2.11}$$

$$\int_0^{\infty} \lambda\, O(\lambda)\, \sin \lambda y \, d\lambda = 0\ , \qquad y > 1, \tag{2.12}$$

where

$$E(\lambda) = A(\lambda) + A(-\lambda) \qquad \textit{and} \qquad O(\lambda) = A(\lambda) - A(-\lambda). \tag{2.13}$$

Our main goal is to find approximate values of $E(\lambda)$ and $O(\lambda)$, since then we approximate the solution by $A(\lambda) = \frac{1}{2}(E(\lambda) + O(\lambda))$.

3. **Description of the method**

We explain the procedure of solving (2.9)-(2.10) in details and the other ones: (2.11)-(2.13), somewhat briefly. For any $t > 1$, integrating both sides of (2.10) with respect to y from 0 to t, gives

$$\int_0^\infty E(\lambda) \sin \lambda t \, d\lambda = 0 \,, \qquad\qquad t > 1.$$

For $0 < t < 1$, we define

$$\varphi(t) = \int_0^\infty E(\lambda) \ \sin \lambda t \ d\lambda. \tag{3.1}$$

Hence for $t > 0$, we have

$$\int_0^\infty E(\lambda) \ \sin \lambda t \ d\lambda = H(t),$$

where

$$H(t) = \begin{cases} \varphi(t), & 0 < t < 1 \\ 0, & t > 1 \end{cases}$$

therefore, using the inverse sine Fourier transform, we obtain

$$E(\lambda) = \frac{2}{\pi} \int_0^\infty H(t) \sin \lambda t \, dt = \frac{2}{\pi} \int_0^1 \varphi(t) \sin \lambda t \, dt. \tag{3.2}$$

Substituting $E(\lambda)$ from (3.2) into (2.10) and using the formula

$$\int_0^\infty \sin(\lambda t) \cos(\lambda y) d\lambda = \frac{t}{t^2 - y^2}$$

(see [1], p.26), we have

$$\int_0^1 \varphi(t) \frac{2t}{t^2 - y^2} \, dt = \pi g_1(y). \tag{3.3}$$

Let

$$s = 2t^2 - 1 \,, \qquad\qquad x = 2y^2 - 1. \tag{3.4}$$

Then the Eq. (3.3) as the Cauchy type singular integral equation

$$\int_{-1}^1 \frac{\psi(s)}{s - x} \, ds = G(x) \,, \tag{3.5}$$

where

$$\varphi(t) = \psi(2t^2 - 1)\ , \qquad G(x) = \pi g_1 \left(y = \sqrt{\frac{x+1}{2}} \right) . \tag{3.6}$$

Using the Fourier-Chebyshev series, we let

$$G(x) \simeq {\sum_{n=0}^{M}}' c_n T_n(x)\ , \qquad c_n = \frac{2}{\pi} \int_{-1}^{1} \frac{G(x) T_n(x)}{\sqrt{1-x^2}}\, dx\, , \tag{3.7}$$

where the symbol ($\sum'$) denotes that the first term in the summation is multiplied by $\frac{1}{2}$. The coefficients c_n can be approximated as

$$c_n \simeq \frac{2}{M+1} \sum_{k=1}^{M+1} G(x_k) T_n(x_k)\, , \qquad n = 0\,, 1\,, \ldots\,, M \tag{3.8}$$

in which $x_k = \cos\left(\frac{(k-\frac{1}{2})\pi}{M+1}\right)$, $k = 1\,, 2\,, \ldots\,, M+1$, are the roots of Chebyshev polynomial $T_{M+1}(x)$.

Using the relation

$$\int_{-1}^{1} \frac{\sqrt{1-s^2} U_{n-1}(s)}{s-x}\, ds = -\pi T_n(x)\, , \qquad n = 1\,, 2\,, \ldots\,, M \tag{3.9}$$

(see [7], Chap.8), along with (3.7), we obtain the approximation

$$\psi(s) \simeq \sum_{n=1}^{M} b_n \sqrt{1-s^2} U_{n-1}(s)\, , \qquad s \in [-1, 1]\, , \tag{3.10}$$

for the Eq. (3.5), where U_n's are the second kind Chebyshev polynomials.

Using (3.10) in (3.5) and (3.9), imply

$$c_0 = 0 \qquad \textit{and} \qquad b_n = -\frac{c_n}{\pi}\, , \qquad n = 1\,, 2\,, \ldots\,, M.$$

Due to (3.7) the value $c_0 = 0$, is equivalent to

$$\int_{-1}^{1} \frac{G(x)}{\sqrt{1-x^2}}\, dx = 0\, , \tag{3.11}$$

which is the solvability condition. Hence we can represent the solution of (3.5) as

$$\psi(s) = -\frac{\sqrt{1-s^2}}{\pi} \sum_{n=1}^{M} c_n U_{n-1}(s)\, , \tag{3.12}$$

which is similar to the solution that is given in [2]. By choosing other kinds of Chebyshev polynomials in (3.7) one can construct other forms of solution.

From (3.6) and (3.12), we obtain

$$\varphi(t) = \psi(2t^2 - 1) = -\frac{2t\sqrt{1-t^2}}{\pi} \sum_{n=1}^{M} c_n U^*_{n-1}(t^2)\,,$$

where U^*_n is the shifted second kind Chebyshev polynomial on [0, 1]. Using the relation $2tU^*_{n-1}(t^2) = U_{2n-1}(t)$, $\varphi(t)$ can be rewritten as

$$\varphi(t) = -\frac{\sqrt{1-t^2}}{\pi} \sum_{n=1}^{M} c_n U_{2n-1}(t). \tag{3.13}$$

On the other hand expanding $\sin(\lambda t)$ based on using Bessel's functions of the first kind (Snyder[10]), we have

$$\sin(\lambda t) = 2\sum_{k=0}^{\infty} (-1)^k J_{2k+1}(\lambda) T_{2k+1}(t)$$

and using the relation $2T_n(t) = U_n(t) - U_{n-2}(t)$ one gets

$$\sin(\lambda t) = \sum_{m=1}^{\infty} (-1)^{m-1} \big[J_{2m-1}(\lambda) + J_{2m+1}(\lambda)\big] U_{2m-1}(t)\,. \tag{3.14}$$

To compute the function $E(\lambda)$ in (3.2), it is convenient to rewrite it as

$$E(\lambda) = \frac{1}{\pi}\int_{-1}^{1} \varphi(t) \sin \lambda t \, dt$$

since the integrand is an even function with respect to t. Hence, using (3.13) and (3.14), yields

$$E(\lambda) \simeq \frac{1}{2\pi} \sum_{n=1}^{M} (-1)^n c_n \big[J_{2n-1}(\lambda) + J_{2n+1}(\lambda)\big]\,, \tag{3.15}$$

by the orthogonality of U_m:s in $[-1, 1]$, where c_n:s are given by (3.8).

We now briefly describe the process of solving (2.11)-(2.12). Considering (2.12) we define another unknown function $\widetilde{\varphi}(y)$ as

$$\widetilde{\varphi}(y) = \int_0^{\infty} \lambda \ O(\lambda) \ \sin \lambda y \ d\lambda, \qquad 0 < y < 1. \tag{3.16}$$

Then using inverse sine Fourier transform, (2.12) and (3.16) return to

$$\lambda O(\lambda) = \frac{2}{\pi}\int_0^1 \widetilde{\varphi}(t) \sin \lambda t dt. \tag{3.17}$$

Differentiating (2.11) with respect to y, gives

$$\int_0^{\infty} \lambda O(\lambda) \cos \lambda y d\lambda = h(y),$$

where $h(y) = \pi \frac{d}{dy}(g_2(y))$ and using (3.17), it will be simplified as

$$\int_0^1 \widetilde{\varphi}(t) \frac{2t}{t^2 - y^2}\, dt = h(y). \tag{3.18}$$

Assuming (3.4), Eq.(3.18) can be written as (3.5), where

$$\widetilde{\varphi}(t) = \psi(2t^2 - 1)\ , \qquad G(x) = h\left(y = \sqrt{\frac{x+1}{2}}\right) . \tag{3.19}$$

To compute $O(\lambda)$ from (3.17), it is convenient to rewrite it in the form

$$O(\lambda) = \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_{-1}^{1} \widetilde{\varphi}(t) \frac{\sin \lambda t}{\lambda}\, dt, \tag{3.20}$$

Dividing both sides of (3.14) by λ and using $\frac{J_n(\lambda)}{\lambda} = \frac{J_{n-1}(\lambda)+J_{n+1}(\lambda)}{2n}$ (Spiegel [11]), yield the expansion

$$\frac{\sin(\lambda t)}{\lambda} = \sum_{m=1}^{\infty} (-1)^m \Big[\frac{J_{2m-2}(\lambda) + J_{2m}(\lambda)}{2(2m-1)} + \frac{J_{2m}(\lambda) + J_{2m+2}(\lambda)}{2(2m+1)}\Big] U_{2m-1}(t). \tag{3.21}$$

Putting (3.21) in (3.20) and using orthogonality of U_m:s we conclude that

$$O(\lambda) \simeq \frac{1}{2\pi} \sum_{n=1}^{M} (-1)^n c_n \Big[\frac{J_{2n-2}(\lambda) + J_{2n}(\lambda)}{2(2n-1)} + \frac{J_{2n}(\lambda) + J_{2n+2}(\lambda)}{2(2n+1)}\Big]. \tag{3.22}$$

4. Numerical results

In this section, some examples are given to demonstrate the theory established in the previous section.

Example 4.1. Let $g_1(y) = \sinh(2y^2 - 1)$. We find the function $E(\lambda)$ satisfying in (2.9)-(2.10).

For this function (3.6) takes the form $G(x) = \pi \sinh(x)$ which is an odd function and (3.11) holds. By choosing $M = 8$ in (3.8) the roots of $T_9(x)$ will be used to obtain

$$c_1 = 3.550999380,\ c_3 = 0.1392883220,\ c_5 = 0.0017056518,\ c_7 = 0.0000100475$$

and the other coefficients are zero. Hence, using (3.15), we obtain

$$\begin{aligned} E(\lambda) \simeq -\frac{1}{2\pi}\big[&c_1(J_1(\lambda) + J_3(\lambda)) + c_3(J_5(\lambda) + J_7(\lambda)) \\ &+ c_5(J_9(\lambda) + J_{11}(\lambda)) + c_7(J_{13}(\lambda) + J_{15}(\lambda))\big]\,, \end{aligned}$$

which satisfies the Eq.(2.10). By defining the error function

$$g_{1,app}(y) = \int_0^{\infty} E(\lambda)\, \cos \lambda y\, d\lambda\ , \qquad 0 < y < 1$$

we report the result in Table 1 to confirm accuracy of the approximate solution.

TABLE 1. Comparison of approximate and exact values of g_1

y	$g_{1,app}(y)$	$g_1(y)$	*Absolute Error*
0	-1.175201182	-1.175201194	1.1 *E*-8
0.01	-1.174892590	-1.174892601	1.1 *E*-8
0.15	-1.106929230	-1.106929219	1.1 *E*-8
0.24	-1.004851564	-1.004851559	4.8 *E*-9
0.35	-0.8288004434	-0.8288004543	1.0 *E*-8
0.5	-0.5210953163	-0.5210953055	1.1 *E*-8
0.62	-0.2332652412	-0.2332652512	9.7 *E*-9
0.73	0.06584748590	0.06584749200	6.2 *E*-9
0.9	0.6604918047	0.6604918021	3.0 *E*-9
0.98	1.056549150	1.056549136	1.1 *E*-8

Example 4.2. Let $g_2(y) = y^7 - (2.1)y^5 + (1.75)y$. Find the function $O(\lambda)$ for which the equations (2.11)-(2.12) are satisfied.

In (3.19), one can get $G(x) = \frac{7\pi}{8}(x^3 - 3x)$ which is an odd function from which (3.11) holds. By setting $M = 3$ in (3.8) the roots of $T_4(x)$ will be used to obtain

$$c_0 = 0\,, \qquad c_1 = -6.185010537\,, \qquad c_2 = 0\,, \qquad c_3 = 0.687223392.$$

Hence,

$$G(x) \simeq c_1 T_1(t) + c_3 T_3(t) \qquad \textit{and} \qquad \varphi(t) \simeq -\frac{\sqrt{1-t^2}}{\pi}(c_1 U_1(t) + c_3 U_3(t))$$

and using (3.22) one gets

$$O(\lambda) \simeq -\frac{1}{4\pi}\left(c_1 J_0(\lambda) + \frac{4}{3}c_1 J_2(\lambda) + \frac{8}{15}(c_1 + c_3)J_4(\lambda) + \frac{12}{35}c_3 J_6(\lambda) + \frac{1}{7}c_3 J_8(\lambda)\right),$$

which satisfies the Eq.(2.12). Table 2 demonstrates the accuracy of this solution by setting

$$g_{2,app}(y) = \int_0^\infty O(\lambda)\, \sin \lambda y \, d\lambda\ , \qquad 0 < y < 1.$$

5. **Conclusion**

The boundary integral value problems with mixed boundary conditions can be formulated as dual integral equations and solving the dual integral equations is connected to solving singular integral equations. Using Fourier-Chebyshev

Table 2. Comparison of approximate and exact values of g_2

y	$g_{2,app}(y)$	$g_2(y)$	*Absolute Error*
0	0	0	0
0.01	0.017499999785	0.0174999997900	5.0 E-12
0.15	0.262342239752	0.262342239844	6.0 E-11
0.24	0.418373713504	0.418373713674	1.7 E-10
0.35	0.602113798921	0.602113799219	2.8 E-10
0.5	0.817187499482	0.8171875	5.2 E-10
0.62	0.927828250655	0.927828251342	6.9 E-10
0.73	0.952628949865	0.952628950661	7.8 E-10
0.9	0.813267899165	0.8132679	8.3 E-10
0.98	0.684891859135	0.68489185997	1.0 E-9

series, we constructed approximate solutions to Cauchy type singular integral equations by using orthogonal Chebyshev polynomials.

We conclude that the proposed method is not only useful to approximate the solution of DIEs but also to approximate the solution of SIE like (3.5). In comparing with the method of Chakrabarti et al. [2], in our method there is no need to solve a system of algebraic equations.

We remark here that the boundary value problem (2.1)-(2.3) also describes the steady-state temperature distribution $u(x, y)$ in the xy plane, due to a given temperature on the segment $|y| < a$ of the y axis and where the rest of the y axis is completely insulated. Another physical problem that is represented by (2.1)-(2.3) is that of the steady irrotational flow of a perfect fluid through the opening $|y| < a$ of an infinite wall along the y axis.

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