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ON THE BOUNDS IN POISSON APPROXIMATION FOR INDEPENDENT GEOMETRIC DISTRIBUTED RANDOM VARIABLES

T. L. HUNG* AND L. T. GIANG

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Abstract. The main purpose of this note is to establish some bounds in Poisson approximation for row-wise arrays of independent geometric distributed random variables using the operator method. Some results related to random sums of independent geometric distributed random variables are also investigated.
Keywords: Poisson approximation, linear operator, geometric random variable, random sums.
MSC(2010): Primary: 60G50; Secondary: 41A17, 41A36.

1. Introduction

Let $(X_{n,j}, j = 1, 2, \cdots, n; n = 1, 2, \cdots)$ be a row-wise triangular array of independent geometric distributed random variables with success probabilities $P(X_{n,j} = k) = p_{n,j}(1-p_{n,j})^k, 0 < p_{n,j} < 1; k = 0, 1, 2, \cdots; j = 1, 2, \cdots, n; n = 1, 2, \cdots$. Let us denote by S_n the number of failures before the nth success in a sequence of independent Bernoulli trials. Then, $S_n = X_{n,1} + X_{n,2} + \cdots + X_{n,n}$. Write $\lambda_n = E(S_n) = \sum_{j=1}^{n}(1 - p_{n,j})p_{n,j}^{-1}$ and suppose that $\lim_{n\to\infty} \lambda_n = \lambda, \quad (0 < \lambda < +\infty)$. We denote by Z_{λ_n} the Poisson random variables with means λ_n.

Up to the present the Poisson approximation for many discrete distributions (notably the Poisson-binomial distribution) has received extensive attention in the literature and many different approaches have been proposed. (see [1, 2, 3, 5, 6, 7, 10, 11, 15, 14] for more details). The problem will be considered in this paper are similar to those encountered in [8]. Actually, based on a linear operator due to A. Renyi in [13], some bounds in Poisson approximation

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for non-random sums and random sums of independent geometric distributed random variables are established.

Let $\mathbb{K}$ denote the class of all real-valued bounded functions f on the set of all non-negative integers $Z_+ = \{0, 1, 2, \cdots\}$. The norm of a function $f \in \mathbb{K}$ is defined by $\|f\| = \sup_{x \in Z_+} |f(x)|$. The Renyi operator associated with random variable X, denoted by A_X, is given by

$$A_X f(x) = E\big(f(X+x)\big) = \sum_{k=0}^{\infty} f(x+k)P(X=k), \forall f \in \mathbb{K}, \forall x \in Z_+. \tag{1.1}$$

(Recall the definition in [13]). It is to be noticed that the linear operator defined in (1.1) is actually a discrete form of the Trotter's operator (see [20] for more details).

Let A_{S_n} and $A_{Z_{\lambda_n}}$, denote the Renyi's operators associated with S_n and Z_{λ_n}, respectively. The main purpose of this note is to establish the upper bounds for $\| A_{S_n} f - A_{Z_{\lambda_n}} f \|$ in Poisson approximation for independent geometric distributed random variables. Some bounds related to $\| A_{S_N} f - A_{Z_{\lambda_{N_n}}} f \|$ in Poisson approximation for random sums of independent geometric distributed random variables are also investigated, with $N_n, n = 1, 2, \cdots$ are positive integer-valued random variables independent of all $X_{n,1}, X_{n,2}, \cdots; n = 1, 2, \cdots$. The results in this paper are extensions of published results in [10], [15, 16, 17, 18, 19, 11, 9]. The present paper is also a continuation of earlier results in [8].

It is to be noticed that in recent years, based on the Stein-Chen method there are many papers related to bounds in Poisson approximation for independent geometric distributed random variables (we refer the reader to [2, 5, 6, 10, 1, 12, 14, 16, 17, 18, 19], and references therein). However, the linear operator used in this paper is very elementary and elegant. The basic idea is very simple and based on elementary properties of a linear operator introduced by Renyi ([13], 1970). The results in this note present a new approach to the Poisson approximation problems for the discrete independent random variables.

2. **Preliminaries**

In the sequel we shall need some properties of Renyi's operator in (1.1). We recall some definitions and notations (see [13] for more details). Let us denote by A_X and A_Y two Renyi's operators associated with two discrete random variables X and Y. Moreover, let α, β be two real numbers and $f, g \in \mathbb{K}$. Then, it is easily seen that

(1) $A_X(\alpha f + \beta g) = \alpha A_X(f) + \beta A_X(g)$.
(2) $\| A_X(f) \| \leqslant \| f \|$.
(3) $\| A_X(f) + A_Y(f) \| \leqslant \| A_X(f) \| + \| A_Y(f) \|$.
(4) $\| A_X A_Y(f) \| \leqslant \| A_Y(f) \|$.

(5) Suppose that $A_{X_1}, A_{X_2}, \cdots, A_{X_n}$ are operators associated with the independent random variables $X_1, X_2, \cdots, X_n$. Then, for $f \in \mathbb{K}$,

$$A_{X_1+X_2+\cdots+X_n}(f) = A_{X_1}A_{X_2}\cdots A_{X_n}(f).$$

(6) Suppose that $A_{X_1}, A_{X_2}, \cdots, A_{X_n}$ and $A_{Y_1}, A_{Y_2}, \cdots, A_{Y_n}$ are operators associated with independent random variables $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$, respectively. Moreover, assume that all random variables $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ are independent. Then, for $f \in \mathbb{K}$,

$$\| A_{\sum_{k=1}^n X_k}(f) - A_{\sum_{k=1}^n Y_k}(f) \| \leqslant \sum_{k=1}^n \| A_{X_k}(f) - A_{Y_k}(f) \|. \tag{2.1}$$

Clearly

$$A_{X_1}A_{X_2}\cdots A_{X_n} - A_{Y_1}A_{Y_2}\cdots A_{Y_n}$$
$$= \sum_{k=1}^n A_{X_1}A_{X_2}\cdots A_{X_{k-1}}(A_{X_k} - A_{Y_k})A_{Y_{k+1}}\cdots A_{Y_n}.$$

Accordingly

$$\| A_{\sum_{k=1}^n X_k}(f) - A_{\sum_{k=1}^n Y_k}(f) \|$$
$$\leq \sum_{k=1}^n \| A_{X_1}\cdots A_{X_{k-1}}(A_{X_k} - A_{Y_k})A_{Y_{k+1}}\cdots A_{Y_n}(f) \|$$
$$\leq \sum_{k=1}^n \| A_{Y_{k+1}}\cdots A_{Y_n}(A_{X_k} - A_{Y_k})(f) \|$$
$$\leq \sum_{k=1}^n \| A_{X_k}(f) - A_{Y_k}(f) \|.$$

(7) $\| A_X^n(f) - A_Y^n(f) \| \leq n \| A_X(f) - A_Y(f) \|$.

(8) Suppose that $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ are independent random variables (in each group), and let $N_n, n = 1, 2, \cdots$ be a sequence of positive integer-valued random variables independent of all X_k and $Y_k, k = 1, 2, \cdots$ Then, for $f \in K$,

$$\| A_{\sum_{j=1}^{N_n} X_k}(f) - A_{\sum_{k=1}^{N_n} Y_k}(f) \| \leqslant \sum_{n=1}^{\infty} P(N_n = n) \sum_{j=1}^n \| A_{X_k}(f) - A_{Y_k}(f) \|. \tag{2.2}$$

Lemma 2.1. *The equation $A_X f(x) = A_Y f(x)$ for $f \in \mathbb{K}, x \in Z_+$, provides that X and Y are identically distributed random variables.*

Let $A_{X_1}, A_{X_2}, \cdots$ be a sequence of Renyi's operators associated with the independent discrete random variables $X_1, X_2, \cdots$, and assume that A_X is a

Renyi's operator associated with the discrete random variable X. The following lemma states one of the most important properties of the Renyi's operator

Lemma 2.2. *A sufficient condition for a sequence of random variables $X_1, X_2, \cdots$ converging in distribution to a random variable X is that*

$$\lim_{n\to\infty} \|A_{X_n}(f) - A_X(f)\| = 0, \quad \text{for} \quad f \in \mathbb{K}.$$

Proof. Since $\lim_{n\to\infty} \|A_{X_n}(f) - A_X(f)\| = 0, \quad$ for $\quad f \in \mathbb{K}$, we conclude that

$$\lim_{n\to\infty} \left| \sum_{k=0}^{\infty} f(x+k)\left(P(X_n = k) - P(X = k)\right) \right| = 0,$$
$$\text{for} \quad f \in \mathbb{K} \quad \text{and for} \quad x \in Z_+.$$

Taking

$$f(x) = \begin{cases} 1, & \text{if} \quad 0 \leqslant x \leqslant t \\ 0, & \text{if} \quad x > t, \end{cases}$$

we obtain

$$\lim_{n\to\infty} \left| \sum_{k=0}^{t} \left(P(X_n = k) - P(X = k)\right) \right| = 0.$$

It follows that, $P(X_n \leqslant t) - P(X \leqslant t) \to 0$ as $n \to +\infty$. We infer that $X_n \xrightarrow{d} X$ as $n \to +\infty$, here and from now, $\xrightarrow{d}$ denotes the convergence in distribution. This finishes the proof. □

3. Results

In this section, based on Renyi's operator-method the theorems 3.1, 3.3, 3.5 and 3.7 are devoted to the discussions on bounds (in term of inequalities (3.1) (3.2) (3.3) and (3.4) in Poisson approximation for independent geometric distributed random variables with parameters $p_{n,j} \in (0,1), j = 1, 2, \cdots, n; n = 1, 2, \cdots$ from row-wise triangular arrays or double arrays and for Poisson random variable Z_{λ_n} with mean $\lambda_n = E(S_n) = \sum_{j=1}^{n} p_{n,j}^{-1}\left(1 - p_{n,j}\right)$. Actually, some upper bounds for $\| A_{S_n} f - A_{Z_{\lambda_n}} f \|$ are established for $f \in \mathbb{K}$. The analogous results related to random sums $S_{N_n} = X_1 + X_2 + \cdots + X_{N_n}$ and $Z_{\lambda_{N_n}}$ are also considered, where N_n is a positive integer-valued random variable independent from all $X_1, X_2, \cdots$ and $\lambda_{N_n} = E(S_{N_n}) = \sum_{n=1}^{\infty} P(N_n = n) E(S_n) = \sum_{n=1}^{\infty} P(N_n = n) \sum_{j=1}^{n} p_{n,j}^{-1}\left(1 - p_{n,j}\right)$. The results considered in this section are somewhat similar to the results in [16, 17, 18] and [19].

Theorem 3.1. *Let* $(X_{n,j}, j = 1, 2, \cdots n; n = 1, 2, \cdots)$ *be a row-wise triangular array of independent, non-identically geometric distributed random variables with probabilities* $p_{n,j} \in (0, 1)$*. Set* $S_n = \sum_{j=1}^{n} X_{n,j}$ *and let us denote by* Z_{λ_n} *the Poisson random variable with mean* $\lambda_n = E(S_n) = \sum_{j=1}^{n} (1 - p_{nj})\, p_{n,j}^{-1}$*. Then, for* $f \in \mathbb{K}$

$$\| A_{S_n} f - A_{Z_{\lambda_n}} f \| \leq 2\, \|f\| \sum_{j=1}^{n} \left[(1 - p_{n,j})^2 + \frac{(1 - p_{n,j})}{p_{n,j}^2} \right]. \tag{3.1}$$

Proof. We first observe that

$$\left\| A_{S_n} f - A_{Z_{\lambda_n}} f \right\| \leq \sum_{j=1}^{n} \left\| A_{X_{n,j}} f - A_{Z_{(1-p_{n,j})p_{n,j}^{-1}}} f \right\|.$$

For $f \in \mathbb{K}$, and for $x \in \mathbb{Z}_+$,

$$\begin{aligned}
&A_{Z_{(1-p_{n,j})p_{n,j}^{-1}}} f(x) - A_{X_{n,j}} f(x) \\
&= \sum_{k=0}^{\infty} f(x+k) \left[P\left(Z_{p_{n,j}} = k\right) - P\left(X_{n,j} = k\right) \right] \\
&= \sum_{k=0}^{\infty} f(x+k) \left[e^{-\left(1-p_{n,j}\right)p_{n,j}^{-1}} \frac{\left[(1 - p_{n,j})\, p_{n,j}^{-1} \right]^k}{k!} - p_{n,j} (1 - p_{n,j})^k \right].
\end{aligned}$$

Hence

$$\begin{aligned}
&\left| A_{Z_{(1-p_{n,j})p_{n,j}^{-1}}} f(x) - A_{X_{n,j}} f(x) \right| \\
&\leq \sum_{k=0}^{\infty} \left| f(x+k) \left(e^{-\left(1-p_{n,j}\right)p_{n,j}^{-1}} \frac{\left[(1-p_{n,j}) p_{n,j}^{-1} \right]^k}{k!} - (1 - p_{n,j})^k p_{n,j} \right) \right| \\
&\leq \sup_{y \in Z_+} |f(y)| \sum_{k=0}^{\infty} \left| e^{-\left(1-p_{n,j}\right)p_{n,j}^{-1}} \frac{\left[(1-p_{n,j}) p_{n,j}^{-1} \right]^k}{k!} - (1 - p_{n,j})^k p_{n,j} \right| \\
&= \|f\| \sum_{k=0}^{\infty} \left| e^{-\left(1-p_{n,j}\right)p_{n,j}^{-1}} \frac{\left[(1-p_{n,j}) p_{n,j}^{-1} \right]^k}{k!} - (1 - p_{n,j})^k p_{n,j} \right| \\
&= \|f\| \left(\left| e^{-\left(1-p_{n,j}\right)p_{n,j}^{-1}} - p_{n,j} \right| + \left| e^{-\left(1-p_{n,j}\right)p_{n,j}^{-1}} (1 - p_{n,j})\, p_{n,j}^{-1} - (1 - p_{n,j})\, p_{n,j} \right| \right. \\
&\qquad \left. + \sum_{k=2}^{\infty} \left| e^{-\left(1-p_{n,j}\right)p_{n,j}^{-1}} \frac{\left[(1-p_{n,j}) p_{n,j}^{-1} \right]^k}{k!} - (1 - p_{n,j})^k p_{n,j} \right| \right) \\
&\leq \|f\| \left[\left(p_{n,j}^{-1} - p_{n,j} \right) + (1 - p_{n,j})\, p_{n,j}^{-1} \left(p_{n,j}^{-1} - p_{n,j}^2 \right) \right. \\
&\qquad \left. + \sum_{k \geq 2} e^{-\left(1-p_{n,j}\right)p_{n,j}^{-1}} \frac{\left[(1-p_{n,j}) p_{n,j}^{-1} \right]^k}{k!} + \sum_{k \geq 2} (1 - p_{n,j})^k p_{n,j} \right]
\end{aligned}$$

$$= \|f\| \left[\left(p_{n,j}^{-1} - p_{n,j}\right) + (1-p_{n,j})\, p_{n,j}^{-1} \left(p_{n,j}^{-1} - p_{n,j}^{2}\right) \right.$$
$$\left. + \left(1 - e^{-(1-p_{n,j})p_{n,j}^{-1}} - (1-p_{n,j})\, p_{n,j}^{-1} e^{-(1-p_{n,j})p_{n,j}^{-1}}\right) + (1-p_{n,j})^2 \right]$$
$$\leq \|f\| \left[2p_{n,j}^2 - 4p_{n,j} + p_{n,j}^{-2} + p_{n,j}^{-1} - (1-p_{n,j})\, p_{n,j}^{-1} + (1-p_{n,j})\, p_{n,j}^{-1} (1-p_{n,j})\, p_{n,j}^{-1} \right]$$
$$= 2\|f\| \left[(1-p_{n,j})^2 + \tfrac{1-p_{n,j}}{p_{n,j}^2} \right].$$

Therefore

$$\left\| A_{X_{n,j}} f - A_{Z_{(1-p_{n,j})p_{n,j}^{-1}}} f \right\|$$
$$= \left\| A_{Z_{p_{n,j}}} f - A_{X_{nj}} f \right\| \leq 2\|f\| \left[(1-p_{n,j})^2 + \frac{(1-p_{n,j})}{p_{n,j}^2} \right].$$

Thus

$$\| A_{S_n} f - A_{Z_{\lambda_n}} f \| \leq 2\|f\| \sum_{j=1}^{n} \left[(1-p_{n,j})^2 + \frac{(1-p_{n,j})}{p_{n,j}^2} \right].$$

The proof is complete. □

Corollary 3.2. *Under the assumptions of Theorem 3.1, from (2.1) with* $k \in \{0, 1, \cdots, n\}$

$$|P(S_n = k) - P(Z_{\lambda_n} = k)| \leq 2 \sum_{j=1}^{n} \left[(1-p_{n,j})^2 + \frac{(1-p_{n,j})}{p_{n,j}^2} \right].$$

Theorem 3.3. *Let* $(X_{n,j}, j = 1, 2, \cdots, n; n = 1, 2, \cdots)$ *be a row-wise triangular array of independent, geometric distributed random variables with parameters* $p_{n,j} \in (0,1)$. *Moreover, we suppose that* $N_n, n = 1, 2, \cdots$ *are independent positive integer-valued random variables independent of all* $X_{n,j}, n = 1, 2, \cdots n; n = 1, 2, \cdots$. *Set* $S_{N_n} = \sum_{j=1}^{N_n} X_{N_n,j}$. *Moreover, let us denote by* $Z_{\lambda_{N_n}}$ *the Poisson random variable with mean*

$$\lambda_{N_n} = E(S_{N_n}) = \sum_{n=1}^{\infty} P(N_n = n) E(S_n) = \sum_{n=1}^{\infty} P(N_n = n) \sum_{j=1}^{n} p_{n,j}^{-1} (1-p_{n,j}).$$

Then, for $f \in \mathbb{K}$

$$\| A_{S_{N_n}} f - A_{Z_{\lambda_{N_n}}} f \| \leq 2\|f\| E\left(\sum_{j=1}^{N_n} \left[(1-p_{N_n,j})^2 + \frac{(1-p_{N_n,j})}{p_{N_n,j}^2} \right] \right). \quad (3.2)$$

Proof. According to the Theorem 3.1, for $f \in \mathbb{K}$, and $x \in \mathbb{Z}_+$,

$$\| A_{S_{N_n}} f - A_{Z_{\lambda_{N_n}}} f \| \leq \sum_{m=1}^{\infty} P\left(N_n = m\right) \left\| A_{S_m}\left(f\right) - A_{Z_{\lambda_m}}\left(f\right) \right\|$$

$$\leq 2 \sum_{m=1}^{\infty} P\left(N_n = m\right) \|f\| \sum_{j=1}^{m} \left[(1 - p_{N_n,j})^2 + \frac{(1-p_{N_n,j})}{p_{N_n,j}^2} \right]$$

$$= 2 \|f\| \sum_{m=1}^{\infty} \left[P\left(N_n = m\right) \left(\sum_{j=1}^{m} \left[(1 - p_{N_n,j})^2 + \frac{(1-p_{N_n,j})}{p_{N_n,j}^2} \right] \right) \right]$$

$$= 2 \|f\| E \left(\sum_{j=1}^{N_n} \left[(1 - p_{N_n,j})^2 + \frac{1-p_{N_n,j}}{p_{N_n,j}^2} \right] \right)$$

Thus, for $f \in \mathbb{K}$

$$\| A_{S_{N_n}} f - A_{X_{\lambda_{N_n}}} \| \leq 2 \| f \| E \left(\sum_{j=1}^{N_n} \left[(1 - p_{N_n,j})^2 + \frac{(1 - p_{N_n,j})}{p_{N_n,j}^2} \right] \right).$$

This finishes the proof. □

Corollary 3.4. *On account of* (3.2), *for* $k \in \{0, 1, \cdots, n\}$,

$$\left| P\left(S_{N_n} = k\right) - P\left(Z_{\lambda_{N_n}} = k\right) \right| \leq 2E \left(\sum_{j=1}^{N_n} \left[(1 - p_{n,j})^2 + \frac{(1 - p_{n,j})}{p_{n,j}^2} \right] \right).$$

Theorem 3.5. *Let* $(X_{i,j}, i = 1, 2, \cdots; j = 1, 2, \cdots)$ *be a double array of independent geometric distributed random variables with probabilities*

$$P\left(X_{i,j} = k\right) = p_{i,j}(1 - p_{i,j})^k, 0 < p_{i,j} < 1, k = 0, 1, 2, \cdots; i, j = 1, 2, \cdots.$$

Assume that for every $i = 1, 2, \cdots$ *the random variables* $X_{i,1}, X_{i,2}, \ldots,$ *are independent, and for every* $j = 1, 2, \cdots$ *the random variables* $X_{1,j}, X_{2,j}, \cdots$ *are independent. Set* $S_{n,m} = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j}$. *Let us denote by* $Z_{\lambda_{n,m}}$ *the Poisson random variable with mean* $\lambda_{n,m} = E(S_{n,m}) = \sum_{i=1}^{n} \sum_{j=1}^{m} \left(1 - p_{i,j}\right) p_{i,j}^{-1}$. *Then, for* $f \in \mathbb{K}$

$$\| A_{S_{n,m}} f - A_{Z_{\lambda_{n,m}}} f \| \leq 2 \|f\| \sum_{i=1}^{n} \sum_{j=1}^{m} \left[(1 - p_{i,j})^2 + \frac{(1 - p_{i,j})}{p_{i,j}^2} \right]. \tag{3.3}$$

Proof. It is easy to check that

$$\| A_{S_{n,m}} f - A_{Z_{\lambda_{n,m}}} f \| \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \| A_{X_{i,j}} f - A_{Z_{(1-p_{i,j})p_{i,j}^{-1}}} f \|.$$

According to the Theorem 3.1, for $f \in \mathbb{K}$, and for all $x \in \mathbb{Z}_+$,

$$\| A_{X_{i,j}} f - A_{Z_{(1-p_{i,j})p_{i,j}^{-1}}} f \| \leq 2\|f\| \left[(1-p_{i,j})^2 + \frac{(1-p_{i,j})}{p_{i,j}^2} \right].$$

Thus

$$\| A_{S_{n,m}} f - A_{Z_{\lambda_{n,m}}} f \| \leq 2\|f\| \sum_{i=1}^{n} \sum_{j=1}^{m} \left[(1-p_{i,j})^2 + \frac{(1-p_{i,j})}{p_{i,j}^2} \right].$$

This finishes the proof. □

Corollary 3.6. *According to the Theorem 3.5, for $r \in \{0, 1, \ldots, n\}$,*

$$\left| P\left(S_{n,m} = r\right) - P\left(Z_{\lambda_{n,m}} = r\right) \right| \leq 2 \sum_{i=1}^{n} \sum_{j=1}^{m} \left[\left(1-p_{i,j}\right)^2 + \frac{(1-p_{i,j})}{p_{i,j}^2} \right].$$

Theorem 3.7. *Let $(X_{i,j}, i = 1, 2, \cdots; j = 1, 2, \cdots)$ be a double array of independent geometric distributed random variables with probabilities*

$$P\left(X_{i,j} = k\right) = p_{i,j}(1-p_{i,j})^k, 0 < p_{i,j} < 1, k = 0, 1, 2, \cdots; i, j = 1, 2, \cdots$$

Assume that for every $i = 1, 2, \cdots$ the random variables $X_{i,1}, X_{i,2}, \cdots$, are independent, and for every $j = 1, 2, \cdots$ the random variables $X_{1,j}, X_{2,j}, \cdots$ are independent. Set $S_{n,m} = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j}$. Moreover, suppose that N_n, M_m are non-negative integer-valued random variables independent of all $(X_{i,j}, i = 1, 2, \cdots; j = 1, 2, \cdots)$. Let us denote by $Z_{\lambda_{N_n M_m}}$ the Poisson random variable with mean

$$\lambda_{N_n, M_m} = E(S_{N_n, M_m}) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} P(N_n = n) P(M_m = m) E(S_{n,m}).$$

Then, for $f \in \mathbb{K}$

(3.4)

$$\| A_{S_{N_n, M_m}} f - A_{Z_{\lambda_{N_n, M_m}}} f \| \leq 2\|f\| E\left(\sum_{i=1}^{N_n} \sum_{j=1}^{M_m} \left[\left(1-p_{i,j}\right)^2 + \frac{(1-p_{i,j})}{p_{i,j}^2} \right] \right).$$

Proof. On account of the definition of Renyi's operator in (1.1), we check at once that

$$\begin{aligned}\left(A_{S_{N_n, M_m}} f\right)(x) &:= E\left(f\left(S_{N_n M_m} + x\right)\right) \\ &= \sum_{n=1}^{\infty} P\left(N_n = n\right) \sum_{m=1}^{\infty} P\left(M_n = m\right) \left(A_{S_{n,m}} f\right)(x)\end{aligned}$$

and

$$\begin{aligned}\left(A_{Z_{\lambda_{N_n, M_m}}} f\right)(x) &:= E\left(f\left(Z_{\lambda_{N_n M_m}} + x\right)\right) \\ &= \sum_{n=1}^{\infty} P\left(N_n = n\right) \sum_{m=1}^{\infty} P\left(M_n = m\right) \left(A_{Z_{\lambda_{nm}}} f\right)(x).\end{aligned}$$

Hence, for $f \in \mathbb{K}$, and for $x \in \mathbb{Z}_+$, we see at once that

$$\begin{aligned}
&\left\| P\left(S_{n,m}=r\right) - P\left(Z_{\lambda_{n,m}}=r\right)\right\| \\
&\leq \sum_{n=1}^{\infty} P\left(N_n=n\right) \sum_{m=1}^{\infty} P\left(M_n=m\right) \left\| A_{S_{nm}} f - A_{Z_{\lambda_{nm}}} f\right\| \\
&\leq 2\left\|f\right\| \sum_{n=1}^{\infty} P\left(N_n=n\right) \sum_{m=1}^{\infty} P\left(M_n=m\right) \left(\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\left(1-p_{i,j}\right)^2 + \frac{\left(1-p_{i,j}\right)}{p_{i,j}^2}\right]\right) \\
&= 2\left\|f\right\| \sum_{n=1}^{\infty} P\left(N_n=n\right) E\left(\sum_{i=1}^{n}\sum_{j=1}^{M_m}\left[\left(1-p_{i,j}\right)^2 + \frac{1-p_{i,j}}{p_{i,j}^2}\right]\right) \\
&= 2\left\|f\right\| E\left(\sum_{i=1}^{N_n}\sum_{j=1}^{M_m}\left[\left(1-p_{i,j}\right)^2 + \frac{\left(1-p_{i,j}\right)}{p_{i,j}^2}\right]\right).
\end{aligned}$$

Thus

$$\| A_{S_{N_n,M_m}} f - A_{Z_{\lambda_{N_n M_m}}} f \| \leq 2\left\|f\right\| E\left(\sum_{i=1}^{N_n}\sum_{j=1}^{M_m}\left[\left(1-p_{i,j}\right)^2 + \frac{\left(1-p_{i,j}\right)}{p_{i,j}^2}\right]\right).$$

This completes the proof. □

Corollary 3.8. *According to the Theorem 3.5, for* $r \in \{0, 1, \cdots, n\}$,

$$\left|P\left(S_{N_n,M_m}=r\right) - P\left(Z_{\lambda_{N_n M_m}}=r\right)\right| \leq 2E\left(\sum_{i=1}^{N_n}\sum_{j=1}^{M_m}\left[\left(1-p_{i,j}\right)^2 + \frac{\left(1-p_{i,j}\right)}{p_{i,j}^2}\right]\right).$$

We conclude this paper with the following comments. The results obtained in this note are illustrations for simplicity and elegant of the Renyi's operator method in Poisson approximation for independent geometric distributed random variables. Especially, this method is likely to be more effective for random vectors in higher dimension spaces.

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