Title:
Which elements of a finite group are non-vanishing?

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WHICH ELEMENTS OF A FINITE GROUP ARE NON-VANISHING?

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Abstract. Let $G$ be a finite group. An element $g \in G$ is called non-vanishing, if for every irreducible complex character $\chi$ of $G$, $\chi(g) \neq 0$. The bi-Cayley graph $BCay(G, T)$ of $G$ with respect to a subset $T \subseteq G$, is an undirected graph with vertex set $G \times \{1, 2\}$ and edge set $\{(x, 1), (tx, 2)\}$ $x \in G, t \in T$. Let $nv(G)$ be the set of all non-vanishing elements of a finite group $G$. We show that $g \in nv(G)$ if and only if the adjacency matrix of $BCay(G, T)$, where $T = Cl(g)$ is the conjugacy class of $g$, is non-singular. We prove that if the commutator subgroup of $G$ has prime order $p$, then

1. $g \in nv(G)$ if and only if $|Cl(g)| < p$.
2. if $p$ is the smallest prime divisor of $|G|$, then $nv(G) = Z(G)$.

Also we show that

(a) if $Cl(g) = \{g, h\}$, then $g \in nv(G)$ if and only if $gh^{-1}$ has odd order,
(b) if $|Cl(g)| \in \{2, 3\}$ and $(o(g), 6) = 1$, then $g \in nv(G)$.

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1. Introduction

Let $G$ be a finite group and $Irr(G)$ be the full set of complex irreducible characters of $G$. A classical theorem of W. Burnside states that every non-linear $\chi \in Irr(G)$ vanishes on some element of $G$. This is equivalent to say that in the character table of $G$, the rows which do not contain the value 0 are precisely those corresponding to linear characters.

The dual question: Which columns of a character table can fail to contain zero? posed by M. Issacs, G. Navarro and T. Wolf [7] in 1999. To investigate the question they introduced the concept of non-vanishing element of a finite group $G$: an element $x \in G$ is called non-vanishing if $\chi(x) \neq 0$ for every
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\(\chi \in \text{Irr}(G)\). Violating the standard duality between characters and conjugacy classes, it is in general not true that the columns not containing the value 0 are precisely those corresponding to conjugacy classes of central elements, as there are finite groups having non-central non-vanishing elements. In fact, a non-vanishing element of \(G\) can even fail to lie in an abelian normal subgroup of \(G\) (see Theorem 5.1 in [7]).

In [7] it is proved that non-vanishing odd order elements of a solvable group \(G\) all lie in a nilpotent normal subgroup of \(G\), i.e. they lie in the Fitting subgroup \(F(G)\). Some authors recently found other sufficient conditions for a non-vanishing element to lie in \(F(G)\), see [3, 5, 6]. In fact a non-vanishing element \(x \in G\) lies in \(F(G)\), when (1) the order of \(x\) is coprime to 6 [3] (2) \(G\) is a nilpotent-by-supersolvable group [5] (3) \(G\) is solvable of order divisible by neither a Fermat nor a Mersenne prime [6].

Issacs et.al. [7] proved that in a nilpotent group every non-vanishing element is central. Also they showed that in a group \(G\) with a normal Sylow \(p\)-subgroup \(P\), every element of \(Z(P)\), the center of \(P\), is non-vanishing. These results encourage some authors to find some groups with non-trivial non-vanishing elements. If \(G\) possesses a non-trivial elementary abelian normal \(p\)-subgroup \(A\) and \(P\) is a Sylow \(p\)-subgroup of \(G\), then all elements of \(Z(P) \cap A\) are non-vanishing in \(G\) [14]. Every irreducible character of \(G\) vanishes only on involutions if and only if \(G = E \times F\), where \(E\) is an elementary abelian 2-group and \(F\) is a Frobenius group with Frobenius complement of order two [2].

Our motivation differs from all previous works. Certainly, every central element of a group \(G\) is non-vanishing because if \(x \in Z(G)\), then \(|\chi(x)|^2 > 1\) for all characters \(\chi \in \text{Irr}(G)\), by [8, Corollary 2.28]. So it is a natural question that when a non-central element is non-vanishing. In this paper, we focus on the size of conjugacy class of non-central elements. We use the concept of bi-Cayley graph of a finite group to establish a relation between non-vanishing elements of a finite group \(G\) and the eigenvalues of a suitable bi-Cayley graph of \(G\) (by an eigenvalue (eigenvector) of a graph we mean an eigenvalue (eigenvector) of the corresponding adjacency matrix).

Let \(S\) be a subset of a group \(G\) not containing the identity element of \(G\). Recall that the Cayley graph \(\Gamma = \text{Cay}(G, S)\) of \(G\) with respect to \(S\) is the graph with vertex set \(G\), where \((x, y)\) is a directed edge if and only if \(yx^{-1} \in S\). Clearly \(\text{Cay}(G, S)\) is undirected if and only if \(G = S^{-1}\), where \(S^{-1} = \{s^{-1} | s \in S\}\).

Now we define a family of undirected bipartite graphs, the bi-Cayley graphs. For a finite group \(G\) and a non-empty subset \(S \subseteq G\), the bi-Cayley graph \(BCay(G, S)\) of \(G\) with respect to \(S\) is the graph with vertex set \(G \times \{1, 2\}\) and edge set \(\{(x, 1), (sx, 2)\} | x \in G, s \in S\). Then \(BCay(G, S)\) is a well-defined bipartite \(|S|\)-regular with bipartition subsets \(G \times \{1\}\) and \(G \times \{2\}\). By [11, p. 1259], \(BCay(G, S)\) is connected if and only if \(G = \langle SS^{-1} \rangle\). Furthermore, if \(1 \in S\) then \(BCay(G, S)\) is connected if and only if \(G = \langle S \rangle\). Also note that if
$S = S^{-1}$ then $\text{BCay}(G, S)$ is isomorphic to the tensor product $\text{Cay}(G, S) \otimes K_2$. Note that the connectivity of $\text{BCay}(G, S)$ is not equivalent to the connectivity of $\text{Cay}(G, S)$.

In Section 2, we compute the spectrum of $\text{BCay}(G, T)$, where $T$ is a conjugacy class of $G$ containing an element $x$. Recall that a graph with non-singular adjacency matrix is called non-singular. Note that a graph $\Gamma$ is non-singular if and only if 0 is not an eigenvalue of $\Gamma$. We prove that $x$ is non-vanishing if and only if $\text{BCay}(G, T)$ is non-singular.

We denote the set of all non-vanishing elements of a finite group $G$ by $\text{nv}(G)$. When $T_x^{-1}$ is a union of conjugacy classes of $\langle T \rangle$, or in particular when $\langle T \rangle$ is abelian, we prove that $x \in \text{nv}(G)$ if and only if $\sum_{t \in T} \chi(t x^{-1}) \neq 0$, for all $\chi \in \text{Irr}(\langle TT^{-1} \rangle)$. In most cases we assume that $\langle TT^{-1} \rangle$ is abelian and characterize non-vanishing elements. We prove that in a finite nilpotent group $G$ with an abelian commutator $p$-subgroup $G'$, $|G| = |\text{nv}(G)| \pmod{p}$, see Corollary 2.11. Also we show that in a finite group $G$ with $|G'| = p$, $p$ a prime, $G$ is nilpotent if and only if every non-central element has exactly $p$ conjugates in $G$, see Theorem 2.12.

In Section 3, we focus on the elements with 2 or 3 conjugates. Using the spectrum of $\text{BCay}(G, T)$, where $T = \{g, h\}$ is a conjugacy class of $G$, we prove that $g \in \text{nv}(G)$ (and so $h \in \text{nv}(G)$) if and only if $g h^{-1}$ is of odd order, see Corollary 3.1. As a result, we prove that every element $x$ with conjugacy class size 2 or 3 and $(o(x), 6) = 1$ is a non-vanishing element, see Corollary 3.5. Also we show that in a finite solvable group $G$ with derived length 2, if $(|G'|, 3) = 1$ then every element with 3 conjugates is non-vanishing, see Corollary 3.4.

2. Main results

The eigenvalues and eigenvectors of Cayley graphs with respect to a union of conjugacy classes were determined by Rô:

**Theorem 2.1.** (See [9, pp. 1-3]) Let $\Gamma = \text{Cay}(G, T)$ be a Cayley graph with respect to $T$. If $T$ is a union of conjugacy classes of $G$, then every eigenvalue of $\Gamma$ is of the form $\lambda_\chi := \sum_{t \in T} \chi(t)/\chi(1)$, for some $\chi \in \text{Irr}(G)$ and the eigenspace of $\Gamma$ corresponding to the eigenvalue $\lambda_\chi$ is generated by the eigenvectors $v_{\chi,i} := (\chi(g_1 g_1^{-1}), \chi(g_2 g_2^{-1}), \ldots, \chi(g_n g_n^{-1}))$, $i = 1, \ldots, n$.

Our terminology and notation will be standard. For the group-theoretic and graph-theoretic terminology not defined here we refer the reader to [8, 1]. In the following proposition, we find a relation between non-vanishing elements of a group $G$ and a non-singular bi-Cayley graph of $G$.

**Proposition 2.2.** Let $T$ be a union of conjugacy classes of $G$. Then $\text{BCay}(G, T)$ is non-singular if and only if for each $\chi \in \text{Irr}(G)$, $\sum_{t \in T} \chi(t) \neq 0$. In particular, if $T$ is a conjugacy class of $G$ containing $x$, then $x \in \text{nv}(G)$ if and only if $\text{BCay}(G, T)$ is non-singular.
Proof. Let $\Gamma = \text{BCay}(G, T)$ and $A$ be the adjacency matrix of $\Gamma$. A fixed chosen ordering $g_1 = 1, g_2, \ldots, g_n$ of elements of $G$ naturally determines the following induced ordering:

$$(g_1, 1), (g_2, 1), \ldots, (g_n, 1), (g_1, 2), (g_2, 2), \ldots, (g_n, 2)$$

of vertices of $\Gamma$. Hence relative to this ordering we have

$$A = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix},$$

where $B$ is the adjacency matrix of $\text{Cay}(G, T)$ and $C = B^\top$, the transposed matrix of $B$, is the adjacency matrix of $\text{Cay}(G, T^{-1})$. Since $T$ is a union of conjugacy classes of $G$, by Theorem 2.1, the vectors

$$v_{\chi, i} = (\chi(g_1 g_1^{-1}), \chi(g_2 g_2^{-1}), \ldots, \chi(g_n g_n^{-1})), \quad i = 1, \ldots, n,$$

are eigenvalues of $B$ corresponding to the eigenvalue $\lambda_\chi := \sum_{t \in T} \chi(t)/\chi(1)$, where $\chi \in \text{Irr}(G)$. Also $v_{\chi, i}, i = 1, \ldots, n$, are eigenvectors of $C$ corresponding to the eigenvalue $\sum_{t \in T} \chi(t^{-1})/\chi(1) = \overline{\lambda_\chi}$, the complex conjugate of $\lambda_\chi$.

On the other hand $\det(xI_{2n} - A) = \det(x^2I_n - CB)$ is the characteristic polynomial of $A$, where $I_n$ is the identity matrix of order $m$. Hence $\lambda$ is an eigenvalue of $A$ if and only if $\lambda^2$ is an eigenvalue of $CB$. Also for each $\chi \in \text{Irr}(G),

$$CBv_{\chi, i} = C\lambda_\chi v_{\chi, i} = \lambda_\chi Cx_{\chi, i} = \lambda_\chi \sum_{i} v_{\chi, i} = |\lambda_\chi|^2 v_{\chi, i},$$

$i = 1, \ldots, n$. Since $CB$ and $B$ have the same number of eigenvalues, this shows that $\lambda$ is an eigenvalue of $B$ if and only if $|\lambda|^2$ is an eigenvalue of $CB$. Since $\Gamma$ is a bipartite graph, for each eigenvalue $\lambda$ of $\Gamma$, $-\lambda$ is also an eigenvalue, with the same multiplicity, see [1, Proposition 3.4.1]. Consequently, $\lambda$ is an eigenvalue of $B$ if and only if $|\lambda|$ and $-|\lambda|$ are eigenvalues of $A$. Hence $\Gamma$ is non-singular if and only if for each $\chi \in \text{Irr}(G)$, $\lambda_\chi \neq 0$ if and only if for each $\chi \in \text{Irr}(G), \sum_{t \in T} \chi(t) \neq 0$. This completes the proof. □

Lemma 2.3. Let $T \subseteq G$, $\Gamma = \text{BCay}(G, T)$ and $H = \langle TT^{-1} \rangle$. Then

1. for each $x \in T$, $Tx^{-1} \subseteq H$, $\text{BCay}(H, Tx^{-1})$ is connected and $\Gamma$ is non-singular if and only if $\text{BCay}(H, Tx^{-1})$ is non-singular,
2. if $T$ is a conjugacy class of $G$, then $H \preceq G'$ and $H \preceq G$.

Proof. Let $x \in T$. Clearly $Tx^{-1} \subseteq H$. On the other hand $\langle Tx^{-1} (Tx^{-1})^{-1} \rangle = \langle TT^{-1} \rangle = H$. Hence, by [4], $\text{BCay}(H, Tx^{-1})$ is a connected bi-Cayley graph. Also $\langle G : H \rangle | \text{BCay}(H, Tx^{-1}) \cong \text{BCay}(G, Tx^{-1})$, see [11, p. 1260], and $\text{BCay}(G, Tx^{-1}) \cong \text{BCay}(G, T)$, by [13, Lemma 2.2]. Hence 0 is an eigenvalue of $\Gamma$ if and only if 0 is an eigenvalue of $\text{BCay}(H, Tx^{-1})$.

Now suppose that $T$ is a conjugacy class of $G$. For each $t_1, t_2 \in T$, there exists $g \in G$ such that $t_1 = g^{-1}tg$. Hence $t_1 t_2^{-1} = g^{-1} t_2 g t_2^{-1} \in G'$ which
shows that $H \leq G'$. Also for each $g \in G$, and $t_1, t_2 \in T$, $g^{-1}t_1t_2^{-1}g = g^{-1}t_1(g^{-1}t_2g)^{-1} \in TT^{-1}$. Hence $H \leq G$. \hfill \Box

Now combining Proposition 2.2 and Lemma 2.3, we have the following result.

**Theorem 2.4.** Let $T = \text{Cl}(x)$ be a conjugacy class of $G$ containing $x$. Then the following statements are equivalent.

1. $x \in \text{nv}(G)$,
2. $\text{BCay}(G, T)$ is non-singular,
3. $\text{BCay}(TT^{-1}, Tx^{-1})$ is non-singular.

In the following corollary, as an application of Theorem 2.4, we obtain a necessary condition for an element to be a non-vanishing element. First we recall that 0 is not an eigenvalue of the complete bipartite graph $K_{m,n}$ if and only if $m + n = 2$, i.e. $m = n = 1$, see [1, 1.5.2].

**Corollary 2.5.** Let $g \in \text{nv}(G)$ and $T$ be a non-central conjugacy class containing $g$. Then $|TT^{-1}| \neq |T|$.

**Proof.** Suppose, for a contradiction, that $|TT^{-1}| = |T|$. Let $H = \langle TT^{-1} \rangle$. Since $|Ty^{-1}| = |T| = |H|, Ty^{-1} = H$, it follows that $\text{BCay}(H, Ty^{-1})$ is isomorphic to the complete bipartite graph $K_{|H|,|H|}$. By Theorem 2.4, $\text{BCay}(H, Ty^{-1})$ is non-singular. So $K_{|H|,|H|}$ is non-singular which implies that $|H| = 1$, a contradiction. \hfill \Box

**Theorem 2.6.** Let $G$ be a group, $T = \text{Cl}(g)$ and $H = \langle TT^{-1} \rangle$. If $Ty^{-1}$ is a union of conjugacy classes of $H$ (or in particular if $H$ is abelian) then

$$g \in \text{nv}(G) \iff \text{for all } \chi \in \text{Irr}(H), \sum_{t \in T}\chi(ty^{-1}) \neq 0.$$ 

**Proof.** Let $\Gamma = \text{BCay}(H, Ty^{-1})$. Then by Theorem 2.4, $g \in \text{nv}(G)$ if and only if $\Gamma$ is non-singular. On the other hand, by Proposition 2.2, $\Gamma$ is non-singular if and only if for each $\chi \in \text{Irr}(H), \sum_{t \in T}\chi(ty^{-1}) \neq 0$. This completes the proof. \hfill \Box

Now we are ready to determine some non-vanishing elements of a finite group. First let us recall the main theorem of [12].

**Theorem 2.7.** There is some vanishing sum $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n = 0$ of $m$-th roots of unity if and only if $n$ is a linear combination, with non-negative integer coefficients, of the prime divisors of $m$.

The following theorem is well-known, see for example [8, Theorem 4.21].

**Theorem 2.8.** Let $G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_t}$, where $t \geq 1$, be a finite abelian group. Then for every $g = (g_1, \ldots, g_t) \in G$, $g_i \in \mathbb{Z}_{n_i}$, and $\chi \in \text{Irr}(G)$, there exist $\chi_i \in \text{Irr}(\mathbb{Z}_{n_i}), i = 1, \ldots, t$, such that $\chi(g) = \chi_1(g_1) \cdots \chi_t(g_t)$. Furthermore, each $\chi_i(g_i)$ is an $n_i$-th root of unity.
Lemma 2.9. Let \( g \) be an element of a finite group \( G \), \( T = \text{Cl}(g) \) and \( H = \langle TT^{-1} \rangle \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_t} \), where \( n_1 \mid n_2 \mid \cdots \mid n_t \). If \( |T| \) is not any linear combination, with non-negative integer coefficients, of prime divisors of \( n_t \), then \( g \in \text{nv}(G) \).

Proof. First note that since \( n_i \mid n_t \), \( i = 1, \ldots, t \), Theorem 2.8 implies that for every \( \chi \in \text{Irr}(H) \) and \( t \in T \), \( \chi(tg^{-1}) \) is an \( n_i \)th root of unity. Now suppose \( |T| \) is not any linear combination, with non-negative integer coefficients, of prime divisors of \( n_t \). Suppose, by contrary, that \( g \in G \setminus \text{nv}(G) \). Then Theorem 2.6 implies that \( \sum_{t \in T} \chi(tg^{-1}) = 0 \), for some \( \chi \in \text{Irr}(H) \). Hence, by Theorem 2.7, \( |T| \) is a linear combination, with non-negative integer coefficients, of the prime divisors of \( n_t \), a contradiction. 

Remark 2.10. Consider the non-abelian group \( G \cong ((\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{Z}_3) \times \mathbb{Z}_2 \) of order 72. One can check, for example by GAP software, that there exists a conjugacy class \( T := \text{Cl}(x) \) in \( G \) of size 6 where \( x \in \text{nv}(G) \). Also \( H := \langle TT^{-1} \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_6 \). This shows that the converse of Lemma 2.9 is not true in general (Here \( H \times K \) denotes the semidirect product of \( H \) by \( K \)). Hence the natural question is that in which groups the converse of Lemma 2.9 is true?

Corollary 2.11. Let \( p \) be a prime, \( g \) an element of a finite group \( G \), \( T = \text{Cl}(g) \) and \( H = \langle TT^{-1} \rangle \) be an abelian \( p \)-group. If \( (|T|, p) = 1 \) then \( g \in \text{nv}(G) \). In particular, in a finite nilpotent group \( G \) with abelian \( p \)-subgroup \( G' \), \( |G| \equiv |\text{nv}(G)| \) (mod \( p \)).

Proof. The first part is a direct consequence of Lemma 2.9. Now let \( G \) be a finite nilpotent group with an abelian \( p \)-subgroup \( G' \). Let \( g \) be a non-central element of \( G \), \( T = \text{Cl}(g) \) and \( H = \langle TT^{-1} \rangle \). By Lemma 2.3, \( H \) is also an abelian \( p \)-group. On the other hand by [7, Theorem B], \( \text{nv}(G) = \mathbb{Z}(G) \), which implies that \( g \in G \setminus \text{nv}(G) \). Hence, by the first part, \( p \mid |T| \). Now the class equation implies that \( |G| \equiv |\mathbb{Z}(G)| \) (mod \( p \)) and so \( |G| \equiv |\text{nv}(G)| \) (mod \( p \)). 

In the following theorem we consider finite groups whose commutator subgroups have prime order.

Theorem 2.12. Let \( p \) be a prime and \( g \) be an element of a finite group \( G \). If \( |G'| = p \), then

1. \( g \in \text{nv}(G) \) if and only if \( |\text{Cl}(g)| < p \) (or equivalently \( |\text{Cl}(g)| \neq p \)),
2. if \( p \) is the smallest prime divisor of \( |G| \), then \( \text{nv}(G) = \mathbb{Z}(G) \),
3. \( G \) is nilpotent if and only if \( |\text{Cl}(g)| = \{1, p\} \), where \( \text{Cl}(G) = \{|\text{Cl}(g)| \mid g \in G\} \).

Proof. (1) Let \( T = \text{Cl}(g) \) be the conjugacy class containing \( g \). Since \( |G'| = p \), \( |T| \leq p \). Also, by Lemma 2.3, \( |H| = p \). Let \( g \in \text{nv}(G) \), then by Corollary 2.5, \( |T| < p \). Conversely, let \( |T| < p \). Then \( p \nmid |T| \) and Corollary 2.11 implies that \( g \in \text{nv}(G) \).
Let \( f : G \to H \) implies that \( f(g) = f(h) \) if and only if \( g = h \) since \( H \). Thus by \([7, \text{Theorem B}], \) \( g \in G \backslash \text{nv}(G) \). So Lemma 2.11 implies that \( p \mid |\text{Cl}(g)| \). On the other hand \( |\text{Cl}(g)| \leq |G'| = p \), which means that \( |\text{Cl}(g)| = p \). Hence \( \text{Cl}(G) = \{1, p\} \).

3. Non-vanishing elements with 2 or 3 conjugates

In this section, we focus on elements with 2 or 3 conjugates. In the following corollary, we give a complete classification of non-vanishing elements with 2 conjugates. Let \( C_n \) be the undirected graph of a cycle with \( n \) vertices. Then the eigenvalues of \( C_n \) are the numbers \( 2\cos(2\pi j/n), j = 0, \ldots, n-1 \), see [1, 1.5.3]. Now it is easy to see that \( C_n \) is non-singular if and only if \( 4 \nmid n \).

**Corollary 3.1.** Let \( G \) be a finite group with an element \( g \) such that \( T := \text{Cl}(g) = \{g, h\} \). Then \( g \in \text{nv}(G) \) if and only if \( gh^{-1} \) has odd order. In particular,

1. if \( \langle g \rangle \cap \langle h \rangle = 1 \), then \( g \in \text{nv}(G) \) if and only if \( g \) is of odd order,
2. if \( g \) is of odd order, then \( g \in \text{nv}(G) \),
3. if \( G' \) is of odd order, then \( g \in \text{nv}(G) \).

**Proof.** Let \( H = \langle TT^{-1} \rangle \), and \( \Gamma = \text{BCay}(G, T) \). By Lemma 2.3, \( \Gamma \cong |G : H|\text{BCay}(H, Tg^{-1}) \), and \( \Sigma := \text{BCay}(H, Tg^{-1}) \) is a connected graph. Indeed since \( |Tg^{-1}| = 2 \), \( \Sigma \) is an undirected cycle with \( 2|H| \) vertices. Now

\[
\begin{align*}
g \in \text{nv}(G) & \iff \Sigma \text{ is non-singular (by Theorem 2.4)} \\
& \iff 4 \nmid 2|H| \text{ (by above discussion)} \\
& \iff |H| \text{ is odd.}
\end{align*}
\]

On the other hand \( H = \langle TT^{-1} \rangle = \langle 1, gh^{-1}, h^{-1}g \rangle = \langle gh^{-1} \rangle \). Hence \( g \in \text{nv}(G) \) if and only if \( gh^{-1} \) is of odd order.

Since \( h^{-1}gh \in T \) and \( h \neq g \), \( h^{-1}gh = g \). So \( g \) and \( h^{-1} \) commute. Also \( g \) and \( h \) have the same order. Hence the following results are straightforward.

If \( \langle g \rangle \cap \langle h \rangle = 1 \) then \( o(gh^{-1}) = o(g) \).

If \( g \) is of odd order \( n \) then, since \( o(gh^{-1}) \mid n \), \( o(gh^{-1}) \) is odd.

If \( |G'| \) is odd then, since \( gh^{-1} \in G' \), \( o(gh^{-1}) \) is odd. \( \square \)

**Lemma 3.2.** Let \( x \in G \) be an element of odd order \( k \) and \( T = \{x, y, z\} \) be the conjugacy class of \( G \) containing \( x \). Then the elements of \( T \) commute and so \( \langle TT^{-1} \rangle = \langle xy^{-1}, zy^{-1} \rangle \) is a normal abelian subgroup of \( G \).
Proof. Let \( a \in T \), \( b \in T \setminus \{a\} \), and suppose, by contrary, that \( ab \neq ba \). Since \( b^{-1}ab \in T \), \( b^{-1}ab = c \). On the other hand \( bab^{-1} \in T \), which implies that \( bab^{-1} = c = b^{-1}ab \). Hence \( b^2a = ab^2 \). Since \( k \) is odd, there exist \( r, s \in \mathbb{Z} \) such that \( 1 = rk + 2s \). So \( ab = ab^r b^2 s = ab^2 = b^s a = bv^{k+2}s a = ba \). This shows that \( \langle TT^{-1} \rangle \) is abelian. Also

\[
\langle TT^{-1} \rangle = \langle 1, xy^{-1}, yx^{-1}, xz^{-1}, zy^{-1} \rangle = \langle xy^{-1}, zy^{-1} \rangle \quad (\text{since } xy^{-1}(zy^{-1})^{-1} = xz^{-1}),
\]

which completes the proof.

\[\square\]

**Theorem 3.3.** Let \( G \) be a finite group, \( x \in G \), \( T := \text{Cl}(x) = \{x, y, z\} \), and \( H := \langle TT^{-1} \rangle \) be abelian. Suppose that

\[H = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_t},\]

where \( n_1 \mid n_2 \mid \cdots \mid n_t \), and for \( j = 1, \ldots, t \), \( \mathbb{Z}_{n_j} = \langle a_j \rangle \) and \( n_j \geq 2 \) and \( t \geq 1 \). Then \( x \in G \setminus \text{nv}(G) \) if and only if for all \( j \in \{1, \ldots, t\} \), there exists \( r_j \in \{0, \ldots, n_j - 1\} \) such that

\[3n_t \mid \left( 3 \sum_{j=1}^{t} b_j k_j r_j - n_t \right) \quad \text{and} \quad 3n_t \mid \left( 3 \sum_{j=1}^{t} b_j l_j r_j + n_t \right),\]

where \( xy^{-1} = (a_1^{b_1}, \ldots, a_t^{b_t}) \), \( zy^{-1} = (a_1^{l_1}, \ldots, a_t^{l_t}) \) and \( b_j = n_t/n_j \), \( j = 1, \ldots, t \).

Proof. We have \( Ty^{-1} \subseteq H \). Since \( H \) is abelian, by Theorem 2.6, \( x \in \text{nv}(G) \) if and only if for each \( \chi \in \text{Irr}(H) \), \( \sum_{t \in T} \chi(ty^{-1}) \neq 0 \); that is \( x \in \text{nv}(G) \) if and only if for each \( \chi \in \text{Irr}(H) \), \( 1 + \chi(xy^{-1}) + \chi(zy^{-1}) \neq 0 \). Thus \( x \in G \setminus \text{nv}(G) \) if and only if there exists \( \chi \in \text{Irr}(H) \) such that \( 1 + \chi(xy^{-1}) + \chi(zy^{-1}) = 0 \).

Suppose that \( \chi \in \text{Irr}(H) \). Then by Theorem 2.8, there exist \( \chi_j \in \text{Irr}(\mathbb{Z}_{n_j}) \), \( j = 1, \ldots, t \), such that for all \( h = (a_1^{m_1}, \ldots, a_t^{m_t}) \in H \), \( \chi(h) = \chi_1(a_1^{m_1}) \chi_2(a_2^{m_2}) \cdots \chi_t(a_t^{m_t}) \). On the other hand, for each \( j = 1, \ldots, t \), there exists \( r_j \in \{0, \ldots, n_j - 1\} \) such that \( \chi_j(a_j^{m_j}) = \exp(2\pi im_j r_j / n_j) \). Let \( xy^{-1} = (a_1^{l_1}, \ldots, a_t^{l_t}) \),
Let $3.2$ an integer $i$. Since $(6)$

\[ j \in \mathbb{N}, \quad i_1, i_2, i_3, \quad 3 \not| 1. \]

Then by Theorem 3.3.1, there exists an integer $k$ such that

\[ 3 | n \cdot k - n. \]

So $3 \not| n$, which implies that $3$ divides $|H|$, a contradiction.

Since, by Lemma 2.3, $H \leq G'$, the second part follows immediately.

**Corollary 3.5.** Let $G$ be a finite group, $x \in G$ and $|\text{Cl}(x)| = 2$ or $3$. If $(o(x), 6) = 1$ then $x \in \text{nv}(G)$.

**Proof.** Since $(6, o(x)) = 1, x$ is of odd order and $3 \not| o(x)$. If $|\text{Cl}(x)| = 2$ then by Corollary 3.1, $x \in \text{nv}(G)$. Now suppose that $|\text{Cl}(x)| = 3$. Then by Lemma 3.2, elements of $T := \text{Cl}(x) = \{x, y, z\}$ commute and $H := \langle TT^{-1} \rangle = \langle xy^{-1}, zy^{-1} \rangle$ is abelian. So every element $h$ of $H$ is of the form $h = x^{i_1}y^{i_2}z^{i_3}$, for some integers $i_1, i_2, i_3$. Since $o(x) = o(y) = o(z), o(h) = o(x)$. This shows that $3 \not| |H|$. Hence, by Corollary 3.4, $x \in \text{nv}(G)$, which completes the proof.
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