ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 42 (2016), No. 5, pp. 1097–1106

Title:

Which elements of a finite group are non-vanishing?

Author(s):

M. Arezoomand and B. Taeri

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 5, pp. 1097–1106
Online ISSN: 1735-8515

WHICH ELEMENTS OF A FINITE GROUP ARE NON-VANISHING?

M. AREZOOMAND AND B. TAERI*

(Communicated by Jamshid Moori)

Abstract. Let G be a finite group. An element $g \in G$ is called non-vanishing, if for every irreducible complex character χ of G, $\chi(g) \neq 0$. The bi-Cayley graph $\mathrm{BCay}(G,T)$ of G with respect to a subset $T \subseteq G$, is an undirected graph with vertex set $G \times \{1,2\}$ and edge set $\{\{(x,1),(tx,2)\} \mid x \in G,\ t \in T\}$. Let $\mathrm{nv}(G)$ be the set of all non-vanishing elements of a finite group G. We show that $g \in \mathrm{nv}(G)$ if and only if the adjacency matrix of $\mathrm{BCay}(G,T)$, where $T = \mathrm{Cl}(g)$ is the conjugacy class of g, is non-singular. We prove that if the commutator subgroup of G has prime order p, then

(1) $g \in \mathrm{nv}(G)$ if and only if $|\mathrm{Cl}(g)| < p$,

(2) if p is the smallest prime divisor of $|G|$, then $\mathrm{nv}(G) = Z(G)$.

Also we show that

(a) if $\mathrm{Cl}(g) = \{g,h\}$, then $g \in \mathrm{nv}(G)$ if and only if gh^{-1} has odd order,

(b) if $|\mathrm{Cl}(g)| \in \{2,3\}$ and $(\mathrm{o}(g),6) = 1$, then $g \in \mathrm{nv}(G)$.

Keywords: Non-vanishing element, character, conjugacy class, Bi-Cayley graph.
MSC(2010): Primary: 20C15; Secondary: 05C25, 05C50.

1. **Introduction**

Let G be a finite group and $\mathrm{Irr}(G)$ be the full set of complex irreducible characters of G. A classical theorem of W. Burnside states that every non-linear $\chi \in \mathrm{Irr}(G)$ vanishes on some element of G. This is equivalent to say that in the character table of G, the rows which do not contain the value 0 are precisely those corresponding to linear characters.

The dual question: Which columns of a character table can fail to contain zero? posed by M. Issacs, G. Navarro and T. Wolf [7] in 1999. To investigate the question they introduced the concept of non-vanishing element of a finite group G: an element $x \in G$ is called non-vanishing if $\chi(x) \neq 0$ for every

Article electronically published on October 31, 2016.
Received: 4 January 2014, Accepted: 2 July 2015.
*Corresponding author.

$\chi \in \mathrm{Irr}(G)$. Violating the standard duality between characters and conjugacy classes, it is in general not true that the columns not containing the value 0 are precisely those corresponding to conjugacy classes of central elements, as there are finite groups having non-central non-vanishing elements. In fact, a non-vanishing element of G can even fail to lie in an abelian normal subgroup of G (see Theorem 5.1 in [7]).

In [7] it is proved that non-vanishing odd order elements of a solvable group G all lie in a nilpotent normal subgroup of G, i.e. they lie in the Fitting subgroup $\mathrm{F}(G)$. Some authors recently found other sufficient conditions for a non-vanishing element to lie in $F(G)$, see [3, 5, 6]. In fact a non-vanishing element $x \in G$ lies in $F(G)$, when (1) the order of x is coprime to 6 [3] (2) G is a nilpotent-by-supersolvable group [5] (3) G is solvable of order divisible by neither a Fermat nor a Mersenne prime [6].

Issacs et.al. [7] proved that in a nilpotent group every non-vanishing element is central. Also they showed that in a group G with a normal Sylow p-subgroup P, every element of $\mathrm{Z}(P)$, the center of P, is non-vanishing. These results encourage some authors to find some groups with non-trivial non-vanishing elements. If G possesses a non-trivial elementary abelian normal p-subgroup A and P is a Sylow p-subgroup of G, then all elements of $Z(P) \cap A$ are non-vanishing in G [14]. Every irreducible character of G vanishes only on involutions if and only if $G = E \times F$, where E is an elementary abelian 2-group and F is a Frobenius group with Frobenius complement of order two [2].

Our motivation differs from all previous works. Certainly, every central element of a group G is non-vanishing because if $x \in \mathrm{Z}(G)$, then $|\chi(x)| = \chi(1) > 0$ for all characters $\chi \in \mathrm{Irr}(G)$, by [8, Corollary 2.28]. So it is a natural question that when a non-central element is non-vanishing. In this paper, we focus on the size of conjugacy class of non-central elements. We use the concept of bi-Cayley graph of a finite group to establish a relation between non-vanishing elements of a finite group G and the eigenvalues of a suitable bi-Cayley graph of G (by an eigenvalue (eigenvector) of a graph we mean eigenvalue (eigenvector) of the corresponding adjacency matrix).

Let S be a subset of a group G not containing the identity element of G. Recall that the Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ of G with respect to S is the graph with vertex set G, where (x, y) is a directed edge if and only if $yx^{-1} \in S$. Clearly $\mathrm{Cay}(G, S)$ is undirected if and only if $S = S^{-1}$, where $S^{-1} = \{s^{-1} \mid s \in S\}$.

Now we define a family of undirected bipartite graphs, the bi-Cayley graphs. For a finite group G and a non-empty subset $S \subseteq G$, the *bi-Cayley graph* $\mathrm{BCay}(G, S)$ of G with respect to S is the graph with vertex set $G \times \{1, 2\}$ and edge set $\{\{(x, 1), (sx, 2)\} \mid x \in G,\ s \in S\}$. Then $\mathrm{BCay}(G, S)$ is a well-defined bipartite $|S|$-regular with bipartition subsets $G \times \{1\}$ and $G \times \{2\}$. By [11, p. 1259], $\mathrm{BCay}(G, S)$ is connected if and only if $G = \langle SS^{-1} \rangle$. Furthermore, if $1 \in S$ then $\mathrm{BCay}(G, S)$ is connected if and only if $G = \langle S \rangle$. Also note that if

$S = S^{-1}$ then $\mathrm{BCay}(G,S)$ is isomorphic to the tensor product $\mathrm{Cay}(G,S) \otimes K_2$. Note that the connectivity of $\mathrm{BCay}(G,S)$ is not equivalent to the connectivity of $\mathrm{Cay}(G,S)$.

In Section 2, we compute the spectrum of $\mathrm{BCay}(G,T)$, where T is a conjugacy class of G containing an element x. Recall that a graph with non-singular adjacency matrix is called *non-singular*. Note that a graph Γ is non-singular if and only if 0 is not an eigenvalue of Γ. We prove that x is non-vanishing if and only if $\mathrm{BCay}(G,T)$ is non-singular.
We denote the set of all non-vanishing elements of a finite group G by $\mathrm{nv}(G)$. When Tx^{-1} is a union of conjugacy classes of $\langle TT^{-1}\rangle$, or in particular when $\langle TT^{-1}\rangle$ is abelian, we prove that $x \in \mathrm{nv}(G)$ if and only if $\sum_{t\in T} \chi(tx^{-1}) \neq 0$, for all $\chi \in \mathrm{Irr}(\langle TT^{-1}\rangle)$. In most cases we assume that $\langle TT^{-1}\rangle$ is abelian and characterize non-vanishing elements. We prove that in a finite nilpotent group G with an abelian commutator p-subgroup G', $|G| \equiv |\mathrm{nv}(G)| \pmod p$, see Corollary 2.11. Also we show that in a finite group G with $|G'| = p$, p a prime, G is nilpotent if and only if every non-central element has exactly p conjugates in G, see Theorem 2.12.

In Section 3, we focus on the elements with 2 or 3 conjugates. Using the spectrum of $\mathrm{BCay}(G,T)$, where $T = \{g, h\}$ is a conjugacy class of G, we prove that $g \in \mathrm{nv}(G)$ (and so $h \in \mathrm{nv}(G)$) if and only if gh^{-1} is of odd order, see Corollary 3.1. As a result, we prove that every element x with conjugacy class size 2 or 3 and $(\mathrm{o}(x), 6) = 1$ is a non-vanishing element, see Corollary 3.5. Also we show that in a finite solvable group G with derived length 2, if $(|G'|, 3) = 1$ then every element with 3 conjugates is non-vanishing, see Corollary 3.4.

2. Main results

The eigenvalues and eigenvectors of Cayley graphs with respect to a union of conjugacy classes were determined by Itô:

Theorem 2.1. *(See [9, pp. 1-3]) Let $\Gamma = \mathrm{Cay}(G,T)$ be a Cayley graph with respect to T. If T is a union of conjugacy classes of G, then every eigenvalue of Γ is of the form $\lambda_\chi := \sum_{t\in T} \chi(t)/\chi(1)$, for some $\chi \in \mathrm{Irr}(G)$ and the eigenspace of Γ corresponding to the eigenvalue λ_χ is generated by the eigenvectors*

$$v_{\chi,i} := (\chi(g_i g_1^{-1}), \chi(g_i g_2^{-1}), \ldots, \chi(g_i g_n^{-1})), \quad i = 1, \ldots, n.$$

Our terminology and notation will be standard. For the group-theoretic and graph-theoretic terminology not defined here we refer the reader to [8, 1]. In the following proposition, we find a relation between non-vanishing elements of a group G and a non-singular bi-Cayley graph of G.

Proposition 2.2. *Let T be a union of conjugacy classes of G. Then $\mathrm{BCay}(G,T)$ is non-singular if and only if for each $\chi \in \mathrm{Irr}(G)$, $\sum_{t\in T} \chi(t) \neq 0$. In particular, if T is a conjugacy class of G containing x, then $x \in \mathrm{nv}(G)$ if and only if $\mathrm{BCay}(G,T)$ is non-singular.*

Proof. Let $\Gamma = \mathrm{BCay}(G, T)$ and A be the adjacency matrix of Γ. A fixed chosen ordering $g_1 = 1, g_2, \ldots, g_n$ of elements of G naturally determines the following induced ordering:

$$(g_1, 1), (g_2, 1), \ldots, (g_n, 1), (g_1, 2), (g_2, 2), \ldots, (g_n, 2)$$

of vertices of Γ. Hence relative to this ordering we have

$$A = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix},$$

where B is the adjacency matrix of $\mathrm{Cay}(G, T)$ and $C = B^{\top}$, the transposed matrix of B, is the adjacency matrix of $\mathrm{Cay}(G, T^{-1})$. Since T is a union of conjugacy classes of G, by Theorem 2.1, the vectors

$$v_{\chi,i} = (\chi(g_i g_1^{-1}), \chi(g_i g_2^{-1}), \ldots, \chi(g_i g_n^{-1})), \quad i = 1, \ldots, n,$$

are eigenvectors of B corresponding to the eigenvalue $\lambda_\chi := \sum_{t \in T} \chi(t)/\chi(1)$, where $\chi \in \mathrm{Irr}(G)$. Also $v_{\chi,i}$, $i = 1, \ldots, n$, are eigenvectors of C corresponding to the eigenvalue $\sum_{t \in T} \chi(t^{-1})/\chi(1) = \overline{\lambda_\chi}$, the complex conjugate of λ_χ.

On the other hand $\det(xI_{2n} - A) = \det(x^2 I_n - CB)$ is the characteristic polynomial of A, where I_m is the identity matrix of order m. Hence λ is an eigenvalue of A if and only if λ^2 is an eigenvalue of CB. Also for each $\chi \in \mathrm{Irr}(G)$,

$$CBv_{\chi,i} = C\lambda_\chi v_{\chi,i} = \lambda_\chi C v_{\chi,i} = \lambda_\chi \overline{\lambda_\chi} v_{\chi,i} = |\lambda_\chi|^2 v_{\chi,i},$$

$i = 1, \ldots, n$. Since CB and B have the same number of eigenvalues, this shows that λ is an eigenvalue of B if and only if $|\lambda|^2$ is an eigenvalue of CB. Since Γ is a bipartite graph, for each eigenvalue λ of Γ, $-\lambda$ is also an eigenvalue, with the same multiplicity, see [1, Proposition 3.4.1]. Consequently, λ is an eigenvalue of B if and only if $|\lambda|$ and $-|\lambda|$ are eigenvalues of A. Hence Γ is non-singular if and only if for each $\chi \in \mathrm{Irr}(G)$, $\lambda_\chi \neq 0$ if and only if for each $\chi \in \mathrm{Irr}(G)$, $\sum_{t \in T} \chi(t) \neq 0$. This completes the proof. □

Lemma 2.3. *Let $T \subseteq G$, $\Gamma = \mathrm{BCay}(G, T)$ and $H = \langle TT^{-1} \rangle$. Then*

(1) *for each $x \in T$, $Tx^{-1} \subseteq H$, $\mathrm{BCay}(H, Tx^{-1})$ is connected and Γ is non-singular if and only if $\mathrm{BCay}(H, Tx^{-1})$ is non-singular,*

(2) *if T is a conjugacy class of G, then $H \leq G'$ and $H \trianglelefteq G$.*

Proof. Let $x \in T$. Clearly $Tx^{-1} \subseteq H$. On the other hand $\langle Tx^{-1}(Tx^{-1})^{-1} \rangle = \langle TT^{-1} \rangle = H$. Hence, by [4], $\mathrm{BCay}(H, Tx^{-1})$ is a connected bi-Cayley graph. Also $|G : H| \mathrm{BCay}(H, Tx^{-1}) \cong \mathrm{BCay}(G, Tx^{-1})$, see [11, p. 1260], and $\mathrm{BCay}(G, Tx^{-1}) \cong \mathrm{BCay}(G, T)$, by [13, Lemma 2.2]. Hence 0 is an eigenvalue of Γ if and only if 0 is an eigenvalue of $\mathrm{BCay}(H, Tx^{-1})$.

Now suppose that T is a conjugacy class of G. For each $t_1, t_2 \in T$, there exists $g \in G$ such that $t_1 = g^{-1} t_2 g$. Hence $t_1 t_2^{-1} = g^{-1} t_2 g t_2^{-1} \in G'$ which shows that $H \leq G'$. Also for each $g \in G$, and $t_1, t_2 \in T$, $g^{-1}t_1t_2^{-1}g = g^{-1}t_1g(g^{-1}t_2g)^{-1} \in TT^{-1}$. Hence $H \trianglelefteq G$. □

Now combining Proposition 2.2 and Lemma 2.3, we have the following result.

Theorem 2.4. *Let $T = \mathrm{Cl}(x)$ be a conjugacy class of G containing x. Then the following statements are equivalent.*

(1) $x \in \mathrm{nv}(G)$,
(2) $\mathrm{BCay}(G, T)$ *is non-singular,*
(3) $\mathrm{BCay}(\langle TT^{-1}\rangle, Tx^{-1})$ *is non-singular.*

In the following corollary, as an application of Theorem 2.4, we obtain a necessary condition for an element to be a non-vanishing element. First we recall that 0 is not an eigenvalue of the complete bipartite graph $K_{m,n}$ if and only if $m + n = 2$, i.e. $m = n = 1$, see [1, 1.5.2].

Corollary 2.5. *Let $g \in \mathrm{nv}(G)$ and T be a non-central conjugacy class containing g. Then $|\langle TT^{-1}\rangle| \neq |T|$.*

Proof. Suppose, for a contradiction, that $|\langle TT^{-1}\rangle| = |T|$. Let $H = \langle TT^{-1}\rangle$. Since $|Tg^{-1}| = |T| = |H|$, $Tg^{-1} = H$, it follows that $\mathrm{BCay}(H, Tg^{-1})$ is isomorphic to the complete bipartite graph $K_{|H|,|H|}$. By Theorem 2.4, $\mathrm{BCay}(H, Tg^{-1})$ is non-singular. So $K_{|H|,|H|}$ is non-singular which implies that $|H| = 1$, a contradiction. □

Theorem 2.6. *Let G be a group, $T = \mathrm{Cl}(g)$ and $H = \langle TT^{-1}\rangle$. If Tg^{-1} is a union of conjugacy classes of H (or in particular if H is abelian) then*

$$g \in \mathrm{nv}(G) \Longleftrightarrow \text{for all } \chi \in \mathrm{Irr}(H), \quad \sum_{t \in T} \chi(tg^{-1}) \neq 0.$$

Proof. Let $\Gamma = \mathrm{BCay}(H, Tg^{-1})$. Then by Theorem 2.4, $g \in \mathrm{nv}(G)$ if and only if Γ is non-singular. On the other hand, by Proposition 2.2, Γ is non-singular if and only if for each $\chi \in \mathrm{Irr}(H)$, $\sum_{t \in T} \chi(tg^{-1}) \neq 0$. This completes the proof. □

Now we are ready to determine some non-vanishing elements of a finite group. First let us recall the main theorem of [12].

Theorem 2.7. *There is some vanishing sum $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n = 0$ of m-th roots of unity if and only if n is a linear combination, with non-negative integer coefficients, of the prime divisors of m.*

The following theorem is well-known, see for example [8, Theorem 4.21].

Theorem 2.8. *Let $G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_t}$, where $t \geq 1$, be a finite abelian group. Then for every $g = (g_1, \ldots, g_t) \in G$, $g_i \in \mathbb{Z}_{n_i}$, and $\chi \in \mathrm{Irr}(G)$, there exist $\chi_i \in \mathrm{Irr}(\mathbb{Z}_{n_i})$, $i = 1, \ldots, t$, such that $\chi(g) = \chi_1(g_1) \ldots \chi_t(g_t)$. Furthermore, each $\chi_i(g_i)$ is an n_ith root of unity.*

Lemma 2.9. *Let g be an element of a finite group G, $T = \mathrm{Cl}(g)$ and $H = \langle TT^{-1}\rangle \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_t}$, where $n_1 \mid n_2 \mid \cdots \mid n_t$. If $|T|$ is not any linear combination, with non-negative integer coefficients, of prime divisors of n_t, then $g \in \mathrm{nv}(G)$.*

Proof. First note that since $n_i \mid n_t$, $i = 1, \dots, t$, Theorem 2.8 implies that for every $\chi \in \mathrm{Irr}(H)$ and $t \in T$, $\chi(tg^{-1})$ is an n_tth root of unity. Now suppose $|T|$ is not any linear combination, with non-negative integer coefficients, of prime divisors of n_t. Suppose, by contrary, that $g \in G \setminus \mathrm{nv}(G)$. Then Theorem 2.6 implies that $\sum_{t\in T} \chi(tg^{-1}) = 0$, for some $\chi \in \mathrm{Irr}(H)$. Hence, by Theorem 2.7, $|T|$ is a linear combination, with non-negative integer coefficients, of the prime divisors of n_t, a contradiction. □

Remark 2.10. Consider the non-abelian group $G \cong ((\mathbb{Z}_2\times\mathbb{Z}_2)\rtimes\mathbb{Z}_9)\rtimes\mathbb{Z}_2$ of order 72. One can check, for example by GAP software, that there exists a conjugacy class $T := \mathrm{Cl}(x)$ in G of size 6 where $x \in \mathrm{nv}(G)$. Also $H := \langle TT^{-1}\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_6$. This shows that the converse of Lemma 2.9 is not true in general (Here $H \rtimes K$ denotes the semidirect product of H by K). Hence the natural question is that in which groups the converse of Lemma 2.9 is true?

Corollary 2.11. *Let p be a prime, g an element of a finite group G, $T = \mathrm{Cl}(g)$ and $H = \langle TT^{-1}\rangle$ be an abelian p-group. If $(|T|, p) = 1$ then $g \in \mathrm{nv}(G)$. In particular, in a finite nilpotent group G with abelian p-subgroup G', $|G| \equiv |\mathrm{nv}(G)| \pmod p$.*

Proof. The first part is a direct consequence of Lemma 2.9. Now let G be a finite nilpotent group with an abelian p-subgroup G'. Let g be a non-central element of G, $T = \mathrm{Cl}(g)$ and $H = \langle TT^{-1}\rangle$. By Lemma 2.3, H is also an abelian p-group. On the other hand by [7, Theorem B], $\mathrm{nv}(G) = \mathrm{Z}(G)$, which implies that $g \in G \setminus \mathrm{nv}(G)$. Hence, by the first part, $p \mid |T|$. Now the class equation implies that $|G| \equiv |\mathrm{Z}(G)| \pmod p$ and so $|G| \equiv |\mathrm{nv}(G)| \pmod p$. □

In the following theorem we consider finite groups whose commutator subgroups have prime order.

Theorem 2.12. *Let p be a prime and g be an element of a finite group G. If $|G'| = p$, then*

(1) *$g \in \mathrm{nv}(G)$ if and only if $|\mathrm{Cl}(g)| < p$ (or equivalently $|\mathrm{Cl}(g)| \neq p$),*
(2) *if p is the smallest prime divisor of $|G|$, then $\mathrm{nv}(G) = \mathrm{Z}(G)$,*
(3) *G is nilpotent if and only if $\mathrm{Cl}(G) = \{1, p\}$, where $\mathrm{Cl}(G) = \{|\mathrm{Cl}(g)| \mid g \in G\}$.*

Proof. (1) Let $T = \mathrm{Cl}(g)$ be the conjugacy class containing g. Since $|G'| = p$, $|T| \leq p$. Also, by Lemma 2.3, $|H| = p$. Let $g \in \mathrm{nv}(G)$, then by Corollary 2.5, $|T| < p$. Conversely, let $|T| < p$. Then $p \nmid |T|$ and Corollary 2.11 implies that $g \in \mathrm{nv}(G)$.

(2) For every $\chi \in \mathrm{Irr}(G)$ and $x \in \mathrm{Z}(G)$, by [8, Corollary 2.28], $|\chi(x)| = \chi(1) > 0$ and so $\mathrm{Z}(G) \subseteq \mathrm{nv}(G)$. Now let $g \in \mathrm{nv}(G)$ and $T = \mathrm{Cl}(g)$. By (1), $|T| < p$. On the other hand $|T|$ divides $|G|$ and p is the smallest prime divisor of $|G|$. This shows that $|T| = 1$, which means that $g \in \mathrm{Z}(G)$. So $\mathrm{nv}(G) \subseteq \mathrm{Z}(G)$. Hence $\mathrm{nv}(G) = \mathrm{Z}(G)$.

(3) It is obvious that if $p = 2$, then G is a nilpotent group of class 2 and $\mathrm{Cl}(G) = \{1, 2\}$. So we may assume that $p > 2$. First let $\mathrm{Cl}(G) = \{1, p\}$. Then by [10, Theorem 1], G is nilpotent. Conversely, let G be nilpotent and g be a non-central element of G. Thus by [7, Theorem B], $g \in G \setminus \mathrm{nv}(G)$. So Lemma 2.11 implies that $p \mid |\mathrm{Cl}(g)|$. On the other hand $|\mathrm{Cl}(g)| \leq |G'| = p$, which means that $|\mathrm{Cl}(g)| = p$. Hence $\mathrm{Cl}(G) = \{1, p\}$. □

3. Non-vanishing elements with 2 or 3 conjugates

In this section, we focus on elements with 2 or 3 conjugates. In the following corollary, we give a complete classification of non-vanishing elements with 2 conjugates. Let C_n be the undirected graph of a cycle with n vertices. Then the eigenvalues of C_n are the numbers $2\cos(2\pi j/n)$, $j = 0, \ldots, n-1$, see [1, 1.5.3]. Now it is easy to see that C_n is non-singular if and only if $4 \nmid n$.

Corollary 3.1. *Let G be a finite group with an element g such that $T := \mathrm{Cl}(g) = \{g, h\}$. Then $g \in \mathrm{nv}(G)$ if and only if gh^{-1} has odd order. In particular,*

(1) *if $\langle g\rangle \cap \langle h\rangle = 1$, then $g \in \mathrm{nv}(G)$ if and only if g is of odd order,*
(2) *if g is of odd order, then $g \in \mathrm{nv}(G)$,*
(3) *if G' is of odd order, then $g \in \mathrm{nv}(G)$.*

Proof. Let $H = \langle TT^{-1}\rangle$, and $\Gamma = \mathrm{BCay}(G, T)$. By Lemma 2.3, $\Gamma \cong |G : H|\mathrm{BCay}(H, Tg^{-1})$, and $\Sigma := \mathrm{BCay}(H, Tg^{-1})$ is a connected graph. Indeed since $|Tg^{-1}| = 2$, Σ is an undirected cycle with $2|H|$ vertices. Now

$$\begin{aligned} g \in \mathrm{nv}(G) &\iff \Sigma \text{ is non-singular (by Theorem 2.4)} \\ &\iff 4 \nmid 2|H| \text{ (by above discussion)} \\ &\iff |H| \text{ is odd.} \end{aligned}$$

On the other hand $H = \langle TT^{-1}\rangle = \langle 1, gh^{-1}, hg^{-1}\rangle = \langle gh^{-1}\rangle$. Hence $g \in \mathrm{nv}(G)$ if and only if gh^{-1} is of odd order.

Since $h^{-1}gh \in T$ and $h \neq g$, $h^{-1}gh = g$. So g and h^{-1} commute. Also g and h have the same order. Hence the following results are straightforward.
If $\langle g\rangle \cap \langle h\rangle = 1$ then $\mathrm{o}(gh^{-1}) = \mathrm{o}(g)$.
If g is of odd order n then, since $\mathrm{o}(gh^{-1}) \mid n$, $\mathrm{o}(gh^{-1})$ is odd.
If $|G'|$ is odd then, since $gh^{-1} \in G'$, $\mathrm{o}(gh^{-1})$ is odd. □

Lemma 3.2. *Let $x \in G$ be an element of odd order k and $T = \{x, y, z\}$ be the conjugacy class of G containing x. Then the elements of T commute and so $\langle TT^{-1}\rangle = \langle xy^{-1}, zy^{-1}\rangle$ is a normal abelian subgroup of G.*

Proof. Let $a \in T$, $b \in T \setminus \{a\}$, and suppose, by contrary, that $ab \neq ba$. Since $b^{-1}ab \in T$, $b^{-1}ab = c$. On the other hand $bab^{-1} \in T$, which implies that $bab^{-1} = c = b^{-1}ab$. Hence $b^2a = ab^2$. Since k is odd, there exist $r, s \in \mathbb{Z}$ such that $1 = rk + 2s$. So $ab = ab^{rk}b^{2s} = ab^{2s} = b^{2s}a = b^{rk+2s}a = ba$. This shows that $\langle TT^{-1} \rangle$ is abelian. Also

$$\begin{aligned}\langle TT^{-1}\rangle &= \langle 1, xy^{-1}, yx^{-1}, xz^{-1}, zx^{-1}, yz^{-1}, zy^{-1}\rangle \\ &= \langle xy^{-1}, zy^{-1}, xz^{-1}\rangle \\ &= \langle xy^{-1}, zy^{-1}\rangle \quad (\text{since } xy^{-1}(zy^{-1})^{-1} = xz^{-1}),\end{aligned}$$

which completes the proof. □

Theorem 3.3. *Let G be a finite group, $x \in G$, $T := \mathrm{Cl}(x) = \{x, y, z\}$, and $H := \langle TT^{-1} \rangle$ be abelian. Suppose that*

$$H = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_t},$$

where $n_1 \mid n_2 \mid \cdots \mid n_t$, and for $j = 1, \ldots, t$, $\mathbb{Z}_{n_j} = \langle a_j \rangle$ and $n_j \geq 2$ and $t \geq 1$. Then $x \in G \setminus \mathrm{nv}(G)$ if and only if for all $j \in \{1, \ldots, t\}$, there exists $r_j \in \{0, \ldots, n_j - 1\}$ such that

$$3n_t \mid \left(3\sum_{j=1}^{t} b_j k_j r_j - n_t\right) \quad \text{and} \quad 3n_t \mid \left(3\sum_{j=1}^{t} b_j l_j r_j + n_t\right),$$

where $xy^{-1} = (a_1^{k_1}, \ldots, a_t^{k_t})$, $zy^{-1} = (a_1^{l_1}, \ldots, a_t^{l_t})$ and $b_j = n_t/n_j$, $j = 1, \ldots, t$.

Proof. We have $Ty^{-1} \subseteq H$. Since H is abelian, by Theorem 2.6, $x \in \mathrm{nv}(G)$ if and only if for each $\chi \in \mathrm{Irr}(H)$, $\sum_{t \in T} \chi(ty^{-1}) \neq 0$; that is $x \in \mathrm{nv}(G)$ if and only if for each $\chi \in \mathrm{Irr}(H)$, $1 + \chi(xy^{-1}) + \chi(zy^{-1}) \neq 0$. Thus $x \in G \setminus \mathrm{nv}(G)$ if and only if there exists $\chi \in \mathrm{Irr}(H)$ such that $1 + \chi(xy^{-1}) + \chi(zy^{-1}) = 0$.

Suppose that $\chi \in \mathrm{Irr}(H)$. Then by Theorem 2.8, there exist $\chi_j \in \mathrm{Irr}(\mathbb{Z}_{n_j})$, $j = 1, \ldots, t$, such that for all $h = (a_1^{m_1}, \ldots, a_t^{m_t}) \in H$, $\chi(h) = \chi_1(a_1^{m_1})\chi_2(a_2^{m_2}) \cdots \chi_t(a_t^{m_t})$. On the other hand, for each $j = 1, \ldots, t$, there exists $r_j \in \{0, \ldots, n_j - 1\}$ such that $\chi_j(a_j^{m_j}) = \exp(2\pi i m_j r_j / n_j)$. Let $xy^{-1} = (a_1^{k_1}, \ldots, a_t^{k_t})$,

$zy^{-1} = (a_1^{l_1}, \dots, a_t^{l_t})$, $\alpha = \sum_{j=1}^{t} k_j r_j / n_j$ and $\beta = \sum_{j=1}^{t} l_j r_j / n_j$. Now

$$\begin{aligned} 1 + \chi(xy^{-1}) + \chi(zy^{-1}) = 0 \quad &\Longleftrightarrow \quad \forall\, i = 1, \dots, t, \exists\, r_i \in \{0, \dots, n_i - 1\} \\ &\qquad 1 + \exp(2\pi i \alpha) = -\exp(2\pi i \beta) \\ &\Longleftrightarrow \quad 1 + \cos(2\pi\alpha) = -\cos(2\pi\beta), \\ &\qquad \sin(2\pi\alpha) = -\sin(2\pi\beta) \\ &\Longleftrightarrow \quad \cos(2\pi\alpha) = \cos(2\pi\beta) = -1/2, \\ &\qquad \sin(2\pi\alpha) = -\sin(2\pi\beta) \\ &\Longleftrightarrow \quad \exists s_1 \in \mathbb{Z},\ 2\pi\alpha = 2\pi/3 + 2\pi s_1, \\ &\qquad \exists s_2 \in \mathbb{Z},\ 2\pi\beta = -2\pi/3 + 2\pi s_2 \\ &\Longleftrightarrow \quad \exists s_1 \in \mathbb{Z},\ \alpha = 1/3 + s_1, \\ &\qquad \exists s_2 \in \mathbb{Z},\ \beta = -1/3 + s_2 \\ &\Longleftrightarrow \quad \alpha - 1/3 \in \mathbb{Z}, \beta + 1/3 \in \mathbb{Z}. \end{aligned}$$

Now since $n_t = n_j b_j$, $j = 1, \dots, t$, we have $\alpha = \left(\sum_{j=1}^{t} b_j k_j r_j\right)/n_t$ and $\beta = \left(\sum_{j=1}^{t} b_j l_j r_j\right)/n_t$. If $3 \mid n_t$, then $\alpha - 1/3 = \left(\sum_{j=1}^{t} b_j k_j r_j - n_t/3\right)/n_t$ and if $3 \nmid n_t$, then $\alpha - 1/3 = \left(3\sum_{j=1}^{t} b_j k_j r_j - n_t\right)/(3n_t)$. Hence in both cases $\alpha - 1/3 \in \mathbb{Z}$ if and only if $3n_t \mid \left(3\sum_{j=1}^{t} b_j k_j r_j - n_t\right)$. Similarly $\beta + 1/3 \in \mathbb{Z}$ if and only if $3n_t \mid \left(3\sum_{j=1}^{t} b_j l_j r_j + n_t\right)$. This completes the proof. □

Corollary 3.4. *Let G be a finite group, $x \in G$ be an element with conjugacy class T of size 3. If $H := \langle TT^{-1} \rangle$ is an abelian group and $3 \nmid |H|$ then $x \in \mathrm{nv}(G)$. In particular, in a solvable group G with abelian commutator subgroup G' of order ≥ 4 and coprime to 3, every element with 3 conjugates is a non-vanishing element of G.*

Proof. Suppose, by contrary, that x is not a non-vanishing element of G. Let $H = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_t}$, where $n_1 \mid n_2 \mid \cdots \mid n_t$, and for $i = 1, \dots, t$, $n_i \geq 2$ and $t \geq 1$. Then by Theorem 3.3, there exists an integer k such that $3n_t \mid 3k - n_t$. So $3 \mid n_t$, which implies that 3 divides $|H|$, a contradiction.

Since, by Lemma 2.3, $H \leq G'$, the second part follows immediately. □

Corollary 3.5. *Let G be a finite group, $x \in G$ and $|\mathrm{Cl}(x)| = 2$ or 3. If $(\mathrm{o}(x), 6) = 1$ then $x \in \mathrm{nv}(G)$.*

Proof. Since $(6, \mathrm{o}(x)) = 1$, x is of odd order and $3 \nmid \mathrm{o}(x)$. If $|\mathrm{Cl}(x)| = 2$ then by Corollary 3.1, $x \in \mathrm{nv}(G)$. Now suppose that $|\mathrm{Cl}(x)| = 3$. Then by Lemma 3.2, elements of $T := \mathrm{Cl}(x) = \{x, y, z\}$ commute and $H := \langle TT^{-1} \rangle = \langle xy^{-1}, zy^{-1} \rangle$ is abelian. So every element h of H is of the form $h = x^{i_1} y^{i_2} z^{i_3}$, for some integers i_1, i_2, i_3. Since $\mathrm{o}(x) = \mathrm{o}(y) = \mathrm{o}(z)$, $\mathrm{o}(h) \mid \mathrm{o}(x)$. This shows that $3 \nmid |H|$. Hence, by Corollary 3.4, $x \in \mathrm{nv}(G)$, which completes the proof. □

References

[1] A. E. Brouwer and W. H. Haemers, Spectra of Graphs, Springer, New York, 2012.

[2] D. Bubboloni, S. Dolfi and P. Spiga, Finite groups whose irreducible characters vanish only on p-elements, *J. Pure Appl. Algebra* **213** (2009), no. 3, 370–376.

[3] S. Dolfi, G. Navvaro, E. Pacifici, L. Sanus and P. H. Tiep, Non-vanishing elements of finite groups, *J. Algebra* **323** (2010), no. 2, 540–545.

[4] S. F. Du and M. Y. Xu, A classification of semisymmetric graphs of order $2pq$, *Comm. Algebra* **28** (2000), no. 6, 2685–2715.

[5] L. He, S. Yu and J. Lu, A result related to non-vanishing elements of finite solvable groups, *Int. J. Algebra* **7** (2013), no. 5-8, 223–227.

[6] L. He, Notes on non-vanishing elements of finite solvable groups, *Bull. Malays. Math. Sci. Soc. (2)* **35** (2012), no. 1, 163–169.

[7] I. M. Isaacs, G. Navarro and T. R. Wolf, Finite group elements where no irreducible character vanishes, *J. Algebra* **222** (1999), no. 2, 413-423.

[8] I. M. Isaacs, Character Theory of Finite Groups, Academic Press, New York-London, 1976.

[9] N. Itô, The spectrum of a conjugacy class graph of a finite group, *Math. J. Okayama Univ.* **26** (1984) 1–10.

[10] N. Itô, On finite groups with given conjugate types I, *Nagoya Math. J.* **6** (1953) 17–28.

[11] W. Jin and W. Liu, A classification of nonabelian simple 3-BCI-groups, *European J. Combin.* **31** (2010), no. 5, 1257–1264.

[12] T. Y. Lam and K. H. Leung, On vanishing sums of roots of unity, *J. Algebra* **224** (2000) 91–109.

[13] Z. P. Lu, C. Q. Wang, and M. Y. Xu, Semisymmetric cubic graphs constructed from bi-Cayley graphs of A_n, *Ars Combin.* **80** (2006) 177–187.

[14] M. Miyamoto, Non-vanishing elements in finite groups, *J. Algebra* **364** (2012) 88–89.

(Majid Arezoomand) Department of Mathematical Sciences, Isfahan University of Technology, P.O. Box 84156-83111, Isfahan, Iran.

E-mail address: arezoomand@math.iut.ac.ir

(Bijan Taeri) Department of Mathematical Sciences, Isfahan University of Technology, P.O. Box 84156-838111, Isfahan, Iran.

E-mail address: b.taeri@cc.iut.ac.ir