ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 42 (2016), No. 5, pp. 1107–1126

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 5, pp. 1107–1126
Online ISSN: 1735-8515

NUMERICAL APPROACH FOR SOLVING A CLASS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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(Communicated by Mohammad Asadzadeh)

Abstract. It is commonly accepted that fractional differential equations play an important role in the explanation of many physical phenomena. For this reason we need a reliable and efficient technique for the solution of fractional differential equations. This paper deals with the numerical solution of a class of fractional differential equation. The fractional derivatives are described based on the Caputo sense. Our main aim is to generalize the Chebyshev cardinal operational matrix to the fractional calculus. In this work, the Chebyshev cardinal functions together with the Chebyshev cardinal operational matrix of fractional derivatives are used for numerical solution of a class of fractional differential equations. The main advantage of this approach is that it reduces fractional problems to a system of algebraic equations. The method is applied to solve nonlinear fractional differential equations. Illustrative examples are included to demonstrate the validity and applicability of the presented technique.
Keywords: Fractional-order differential equation, operational matrix of fractional derivative, Caputo derivative, Chebyshev cardinal function, collocation method.
MSC(2010): Primary: 34A08; Secondary: 65M70, 65L60.

1. **Introduction**

Fractional differential equations have been found to be effective to describe some physical phenomena such as damping laws, electromagnetic, acoustics, viscoelasticity, electroanalytical chemistry, neuron modeling, diffusion processing and material sciences [3, 11, 13, 28, 32, 38].

The treatment of models of the above mentioned phenomena takes different facets. For example, existence and uniqueness of solutions have been investigated in [11, 21, 33, 34].

Article electronically published on October 31, 2016.
Received: 10 January 2015, Accepted: 5 July 2015.
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In the recent decades, some attempts have been made to find analytical and numerical solutions for the fractional problems. These attempts have included finite difference methods [27, 30, 39], collocation– shooting methods ([1, 9, 37]), spline and B-spline collocation methods [22, 26], Adomian decomposition method [10, 40], flatlet oblique multiwavelets method [18], variational iteration methods [17, 34], homotopy analysis methods [16, 20, 33] and etc.

Interpolation approximate base function have received considerable attention in dealing with various problems. The main characteristic behind this work using this technique is that it reduces fractional problems to those of solving a system of algebraic equations thus greatly simplifying the problem. In this method, a Chebyshev cardinal function is used for numerical solution of differential equations, with the goal of obtaining efficient computational solutions. Several papers have appeared in the literature concerned with the application of Chebyshev cardinal functions [12, 19, 23, 24, 25].

In the present paper we extend the application of Chebyshev cardinal functions to solve a nonlinear fractional differential equation.

Consider the nonlinear multi-order fractional differential equation

$$F(y(x), D^{(\alpha)}y(x), D^{(\beta_1)}y(x), \ldots, D^{(\beta_m)}y(x)) = g(x), \tag{1.1}$$

with boundary or supplementary conditions

$$H_i(y(\xi_i), y'(\xi_i)) = d_i, \quad i = 0, 1, \tag{1.2}$$

where F is a multivariable function and $g(x)$ is a known function, $\xi_i \in [0, 1],\ i = 0,\ 1,\ 1 < \alpha \leq 2,\ 0 < \max\{\beta_i,\ i = 1, \ldots, m\} \leq 1,\ H_i$ are linear combinations of $y(x), y'(x)$ and $D^{(\alpha)}, D^{(\beta_i)}$ denote the Caputo fractional derivative of order α and β_i respectively and $y(x) \in L^2[0, 1]$.

The existence and uniqueness and continuous dependence of the solution of proposed problem are discussed in [2, 31]. We apply the operational matrix of fractional derivatives to solve nonlinear multi-order fractional differential equations.

We recall the existence and uniqueness of a special case of (1.1) from [31], and we propose some stability analysis, convergence analysis, accuracy order of

$$D^{(\alpha)}y(x) = f(x, y(x), D^{(\beta)}y(x)), \qquad 1 < \alpha \leq 2,\ 0 \leq \beta \leq 1, \tag{1.3}$$

with initial conditions

$$y(0) = y_0, \qquad y'(0) = y_1 \tag{1.4}$$

or boundary conditions

$$y(0) = y_0, \qquad y(1) = y_1. \tag{1.5}$$

Our main aim is to generalize Chebyshev cardinal operational matrix to fractional calculus. It is worthy to mention here that, the method based on using the operational matrix of an interpolate function for solving differential equations is computer oriented.

The rest of the paper is organized as follows: Basic concepts of fractional differential problems are discussed in Section 2. Section 3 is devoted to the analysis of the methods and the construction of operational matrix for fractional derivative. Application of proposed methods for fractional problems are given in Section 4. In Section 5, we express existence and uniqueness and we discuss stability analysis, convergence analysis, accuracy order for class of nonlinear multi-order fractional differential equation. The numerical results for confirming effectively of the proposed methods are given in Section 6.

2. Concepts of fractional problems

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1. A real function $f(x),\ x>0$, is said to be in the space $C_\mu,\ \mu\in\mathbb{R}$, if there exists a real number $p>\mu$ such that $f(x)=x^p f_1(x)$, where $f_1(x)\in C[0,1)$. Clearly $C_\mu\subset C_\beta$ if $\beta\le\mu$.

Definition 2.2. A function $f(x),\ x>0$, is said to be in the space $C_\mu^m,\ m\in\mathbb{N}\cup\{0\}$, if $f^{(m)}\in C_\mu$.

Definition 2.3. The left sided Riemann–Liouville fractional integral operator of order $\alpha\ge 0$ of a function $f\in C_\mu,\quad \mu\ge -1$, is defined in [29] as follows:

$$J^{(\alpha)}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{f(t)}{(x-t)^{1-\alpha}}dt,\quad \alpha>0,\qquad x>0,$$

$$J^{(0)}f(x)=f(x). \tag{2.1}$$

Definition 2.4. Let $f\in C_{-1}^m,\ m\in\mathbb{N}\cup\{0\}$. The Caputo fractional derivative of $f(x)$ is defined as in [29]:

$$D^{(\alpha)}f(x)=\begin{cases} J^{(m-\alpha)}f^{(m)}(x), & m-1<\alpha<m,\ m\in N,\\ \frac{D^m f(x)}{Dx^m}, & \alpha=m.\end{cases} \tag{2.2}$$

It can be shown that [4, 8, 29, 36]:

(2.3)

1. $J^{(\alpha)}J^{(\nu)}f=J^{(\alpha+\nu)}f,\quad \alpha,\nu>0,\ f\in C\mu,\ \mu>0.$
2. $J^{(\alpha)}x^\gamma=\dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma},\quad \alpha>0,\gamma>-1,\ x>0.$
3. $J^{(\alpha)}D^{(\alpha)}f(x)=f(x)-\displaystyle\sum_{k=0}^{m-1}f^{(k)}(0^+)\frac{x^k}{k!},\quad x>0,\quad m-1<\alpha\le m.$
4. $D^{(\alpha)}J^{(\alpha)}f(x)=f(x),\quad x>0,\quad m-1<\alpha\le m,$

5. $D^{(\alpha)}C = 0$, C is constant,
6. $D^{(\alpha)}x^{\beta} = 0$, $\beta \in \mathbb{N}_0 \quad \beta < [\alpha]$, $\mathbb{N}_0 = \{0, 1, \ldots\}$,
7. $D^{(\alpha)}x^{\beta} = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}$, $\beta \in \mathbb{N}_0 \quad \beta > [\alpha]$,
8. $D^{(m+\alpha)}f(x) = D^{(\alpha)}[D^{(m)}f(x)] \neq D^{(m)}[D^{(\alpha)}f(x)]$, $m \in \mathbb{N}$, $[\alpha] \in \mathbb{Z}$,
9. $J^{(\alpha)}f(x) = D^{(-\alpha)}f(x)$, $\alpha > 0$.
10. $D^{(\alpha)}J^{(\beta)}f(x) = D^{(\alpha-\beta)}f(x)$,
11. $J^{(\alpha)}D^{(\beta)}f(x) = J^{(\alpha)}D^{(\alpha)}(J^{(\alpha-\beta)}f(x))$

$$= D^{(\beta-\alpha)}f(x) - \sum_{k=0}^{|m-n|\text{or}|m-n-1|} \frac{f^{(k+|\beta-\alpha|)}(0^+)x^k}{k!}, n \le \alpha < n+1, m \le \beta < m+1, \beta \le \alpha.$$

The Caputo fractional derivative is considered here because, it allows traditional initial and boundary conditions to be included in the formulation of the problem.

3. **Analysis of the methods**

In this section, we first present a brief review of the Chebyshev cardinal functions for solving fractional differential equations.

Chebyshev cardinal functions of order N in $[-1, 1]$ are defined as [7]:

$$\phi_j(x) = \frac{T_{N+1}(x)}{T'_{N+1}(x_j)(x - x_j)}, \qquad j = 1, 2, ..., N+1, \tag{3.1}$$

where $T_{N+1}(x)$ is the first kind Chebyshev function of order $N+1$ in $[-1, 1]$ defined by

$$T_{N+1}(x) = \cos((N+1)\arccos(x)) \tag{3.2}$$

and x_j, $j = 1, 2, ..., N+1$, are the zeros of $T_{N+1}(x)$ defined by $\cos((2j-1)/(2N+2))$, $j = 1, 2, \ldots, N+1$. We apply variable changing $t = (x+1)L/2$ to use these functions on $[0, L]$. Now any function $f(t)$ on $[0, L]$ can be approximated as

$$f(t) = \sum_{j=1}^{N+1} f(t_j)\phi_j(t) = F^T\Phi_N(t), \tag{3.3}$$

where t_j, $j = 1, 2, \ldots, N+1$, are the shifted points of x_j, $j = 1, 2, \ldots, N+1$, by transforming $t = (x+1)L/2$ (here we choose t_j so that, $t_1 < t_2 < \ldots < t_{N+1}$),

$$F = [f(t_1),\ f(t_2),\ ..., f(t_{N+1})]^T, \qquad \Phi_N(t) = [\phi_1(t),\ \phi_2(t),\ ..., \phi_{N+1}(t)]^T. \tag{3.4}$$

Note that the functions $\phi_j(t)$ satisfy the relation

$$\phi_j(t_i) = \delta_{j,i} = \begin{cases} 1, & j = i, \\ 0, & j \neq i \end{cases}, \quad j, i = 1, \ldots, N+1.$$

So we have

$$\Phi_N(t_i) = e_i, \qquad i = 1, \dots, N+1, \tag{3.5}$$

where e_i is the ith column of unit matrix of order $N+1$.

Theorem 3.1. *Let $y(t) \in C^{N+1}[0, L]$ and $P_N(t)$ be polynomial interpolation of $y(t)$ at the points t_i, $i = 1, \dots, N+1$ (zeros of Chebyshev polynomial of degree $N+1$), then*

$$e_N = \max_{0 \le t \le L} |y(t) - P_N(t)| \le \frac{M_{N+1}}{2^N (N+1)!}, \tag{3.6}$$

where $M_{N+1} = \max |y^{(N+1)}(\xi)|$, $\xi \in [0, L]$. Thus $P_N(t) \to y(t)$ as N tends to infinity [35].

Definition 3.2. [15] Let $M_n : \mathbb{R}^n \to P_{n-1}$ be the linear map associating to each vector $u^T = [u_1, u_2, \dots, u_k] \in \mathbb{R}^n$ and

$$p(x) = \sum_{k=1}^{n} u_k x^{k-1} \in P_{n-1}, \quad n \ge 2.$$

For any $p \in P_{n-1}$, we shall write $u = M_n^{-1} p$, where M^{-1} is the inverse map of M_n. We define the condition of the map M_n, relative to the compact interval $[a, b]$, by [15]

$$Cond_\infty M_n = ||M_n||_\infty ||M_n^{-1}||_\infty, \tag{3.7}$$

where the norms are $||u||_\infty = \max\limits_{1<k<n} |u_k|$ (in $\mathbb{R}^n$) and $||p||_\infty = \max\limits_{a<x<b} |p(x)|$ (in $P_{n-1}[a, b]$).

Definition 3.3. [15] The Chebyshev polynomial T_m, adjusted to the interval $[a, b]$, will be denoted by $T_m[a, b]$,

$$T_m[a, b](x) = T_m(\frac{2x - a - b}{b - a}), \quad a \le x \le b.$$

Relative to any such interval $[a, b]$, the norm of the map M_n is easily seen to be

$$||M_n||_\infty = \begin{cases} \frac{b^n - 1}{b - 1}, & b \ne 1, \\ n, & b = 1. \end{cases}$$

More delicate is the determination of $||M_n^{-1}||_\infty$, as this amounts to finding the norms of the linear functionals $\lambda_k : p \to p^{(k-1)}(0)/(k-1)!$, $p \in P_{n-1}[a, b]$, $k = 1, 2, \dots, n$. Indeed

$$||M_n^{-1}||_\infty = \max_{1 \le k \le n} ||\lambda_k||_\infty.$$

Theorem 3.4. *The condition number (3.7) on $[-w, w]$ is given by*

$$Cond_{\infty} M_n = \frac{w^n - 1}{w - 1} \max\{||u_{T_{n-1}(x/w)}||_{\infty}, ||u_{T_{n-2}(x/w)}||_{\infty}\}, \tag{3.8}$$

where $\frac{w^n-1}{w-1}$ (here and in the sequel) is to be interpreted as having the value n if $w = 1$ (for more details see [15]).

We can get good approximate function $f \in L^2[0, 1]$ using Chebyshev cardinal functions by small N where N is the number of Chebyshev cardinal basis. But for large values of N, the expansion coefficients grows like $(1+\sqrt{2})^{2n}$ ($(1+\sqrt{2})^n$ on $L^2[-1, 1]$) and so the condition number is large, in this case. Therefore, we use this expansion for small values of N (see [5, 6, 15]).

3.1. **The operational matrix of derivative.** The differentiation of vector Φ_N in (3.4) can be expressed as

$$\Phi'_N = \mathbf{D}\Phi_N, \tag{3.9}$$

where $\mathbf{D}$ is $(N + 1) \times (N + 1)$ operational matrix of derivative for Chebyshev cardinal functions.

It is shown [23] that the matrix $\mathbf{D}$ is in the form

$$\mathbf{D} = \begin{pmatrix} \phi'_1(t_1) & \dots & \phi'_1(t_{N+1}) \\ \vdots & \vdots & \vdots \\ \phi'_{N+1}(t_1) & \dots & \phi'_{N+1}(t_{N+1}) \end{pmatrix}, \tag{3.10}$$

where

$$\phi'_j(t_j) = \sum_{\substack{i=1 \\ i \neq j}}^{N+1} \frac{1}{t_j - t_i}, \qquad j = 1, \dots, N+1,$$

$$\phi'_j(t_k) = \frac{\beta}{T'_{N+1}(t_j)} \prod_{\substack{l=1 \\ l \neq k,j}}^{N+1} (t_k - t_l), \qquad j, k = 1, \dots, N+1, \quad j \neq k \tag{3.11}$$

and $\beta = 2^{2N+1}/L^{N+1}$. Note that

$$\frac{T_{N+1}(t)}{t - t_j} = \beta \times \prod_{\substack{k=1 \\ k \neq j}}^{N+1} (t - t_k). \tag{3.12}$$

3.2. **The operational matrix of fractional derivative.** The fractional differentiation of vector $\Phi_N(t)$ in (3.4) can be expressed as

$$D^{(\alpha)}\Phi_N = \mathbf{D}_{\alpha}\Phi_N, \tag{3.13}$$

where $\mathbf{D}_\alpha$ is $(N+1)\times(N+1)$ operational matrix of fractional derivative for Chebyshev cardinal functions. The matrix $\mathbf{D}_\alpha$ can be obtained by the following process. Let

$$D^{(\alpha)}\Phi_N(t)=[\phi_1^{(\alpha)}(t),\ \phi_2^{(\alpha)}(t),\ ...,\phi_{N+1}^{(\alpha)}(t)]^T. \tag{3.14}$$

Using Eqs. (2.2), (2.3), (3.4) and (3.12) the function $\phi_j^{(\alpha)}(t)$ can be approximated by two methods as

$$\phi_j^{(\alpha)}(t)=\beta\times\frac{1}{T'_{N+1}(t_j)}(\prod_{\substack{k=1\\k\neq j}}^{N+1}(t-t_k))^{(\alpha)}. \tag{3.15}$$

First method: We can expand $\prod_{\substack{k=1\\k\neq j}}^{N+1}(t-t_k)$ as

$$\prod_{\substack{k=1\\k\neq j}}^{N+1}(t-t_k)=t^N-(\sum_{\substack{k_1\neq j\\1\leq k_1\leq N+1}}t_{k_1})t^{N-1}+(\sum_{\substack{k_1,k_2\neq j\\1\leq k_1<k_2\leq N+1}}t_{k_1}t_{k_2})t^{N-2}-\ldots+(-1)^N\prod_{\substack{k=1\\k\neq j}}^{N+1}t_k, \quad j=1,2,\ldots,N+1. \tag{3.16}$$

Lemma 3.5. *Let $\phi_n(t)$ be a Chebyshev cardinal function such that $n<\alpha$, then $D^\alpha\phi_n(t)=0$.*

Proof. Using Eqs.(2.3) in Eq.(3.16) the lemma can be proved. □

For $0<\alpha<1$ using (3.16), we get

$$\begin{aligned}\phi_j^{(\alpha)}(t)&=\beta\times\frac{1}{T'_{N+1}(t_j)}(\prod_{\substack{k=1\\k\neq j}}^{N+1}(t-t_k))^{(\alpha)}=\frac{\beta}{T'_{N+1}(t_j)\Gamma(N+1-\alpha)}\\&\times[N!t^{N-\alpha}-(N-\alpha)(N-1)!(\sum_{\substack{k_1\neq j\\1\leq k_1\leq N+1}}t_{k_1})t^{N-1-\alpha}\\&+(N-\alpha)(N-\alpha-1)(N-2)!(\sum_{\substack{k_1,k_2\neq j\\1\leq k_1<k_2\leq N+1}}t_{k_1}t_{k_2})t^{N-2-\alpha}-\ldots\\&+(-1)^{(N-1)}\prod_{k=0}^{N-2}(N-\alpha-k)(\sum_{\substack{k_1,k_2,\ldots k_{(N-1)}\neq j\\1\leq k_1<k_2<\ldots<k_{(N-1)}\leq N+1}}t_{k_1}t_{k_2}\ldots t_{k_{(N-1)}})t^{1-\alpha}]\\&\qquad j=1,2,\ldots,N+1.\end{aligned} \tag{3.17}$$

Any function $\phi_j^{(\alpha)}(t)$, using (3.3) can be approximated as

$$\phi_j^{(\alpha)}(t) = \sum_{k=1}^{N+1} \phi_j^{(\alpha)}(t_k)\phi_k(t). \tag{3.18}$$

By comparing (3.13) and (3.18), we get

$$\mathbf{D}_\alpha = \begin{pmatrix} \phi_1^{(\alpha)}(t_1) & \dots & \phi_1^{(\alpha)}(t_{N+1}) \\ \vdots & \ddots & \vdots \\ \phi_{N+1}^{(\alpha)}(t_1) & \dots & \phi_{N+1}^{(\alpha)}(t_{N+1}) \end{pmatrix}, \tag{3.19}$$

where the entries of the matrix $\mathbf{D}_\alpha$ can be found using Eq. (3.17).

Second method: Let

$$\mathbf{T} = [1, t, t^2, \dots, t^N]^T, \tag{3.20}$$

then (3.4) results in

$$\Phi_N(t) = [\phi_1(t),\ \phi_2(t),\ ..., \phi_{N+1}(t)]^T = \mathbf{A}.\ \mathbf{T}, \tag{3.21}$$

where $\mathbf{A}$ is $(N+1)\times(N+1)$ operational matrix of coefficient for Chebyshev cardinal functions as follows

(3.22)

$$\mathbf{A} = \beta \times \begin{pmatrix} (-1)^N \dfrac{1}{T'_{N+1}(t_1)} \prod\limits_{\substack{k=1\\k\neq 1}}^{N+1} t_k, & \dots -(\dfrac{1}{T'_{N+1}(t_1)} \sum\limits_{\substack{k_1\neq 1\\1\leq k_1\leq N+1}} t_{k_1}), & \frac{1}{T'_{N+1}(t_1)} \\ (-1)^N \dfrac{1}{T'_{N+1}(t_2)} \prod\limits_{\substack{k=1\\k\neq 2}}^{N+1} t_k, & \dots -(\dfrac{1}{T'_{N+1}(t_2)} \sum\limits_{\substack{k_1\neq 2\\1\leq k_1\leq N+1}} t_{k_1}), & \frac{1}{T'_{N+1}(t_2)} \\ \vdots & \ddots & \vdots \\ (-1)^N \dfrac{1}{T'_{N+1}(t_{N+1})} \prod\limits_{\substack{k=1\\k\neq N+1}}^{N+1} t_k, & \dots -(\dfrac{1}{T'_{N+1}(t_{N+1})} \sum\limits_{\substack{k_1\neq N+1\\1\leq k_1\leq N+1}} t_{k_1}), & \frac{1}{T'_{N+1}(t_{N+1})} \end{pmatrix}.$$

Because of orthogonality of $\phi_j(t), j = 1, \dots, N+1$, this matrix is invertible. From (2.3) and for $0 < \alpha \leq 1$, we get

(3.23)

$$D^\alpha\ \mathbf{T} = [0, \frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}, \frac{\Gamma(3)}{\Gamma(3-\alpha)}t^{2-\alpha}, \dots, \frac{\Gamma(N+1)}{\Gamma(N+1-\alpha)}t^{N-\alpha}]^T = t^{-\alpha}\ \mathbf{D}_1.\ \mathbf{T},$$

where $\mathbf{D}_1$ is $(N+1)\times(N+1)$ matrix of the following form

$$\mathbf{D}_1 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)} & 0 & \dots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\alpha)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & \frac{\Gamma(N+1)}{\Gamma(N+1-\alpha)} \end{pmatrix}. \tag{3.24}$$

If $1 < \alpha \leq 2$ then the second row of $\mathbf{D}_1$ is zero and ... . Using (3.21) we have

$$D^{\alpha}\Phi_N(t) = \mathbf{A}.D^{\alpha}\ \mathbf{T} = t^{-\alpha}\ \mathbf{A}.\ \mathbf{D}_1.\ \mathbf{T}. \tag{3.25}$$

Note that $\mathbf{A}$ is invertible, so

$$D^{\alpha}\Phi_N(t) = t^{-\alpha}\ \mathbf{A}.\ \mathbf{D}_1.\mathbf{A}^{-1}.\ \mathbf{A}.\ \mathbf{T} = t^{-\alpha}\ \mathbf{A}.\ \mathbf{D}_1.\ \mathbf{A}^{-1}.\Phi_N(t). \tag{3.26}$$

Hence

$$D^{\alpha}\Phi_N(t) = \mathbf{D}_{\alpha}.\Phi_N(t), \tag{3.27}$$

where $\mathbf{D}_{\alpha} = t^{-\alpha}\ \mathbf{A}.\ \mathbf{D}_1.\ \mathbf{A}^{-1}$.

4. Application of the operational matrix of fractional derivative

In this section, In order to use Chebyshev cardinal functions for Eq. (1.1), we first approximate $y(x)$, $g(x), D^{(\alpha)}y(x)$ and $D^{(\beta_j)}y(x)$, for $j = 0, \ldots, m$ from (3.3) and (3.18) on the interval $[0, 1]$ as follows

(4.1)

$$\begin{aligned}
&y(x) \simeq \sum_{j=1}^{N+1} c_j\phi_j(x) = C^T\Phi_N(x),\\
&g(x) \simeq \sum_{j=1}^{N+1} g_j\phi_j(x) = G^T\Phi_N(x),\\
&D^{(\alpha)}y(x) \simeq D^{(\alpha)}(C^T\Phi_N(x)) = C^T D^{(\alpha)}(\Phi_N(x)) = C^T\mathbf{D}_{\alpha}\Phi_N(x),\\
&D^{(\beta_j)}y(x) \simeq D^{(\beta_j)}(C^T\Phi_N(x)) = C^T D^{(\beta_j)}(\Phi_N(x)) = C^T\mathbf{D}_{\beta_j}\Phi_N(x), \quad j = 1, \ldots, m,
\end{aligned}$$

where $G = [g_1, \ldots, g_{N+1}]^T$, $g_j = g(t_j)$, $j = 1, \ldots, N+1$, $C = [c_1, \ldots, c_{N+1}]^T$ is an unknown vector and $N > 1$. Employing (4.1) in (1.1) we get

(4.2)

$$\begin{aligned}
R_{N+1}(x) = {} & F(C^T\Phi_N(x), C^T\mathbf{D}_{\alpha}\Phi_N(x), C^T\ \mathbf{D}_{\beta_1}\Phi_N(x), \ldots, C^T\ \mathbf{D}_{\beta_m}\Phi_N(x)) - G^T\Phi_N(x)\\
& \cong 0.
\end{aligned}$$

Collocating Eq. (4.2) in the points $t_i, i = 3, \ldots, N+1$ and using Eq.(3.5), we get

$$R_{N+1}(t_i) = F(C^T e_i, C^T\mathbf{D}_{\alpha}e_i, C^T\ \mathbf{D}_{\beta_1}e_i, \ldots, C^T\ \mathbf{D}_{\beta_m}e_i) - G^T e_i. \tag{4.3}$$

Also, by substituting Eqs. (3.9) and (4.1) in Eq. (1.2) we obtain

$$H_i(C^T\Phi_N(\xi_i), C^T\mathbf{D}\Phi_N(\xi_i)) = d_i, \quad i = 0,\ 1. \tag{4.4}$$

Equation (4.3) together with equation (4.4) gives a system of equations with $N+1$ set of algebraic equations, which can be solved to find c_i, $i = 1, \ldots, N+1$. Consequently, the unknown function $y(x)$ given in Eq. (4.1) can be calculated.

5. Main results

The aim of this section is to analyze the numerical scheme (1.1) with special cases (1.3)-(1.5).

5.1. **Existence and uniqueness.** We consider the space ß $= \{y(t) : y(t) \in C[0,1], D^{(\beta)}y(t) \in C[0,1]\}$ furnished with the norm $||y(t)|| = \max_{t\in C[0,1]} |y(t)| + \max_{t\in C[0,1]} |D^{(\beta)}y(t)|$. The space ß is a Banach space [41].

Theorem 5.1. *(Theorem 3.2 in [31]) Let $f : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be continuous and there exists a function $\mu : [0,1] \to [0,\infty]$, such that*

$$|f(x,y,z)| \le \mu(t) + a_1|y| + a_2|z|, \quad a_1, a_2 \ge 0, \quad a_1 + a_2 \le m, \tag{5.1}$$

where $m = \min\{\frac{\Gamma(\alpha+1)}{2}, \frac{\Gamma(\alpha)\Gamma(2-\beta)+\Gamma(\alpha-\beta+1)}{4\Gamma(\alpha-\beta+1)\Gamma(\alpha)\Gamma(2-\beta)}\}$. Then, the boundary value problem (1.3)-(1.5) has a solution.

Theorem 5.2. *(Theorem 3.3 in [31]) Let $f : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be continuous. If f satisfies Lipschitz condition with respect to the second and third variables as*

$$|f(x,y,z) - f(x,y_1,z_1)| \le k(|y-y_1| + |z-z_1|), \text{ for each } x \in [0,1]\ y, y_1, z, z_1 \in \mathbb{R}\ k < 1, \tag{5.2}$$

then there exists a unique solution of the boundary value problem (1.3), (1.5) such that $y(x)$ is the solution of integral equation

$$\begin{aligned} y(x) &= J^{(\alpha)}(f(x,y(x),D^{(\beta)}y(x))) - xJ^{(\alpha)}(f(1,y(1),D^{(\beta)}y(1))) + (y_1-y_0)x + y_0, \\ &= \int_0^1 G(x,s)f(s,y(s),D^{(\beta)}y(s))ds + (y_1-y_0)x + y_0, \end{aligned} \tag{5.3}$$

where $G(x,s)$ is the Green function, given by

$$G(x,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (x-s)^{\alpha-1} - x(1-s)^{\alpha-1} & 0 \le s \le x \\ -x(1-s)^{\alpha-1} & x \le s \le 1. \end{cases} \tag{5.4}$$

Theorem 5.3. *Under the hypothesis of theorem 5.1 with $m = \min\{\frac{\Gamma(\alpha+1)}{2}, \frac{\Gamma(\alpha-\beta+1)}{4}\}$ the initial value problem (1.3)–(1.4) has a solution.*

Theorem 5.4. *Under the hypothesis of theorem 5.2 with $k \le \min\{\Gamma(\alpha+1), \Gamma(\alpha-\beta+1)\}$ the initial value problem (1.3)–(1.4) has a unique solution.*

The proofs of Theorem 5.3 and 5.4 are similar to those of Theorem 5.1 and 5.2.

5.2. **Stability analysis.** The sufficient conditions for the local asymptotical stability of (1.3) are discussed in this part.

Definition 5.5. we define an operator $\mathfrak{A} : ß \to ß$ by

(5.5)
$$\begin{aligned}\mathfrak{A}y(x) &= J^{(\alpha)}(f(x, y(x), D^{(\beta)}y(x))) - xJ^{(\alpha)}(f(1, y(1), D^{(\beta)}y(1))) + (y_1 - y_0)x + y_0,\\ &= \int_0^1 G(x,s)f(s, y(s), D^{(\beta)}y(s))ds + (y_1 - y_0)x + y_0.\end{aligned}$$

Definition 5.6. The equilibrium $y^* = 0$ of nonlinear fractional differential equation (1.3) is said to be locally asymptotically stable if $\exists\delta > 0$ such that $\forall y_a \in K$, one has

$$\lim_{x\to\infty} ||y(x, y_a)|| = 0, \tag{5.6}$$

where $K = \{y : ||y|| < \delta\}$ and $y(x, y_a)$ denotes the solution of (1.3) with initial or boundary conditions.

By assuming that $0 < \alpha - \beta < 1$, $u_1(x) = y(x)$, $u_2(x) = D^{(\beta)}y(x)$, we can reduce (1.3) to the system of fractional differential equation as follows

$$\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}^{(\alpha-\beta)} = \begin{pmatrix} \int_0^1 \hat{G}(x,s)f(s, u_1(s), u_2(s))ds + \frac{(u_1(1)-u_1(0))x^{1-\alpha+\beta}}{\Gamma(2-\alpha+\beta)} \\ f(x, u_1(x), u_2(x)) \end{pmatrix},$$

$$U^{(\alpha-\beta)} = F(x, U), \tag{5.7}$$

where $U = (u_1, u_2)^T$ and $\hat{G}(x, s)$ is obtained as follows:

$$\begin{aligned}D^{(\alpha-\beta)}(\mathfrak{A}y)(x) &= J^{(1-\alpha+\beta)}(D\mathfrak{A}y)(x)\\ &= J^{(1-\alpha+\beta)}(J^{(\alpha-1)}f(x, y(x), D^{(\beta)}y(x))\\ &\quad -J^{(\alpha)}(f(1, y(1), D^{(\beta)}y(1))) + (y_1 - y_0))\\ &= J^{(\beta)}f(x, y(x), D^{(\beta)}y(x)) - (J^{(\alpha)}(f(1, y(1), D^{(\beta)}y(1)))\\ &\quad +(y_1 - y_0))\frac{x^{1-\alpha+\beta}}{\Gamma(2-\alpha+\beta)}\\ &= \int_0^1 \hat{G}(x,s)f(s, y(s), D^{(\beta)}y(s))ds + \frac{(y_1 - y_0)x^{1-\alpha+\beta}}{\Gamma(2-\alpha+\beta)}, \end{aligned} \tag{5.8}$$

where $\hat{G}(x, s)$, (with respect to x) is of order $(\alpha - \beta)$ as

$$\hat{G}(x,s) = \begin{cases} \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} - \frac{(1-s)^{\alpha-1}x^{1-\alpha+\beta}}{\Gamma(\alpha)\Gamma(2-\alpha+\beta)}, & 0 \le s \le x, \\ -\frac{(1-s)^{\alpha-1}x^{1-\alpha+\beta}}{\Gamma(\alpha)\Gamma(2-\alpha+\beta)}, & x \le s \le 1 \end{cases}. \tag{5.9}$$

If $1 \le \alpha - \beta < 2$ then, we get $0 \le \alpha - \beta - 1 < 1$ and we continue the similar process of (5.7)-(5.9) as follows

$$\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}^{(\alpha-\beta-1)} = \begin{pmatrix} \int_0^1 \hat{G}(x,s)f(s, u_1(s), u_2(s))ds + \frac{(u_1(1)-u_1(0))x^{-\alpha+\beta}}{\Gamma(1-\alpha+\beta)} \\ \int f(x, u_1(x), u_2(x))dx \end{pmatrix},$$

$$U^{(\alpha-\beta-1)} = F(x, U), \tag{5.10}$$

where $\hat{G}(x,s)$, (with respect to x) is of order $(\alpha-\beta-1)$ as

$$\hat{G}(x,s)=\begin{cases} \frac{(x-s)^{\beta-2}}{\Gamma(\beta)}-\frac{(1-s)^{\alpha-1}x^{-\alpha+\beta}}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)}, & 0\le s\le x,\\ -\frac{(1-s)^{\alpha-1}x^{-\alpha+\beta}}{\Gamma(\alpha)\Gamma(1-\alpha+\beta)}, & x\le s\le 1\end{cases}. \tag{5.11}$$

Theorem 5.7. *The equilibrium $U^*=0$ of autonomous nonlinear fractional differential equation of (5.7) or (5.10) with $\nabla F(U)\in C([0,1]\times[0,1]\times[0,1])$ and $(\alpha-\beta)\in(0,1]$ or $(\alpha-\beta-1)\in(0,1]$ is locally asymptotically stable if $Re(\Lambda)<0$ where Λ is eigenvalues of the Jacobian matrix ∇F.*

The proof of this theorem is similar to the proof of Theorem 3.2 in [14].

5.3. Convergence analysis, accuracy order of the proposed method.

Theorem 5.8. *Let $e_{N+1}(x)=y(x)-y_{N+1}(x)$ be the error function of Chebyshev cardinal approximation, where $y(x)$ is the exact solution of (1.3) and $y_{N+1}(x)=\sum_{i=1}^{N+1}c_i\phi_i(x)=C^T\Phi_N(x)$ is the Chebyshev cardinal approximation for $y(x)$. Under the hypothesis of Theorems 5.1, 5.2 or 5.3, 5.4, $e_{N+1}(x)\to 0$ as $N\to\infty$ for (1.3), (1.5) or (1.3)–(1.4), respectively.*

Proof. Using Eqs. (4.1), (5.3) and (5.4) we have

$$\begin{aligned}|e_{N+1}(x)| &= |\int_0^1 G(x,s)f(s,y(s),D^{(\beta)}y(s))ds+(y_1-y_0)x+y_0-C^T\Phi_N(x)|\\ &= |\int_0^1 G(x,s)[f(s,y(s),D^{(\beta)}y(s))-f(s,y_{N+1}(s),D^{(\beta)}y_{N+1}(s))\\ &\quad +f(s,y_{N+1}(s),D^{(\beta)}y_{N+1}(s))]ds+(y_1-y_0)x+y_0-C^T\Phi_N(x)|\\ &= |\int_0^1 G(x,s)\left(f(s,y(s),D^{(\beta)}y(s))-f(s,y_{N+1}(s),D^{(\beta)}y_{N+1}(s))\right)ds\\ &\quad +\int_0^1 G(x,s)f(s,y_{N+1}(s),D^{(\beta)}y_{N+1}(s))ds+(y_1-y_0)x+y_0-C^T\Phi_N(x)|.\end{aligned} \tag{5.12}$$

Using Eq. (5.2) we get

$$|e_{N+1}(x)| \le k||e_{N+1}(x)|| \left(\int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} ds + x \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right)$$
$$+ |\int_0^1 G(x,s) f(s, C^T\Phi_N(s), C^T \mathbf{D}_\beta \Phi_N(s)) ds + (y_1 - y_0)x + y_0 - C^T\Phi_N(x)|$$
$$\le \quad k||e_{N+1}(x)|| \left(\frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x}{\Gamma(\alpha+1)} \right)$$
$$+ |\int_0^1 G(x,s) f(s, C^T\Phi_N(s), C^T \mathbf{D}_\beta \Phi_N(s)) ds + (y_1 - y_0)x + y_0 - C^T\Phi_N(x)|$$
$$\le \quad k||e_{N+1}(x)|| \frac{2}{\Gamma(\alpha+1)} + | \int_0^1 G(x,s) f(s, C^T\Phi_N(s), C^T \mathbf{D}_\beta \Phi_N(s)) ds$$
$$+(y_1 - y_0)x + y_0 - C^T\Phi_N(x)|. \tag{5.13}$$

On the other hand we have

$$R_{N+1}(x) = f(x, C^T\Phi_N(x), C^T \ \mathbf{D}_\beta \Phi_N(x)) - C^T \ \mathbf{D}_\alpha \Phi_N(x)) \cong 0. \tag{5.14}$$

Employing $J^{(\alpha)}$ on $R_{N+1}(x)$ and using Eq. (5.3) we have

(5.15)

$$|J^{(\alpha)} R_{N+1}(x)| \quad = \quad | \int_0^1 G(x,s) f(s, C^T\Phi_N(s), C^T \mathbf{D}_\beta \Phi_N(s)) ds$$
$$+(y_{N+1}(1) - y_{N+1}(0))x + y_{N+1}(0) - y_{N+1}(x)|.$$

Using (5.15) in (5.13) and assuming $y_{N+1}(0) = y_0, \quad y_{N+1}(1) = y_1$ we get

$$|e_{N+1}(x)| \quad \le \quad k||e_{N+1}(x)|| \frac{2}{\Gamma(\alpha+1)} + |J^{(\alpha)} R_{N+1}(x)|. \tag{5.16}$$

On the other hand, we have

$$|D^{(\beta)} e_{N+1}(x)| \le \int_0^1 |\tilde{G}(x,s)||f(s, y(s), D^{(\beta)} y(x)) ds - f(s, y_{N+1}(s), D^{(\beta)} y_{N+1}(x))| ds$$
$$\le ||e_{N+1}(x)|| (\int_0^1 \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} ds + \frac{x^{1-\beta}}{\Gamma(2-\beta)} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds)$$
$$= ||e_{N+1}(x)|| (\frac{x^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{x^{1-\beta}}{\Gamma(\alpha+1)\Gamma(2-\beta)})$$
$$\le \rho ||e_{N+1}(x)||, \tag{5.17}$$

where $\rho < 1$ and $\tilde{G}(x,s)$ is defined by Eq. (3.6) in [31]. Thus, we have

$$||e_{N+1}(x)|| \quad \le \quad (1 - \rho - \frac{2k}{\Gamma(\alpha+1)})^{-1} |J^{(\alpha)} R_{N+1}(x)|. \tag{5.18}$$

If we set $x = t_i, \quad i = 1, \ldots, N+1$, then our aim is to have $R_{N+1}(t_i) \le 10^{-r_i}$, where r_i is any positive integer. If we prescribe, $\max r_i = r$, then we increase N

as long as the following inequality holds at each point t_i:

$$|R_{N+1}(t_i)| \quad \leq \quad 10^{-r}, \tag{5.19}$$

in other words, by increasing N the error function $R_{N+1}(t_i)$ approaches zero. If $R_{N+1}(t_i) \to 0$ when N is sufficiently large enough, then the error decreases.

□

In this part we present accuracy order of numerical approach for solving (1.3). Let $y(x)$ is the exact solution of (1.3) and $y_{N+1}(x) = \sum_{i=1}^{N+1} c_i\phi_i(x) = C^T\Phi_N(x)$ is the Chebyshev cardinal approximation. So, by using (3.2), (5.2) for sufficiently large enough N, we have

$$|D^{(\alpha)}y(x) - D^{(\alpha)}y_{N+1}(x)| = |D^{(\alpha)}y(x) - \mathbf{D}_\alpha y_{N+1}(x) + \mathbf{D}_\alpha y_{N+1}(x) - D^{(\alpha)}y_{N+1}(x)|. \tag{5.20}$$

Thus

$$\begin{aligned}
&|D^{(\alpha)}y(x) - \mathbf{D}_\alpha y_{N+1}(x)|\\
&= |f(x, y(x), D^{(\beta)}y(x)) - f(x, y_{N+1}(x), \mathbf{D}_\beta y_{N+1}(x))|\\
&\leq k[|y(x) - y_{N+1}(x)| + |D^{(\beta)}y(x) - D^{(\beta)}y_{N+1}(x) + D^{(\beta)}y_{N+1}(x) - \mathbf{D}_\beta y_{N+1}(x)|]\\
&\leq k[\frac{|y^{(N+1)}(x)|}{2^N(N+1)!} + \ldots + |D^{(\beta)}[y(x) - y_{N+1}(x)]| + |D^{(\beta)}y_{N+1}(x) - \mathbf{D}_\beta y_{N+1}(x)|].
\end{aligned} \tag{5.21}$$

On the other hand, by using (2.3), (3.16), we have

$$|D^{(\alpha)}y_{N+1}(x) - \mathbf{D}_\alpha y_{N+1}(x)| = |\sum_{k=1}^{N+1} c_k(\phi_k^{(\alpha)}(x) - \sum_{j=1}^{N+1} \phi_k^{(\alpha)}(t_j)\phi_j(x))|$$

$$\leq \frac{\sum_{k=1}^{N+1} |c_k D^{N+1}(\phi_k^{(\alpha)}(x))|}{2^N(N+1)!} + \ldots. \tag{5.22}$$

If $\alpha = 1+\theta, \quad 0 \leq \theta < 1$, it is shown in Lemma 3.2 in [22] that

$$\begin{aligned}
D^{N+1}(\phi_k^{(\alpha)}(x)) &= D^{N+1}[D(\phi_k^{(\theta)}(x)) + \frac{\phi_k(0)x^{-2-\theta}}{\Gamma(-2-\theta)} - \sum_{h=0}^{1}\frac{\phi_k^{(h)}(0)x^{h-\theta}}{\Gamma(h+1-\theta)}]\\
&= D^{N+2}(\phi_k^{(\theta)}(x)) - \frac{S(-2-\theta-N, N+1)\phi_k(0)x^{-3-\theta-N}}{\Gamma(-2-\theta)}\\
&\quad + \sum_{h=0}^{1}\frac{\phi_k^{(h)}(0)S(k-\theta-N, N+1)x^{h-\theta-N-1}}{\Gamma(h+1-\theta)}, \quad k = 1, \ldots, N+1
\end{aligned} \tag{5.23}$$

and

$$D^{N+2}(\phi_k^{(\theta)}(x)) = \phi_k^{(N+2+\theta)}(x) - \frac{\phi_k(0)x^{-2(N+2)-\theta}}{\Gamma(-3-2N-\theta)} + \sum_{p=0}^{N+2}\frac{\phi_k^{(p)}(0)x^{p-N-1-\theta}}{\Gamma(p-N-\theta)}$$

$$= -\frac{\phi_k(0)x^{-2(N+2)-\theta}}{\Gamma(-3-2N-\theta)} + \sum_{p=0}^{N+2}\frac{\phi_k^{(p)}(0)x^{p-N-1-\theta}}{\Gamma(p-N-\theta)}$$

$$k = 1,\dots,N+1, \tag{5.24}$$

where $S(z,n) = z(z+1)\dots(z+n-1)$. However, by using (2.3) and employing $J^{(\alpha)}$ in (5.22), we get

$$|J^{(\alpha)}(D^{(\alpha)}y_{N+1}(x) - \mathbf{D}_\alpha y_{N+1}(x))|$$

$$\leq \frac{1}{2^N(N+1)!}\sum_{j=1}^{N+1}|c_j[-\frac{\phi_j(0)x^{-2(N+2)}}{\Gamma(-2N-1)} + \sum_{p=0}^{N+2}\frac{\phi_j^{(p)}(0)x^{p-N+1}}{\Gamma(p+1-N)} + \dots]|$$

$$\leq \frac{1}{2^N(N+1)!}|[\frac{y(0)x^{-2(N+1)}}{\Gamma(-2N-1)} + \sum_{p=0}^{N+2}\frac{y^{(p)}(0)x^{p-N}}{\Gamma(p+1-N)} + \dots]|,$$

$$\leq \frac{1}{2^N(N+1)!}|(y^{(N)}(0) + y^{(N+1)}(0)x + y^{(N+2)}(0)\frac{x^2}{2!})|$$

$$\leq \frac{1}{2^N(N+1)!}(|y^{(N)}(x)| + |\epsilon(N)|), \tag{5.25}$$

where $\epsilon(N) = \sum_{p=N+3}^{\infty}\frac{y^{(p)}(0)x^{p-N+1}}{\Gamma(p+1-N)} \to 0$ as $N \to \infty$, $\frac{1}{\Gamma(-2N-1)} \approx 0$ and $\frac{1}{\Gamma(p+1-N)} \approx 0$, $p = 0,\dots,N-2$. However, by using (2.3) in (5.21)

$$|e_{N+1}(x)| \leq kJ^{(\alpha)}[\frac{|y^{(N+1)}(x)|}{2^N(N+1)!} + \dots + |D^{(\beta)}[y(x) - y_{N+1}(x)]| + \dots]$$

$$\leq k[\frac{|y^{(N+1-\alpha)}(x)|}{2^N(N+1)!} + \dots + |D^{(\beta-\alpha)}[\frac{|y^{(N+1)}(x)|}{2^N(N+1)!}]$$

$$-\sum_{k=0}^{1}\frac{|y^{(k+N+1+\beta-\alpha)}(x)|_{x=0}x^k}{2^N(N+1)!k!}| + \dots]$$

$$\leq k[\frac{|y^{(N+1-\alpha)}(x)|}{2^N(N+1)!} + \dots + \frac{|y^{(N+1+\beta-\alpha)}(x)|}{2^N(N+1)!} + \dots]. \tag{5.26}$$

Thus

$$|e_{N+1}(x)| \leq k\frac{|y^{(N+1-\alpha)}(\xi_x)|}{2^N(N+1)!}, \tag{5.27}$$

where $\xi_x \in [0,1]$. Eqs (5.25) and (5.27) show that for $y(x) \in C^N_{-1}$, we get an exponentially convergence approximation.

6. Numerical examples

In this section we give a computational results of numerical experiments with methods based on preceding sections, to support our theoretical discussion. It should be noted that in Examples 6.1 and 6.3 the exact solution $y(x)$ does not belong to C^N_{-1}, $N \geq 1$. So, we have not exponential convergence. But for Examples 6.2, 6.4 and 6.5 $y(x)$ belongs to C^N_{-1}, $N \geq 1$, so we get exponentially convergence.

Example 6.1. Consider the nonlinear fractional differential equation:

$$D^{\frac{4}{3}}y(x) + D^{\frac{1}{2}}y(x) + y(x)^2 = \frac{9\Gamma\left(\frac{5}{6}\right)\sqrt[6]{x}}{4\sqrt{\pi}} + \frac{3}{4}\sqrt{\pi}x + x^3, \qquad x \in [0,1],$$

$$y(0) = 0, \qquad y(1) = 1. \tag{6.1}$$

The exact solution of this problem is $y(x) = x\sqrt{x}$. Table 1 shows the L_2 and L_∞ errors for the method presented in Section 4 for different values of N.

TABLE 1. L_2 and L_∞ errors using presented method for Example 6.1

$N+1$	L_2 error	L_∞ error
2	$1.0 \times e-1$	$1.4 \times e-1$
4	$1.3 \times e-2$	$2.3 \times e-2$
8	$2.2 \times e-3$	$2.9 \times e-3$
10	$1.3 \times e-3$	$2.3 \times e-3$

Example 6.2. Consider nonlinear boundary value problem

$$xD^{\frac{5}{4}}y(x) + (1+2x^2)D^{\frac{1}{3}}y(x)(y(x)) = \frac{243}{55\Gamma(\frac{2}{3})}x^{\frac{29}{3}} + \frac{243}{110\Gamma(\frac{2}{3})}x^{\frac{23}{3}}$$

$$+\frac{243}{55\Gamma(\frac{2}{3})}x^{\frac{17}{3}} + \frac{512}{77\Gamma(\frac{3}{4})}x^{\frac{15}{4}} + \frac{243}{110\Gamma(\frac{2}{3})}x^{\frac{11}{3}},$$

$$y(0) = 1, \qquad y(1) = 2. \tag{6.2}$$

The exact solution is

$$y(x) = x^4 + 1. \tag{6.3}$$

Table 2 shows the L_2 and L_∞ errors for the method presented in Section 4 for different values of N.

TABLE 2. L_2 and L_∞ errors using presented method for Example 6.2

$N+1$	L_2 error	L_∞ error
2	$3.3 \times e-1$	$4.7 \times e-1$
4	$6.3 \times e-2$	$8.7 \times e-2$
8	$1.7 \times e-6$	$3.4 \times e-6$
10	$1.8 \times e-8$	$9.5 \times e-8$

Example 6.3. Consider the fractional differential equation:

(6.4)

$$\begin{aligned} 4(x+1)D^{\frac{4}{3}}y(x)+4D^{\frac{1}{4}}y(x) &+ \frac{1}{\sqrt{x+1}}y(x) \\ &= -2\sqrt{x}+2\arcsin\left(\frac{-1+x}{x+1}\right)+\pi+\frac{\sqrt{\pi\,(x+1)}}{\sqrt{x+1}}, \\ & \quad y(0)=\sqrt{\pi}, \qquad y(1)=\sqrt{2\pi}. \end{aligned}$$

The exact solution is $y(x)=\sqrt{\pi(x+1)}$. The L_2 and L_∞ errors are obtained in Table 4 for different values of N using presented method in Section 4.

TABLE 3. L_∞ and L_2 errors using presented method for Example 6.3

$N+1$	$L_2 error$	$L_\infty error$
3	$5.1 \times e-3$	$7.4 \times e-3$
4	$3.6 \times e-4$	$7.0 \times e-4$
5	$7.9 \times e-5$	$1.1 \times e-4$
6	$6.5 \times e-6$	$9.7 \times e-6$
7	$1.6 \times e-6$	$2.3 \times e-6$
8	$1.3 \times e-7$	$2.7 \times e-7$

Example 6.4. Consider the nonlinear fractional differential equation:

$$D^{\frac{3}{2}}y(x)+D^{\frac{1}{2}}y(x)+(y(x))^2=\frac{32768}{715}\frac{x^{\frac{15}{2}}}{\sqrt{\pi}}+\frac{65536}{12155}\frac{x^{\frac{17}{2}}}{\sqrt{\pi}}+x^{18}$$

$$y(0)=0, \qquad y(1)=1. \tag{6.5}$$

The exact solution is $y(x)=x^9$.

Table 3 shows the L_∞ and L_2 errors that obtains for different values of N.

TABLE 4. L_∞ and L_2 errors using presented method for Example 4

$N+1$	$L_2 error$	$L_\infty error$
7	$7.3 \times e-3$	$8.2 \times e-3$
8	$5.4 \times e-4$	$1.2 \times e-3$
9	$4.1 \times e-5$	$7.0 \times e-5$
10	$5.6 \times e-35$	$9.1 \times e-35$

Example 6.5. Consider the fractional differential equation:

$$y''(x) + \Gamma(\frac{4}{5})(x)^{\frac{6}{5}} D^{\frac{6}{5}} y(x) + \frac{11}{9}\Gamma(\frac{5}{6})(x)^{\frac{1}{6}} D^{\frac{1}{6}} y(x) - (y'(x))^2 = 2 + \frac{1}{10}x^2$$

$$y(0) = 1, \quad y(1) = 2. \tag{6.6}$$

The exact solution is $y(x) = x^2 + 1$.

Figure 1 shows the plot of error with $N = 3$ using the method presented in section 4.

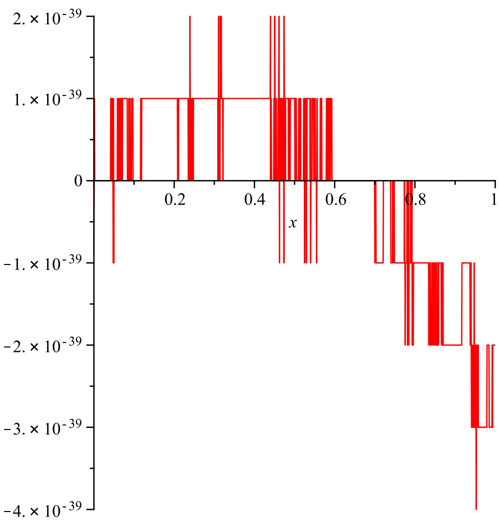

Figure 1. Plot of error for $y(x)$ with $N = 3$ for Example 6.5

7. Conclusion

In this paper we presented a numerical scheme for solving the nonlinear fractional differential equation. The Chebyshev cardinal functions was employed. The obtained results showed that this approach can solve the problem effectively.

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