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MULTIPLICATION OPERATORS ON BANACH MODULES OVER SPECTRALLY SEPARABLE ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a commutative Banach algebra and $\mathscr{X}$ be a left Banach $\mathcal{A}$-module. We study the set $\mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$ of all elements in $\mathcal{A}$ which induce a decomposable multiplication operator on $\mathscr{X}$. In the case $\mathscr{X} = \mathcal{A}$, $\mathrm{Dec}_{\mathcal{A}}(\mathcal{A})$ is the well-known Apostol algebra of $\mathcal{A}$. We show that $\mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$ is intimately related with the largest spectrally separable subalgebra of $\mathcal{A}$ and in this context we give some results which are related to an open question if Apostol algebra is regular for any commutative algebra $\mathcal{A}$.

Keywords: Commutative Banach algebra, decomposable multiplication operator, spectrally separable algebra.
MSC(2010): Primary 46J05; Secondary: 47B48, 46H25, 47B40.

1. Introduction

In their famous monograph [6], Colojoară and Foiaş have shown that every element a in a semisimple regular commutative complex Banach algebra $\mathcal{A}$ induces a multiplication operator $L_a : \mathcal{A} \to \mathcal{A}$ which is decomposable in the sense of Foiaş (see §2, Chapter 6 in [6] for more details). In [7], Frunză proved the converse, i.e., a semisimple commutative Banach algebra has to be regular if all multiplication operators on the algebra are decomposable. This result indicates close connection between the structure of a commutative Banach algebra and the local spectral properties of multiplication operators on it.

Let $\mathcal{A}$ be a commutative Banach algebra and $\mathscr{X}$ be a left Banach $\mathcal{A}$-module. Denote by $\mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$ the set of all elements in $\mathcal{A}$ for which the corresponding multiplication operator on $\mathscr{X}$ is decomposable. For instance, consider $\mathcal{A}$ as a left Banach module over itself through the usual multiplication in $\mathcal{A}$. Then the above result of Colojoară, Foiaş, and Frunză may be formulated in the following way.

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Theorem 1.1. *A semisimple commutative Banach algebra $\mathcal{A}$ is regular if and only if* $\mathrm{Dec}_{\mathcal{A}}(\mathcal{A}) = \mathcal{A}$.

Another classical result in this direction is the following theorem by Apostol. Let $\mathscr{X}$ be a complex Banach space and $\mathcal{B}$ be a closed subalgebra of $B(\mathscr{X})$, the Banach algebra of all bounded linear operators on $\mathscr{X}$, such that the identity operator I is in $\mathcal{B}$. Denote by $\mathcal{B}'$ the commutant of $\mathcal{B}$ in $B(\mathscr{X})$ and set $\mathcal{A} := \mathcal{B} \cap \mathcal{B}'$. Then $\mathcal{A}$ is a closed commutative subalgebra of $B(\mathscr{X})$ containing I. It is clear that $\mathcal{B}$ is a left Banach $\mathcal{A}$-module for the usual multiplication of operators. Recall that a subalgebra $\mathcal{S} \subseteq B(\mathscr{X})$ is said to be a *full subalgebra* if $T^{-1} \in \mathcal{S}$ whenever $T \in \mathcal{S}$ is invertible in $B(\mathscr{X})$.

Theorem 1.2 ([2], Theorem 3.6)**.** *The set* $\mathrm{Dec}_{\mathcal{A}}(\mathcal{B})$ *is a closed full subalgebra of* $B(\mathscr{X})$.

It is an immediate consequence of Theorem 1.2 that $\mathrm{Dec}_{\mathcal{A}}(\mathcal{A})$ is a closed full subalgebra of $\mathcal{A}$ whenever $\mathcal{A}$ is a closed commutative subalgebra of $B(\mathscr{X})$ containing I.

Our aim is to study $\mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$. The paper is organized as follows. In Section 2 we introduce notation and we present some background of the theory of commutative Banach algebras and of the local spectral theory of operators. Some basic properties of $\mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$ are presented in Section 3. The intimate relation between decomposable multiplication operators and spectrally separable algebras is outlined in Sections 4 and 5.

2. **Preliminaries**

Let $\mathcal{A}$ be a commutative Banach algebra. The *character space* of $\mathcal{A}$ is the set $\Sigma(\mathcal{A})$ of all non-zero multiplicative linear functionals on $\mathcal{A}$. As it is well known, every character is continuous with norm less than or equal to 1 ([9, Lemma 2.1.5]). Hence, $\Sigma(\mathcal{A})$ is a subset of the closed unit ball of $\mathcal{A}^*$, the topological dual of $\mathcal{A}$. The *Gelfand topology* on $\Sigma(\mathcal{A})$ is defined to be the relative weak-* topology on $\Sigma(\mathcal{A})$, which is considered as a subset of the closed unit ball of $\mathcal{A}^*$.

Another important topology on $\Sigma(\mathcal{A})$ is the *hk-topology* which is defined as follows. For a non-empty subset $E \subseteq \Sigma(\mathcal{A})$, *kernel* is defined as $\mathrm{k}(E) = \cap_{\varphi \in E} \ker \varphi$, and the kernel of the empty set is the whole algebra $\mathcal{A}$. The *hull* of a subset $\mathcal{U} \subseteq \mathcal{A}$ is $\mathrm{h}(\mathcal{U}) = \{\varphi \in \Sigma(\mathcal{A});\ \mathcal{U} \subseteq \ker \varphi\}$. Then the hk-topology on $\Sigma(\mathcal{A})$ is given by the closure operation

$$E \mapsto \mathrm{hk}(E) \qquad (E \subseteq \Sigma(\mathcal{A})).$$

The Gelfand topology is stronger than the hk-topology; if both topologies coincide the algebra is said to be *regular* (see [9, §4.2]). It is well known that $\mathcal{A}$ is regular if and only if the hk-topology is Hausdorff.

For each $a \in \mathcal{A}$, the *Gelfand transform* $\widehat{a}$ is given by $\widehat{a}(\varphi) = \varphi(a)$ $(\varphi \in \Sigma(\mathcal{A}))$. The mapping $a \mapsto \widehat{a}$ is the *Gelfand representation* and its kernel is the *radical* of $\mathcal{A}$. If the Gelfand representation is injective, $\mathcal{A}$ is said to be *semisimple*.

The *co-zero set* of $a \in \mathcal{A}$ is $\omega(a) = \{\varphi \in \Sigma(\mathcal{A});\ \varphi(a) \neq 0\}$. Note that this is a hk-open subset of $\Sigma(\mathcal{A})$. Its closure in the Gelfand topology is $\operatorname{supp} \widehat{a}$, the *support* of the Gelfand transform of a.

Let $\mathcal{A}$ be a commutative Banach algebra and $\mathscr{X}$ be a Banach space. A *representation* of $\mathcal{A}$ on $\mathscr{X}$ is an algebra homomorphism $\pi :\ \mathcal{A} \to B(\mathscr{X})$. Each representation π of $\mathcal{A}$ on $\mathscr{X}$ defines a structure of a left $\mathcal{A}$-module on $\mathscr{X}$ if the module multiplication is given by

$$a \cdot x := \pi(a)x \qquad (a \in \mathcal{A},\ x \in \mathscr{X}).$$

In this paper, it is always assumed that $\pi(1) = I$ if $\mathcal{A}$ is unital and 1 is its unit.

On the other hand, if $\mathscr{X}$ is a left $\mathcal{A}$-module, then the *corresponding representation* of $\mathcal{A}$ on $\mathscr{X}$ is the algebra homomorphism π of $\mathcal{A}$ into $B(\mathscr{X})$ that is defined by the above equality. If the corresponding representation of a left $\mathcal{A}$-module $\mathscr{X}$ is bounded, $\mathscr{X}$ is called a left *Banach $\mathcal{A}$-module*. The algebra $\mathcal{A}$ itself, considered as a left $\mathcal{A}$-module through the usual multiplication, is a left Banach module; the corresponding representation is denoted by λ.

For a left Banach $\mathcal{A}$-module $\mathscr{X}$, let $\mathscr{X}^*$ denote its topological dual. Then $\mathscr{X}^*$ is a left Banach $\mathcal{A}$-module for a module multiplication that is given through

$$\langle a \cdot \xi, x\rangle = \langle \xi, a \cdot x\rangle \qquad (a \in \mathcal{A},\ x \in \mathscr{X},\ \xi \in \mathscr{X}^*),$$

where $\langle \cdot, \cdot \rangle$ denotes the usual pairing between $\mathscr{X}$ and $\mathscr{X}^*$. If $\mathscr{X}^*$ has this module structure, then it is called the *dual module* of $\mathscr{X}$. Note that in the case of a non-commutative algebra the dual module of a given left module has a structure of a right module.

Let $\mathscr{X}$ be a left Banach $\mathcal{A}$-module with the corresponding representation π. The *annihilator* of $\emptyset \neq \mathscr{M} \subseteq \mathscr{X}$ is

$$\operatorname{ann}_\pi(\mathscr{M}) := \{a \in \mathcal{A};\ \pi(a)x = 0,\ \forall\ x \in \mathscr{M}\}.$$

In particular, the annihilator of a vector $x \in \mathscr{X}$ is given by

$$\operatorname{ann}_\pi(x) := \{a \in \mathcal{A};\ \pi(a)x = 0\}.$$

It is easily seen that an annihilator is a closed ideal of $\mathcal{A}$. The *Arveson spectrum* of π is defined as

$$\operatorname{sp}(\pi) := \{\varphi \in \Sigma(\mathcal{A});\ \operatorname{ann}_\pi(\mathscr{X}) \subseteq \ker \varphi\} = \operatorname{h}(\operatorname{ann}_\pi(\mathscr{X}))$$

and the *local Arveson spectrum* of π at x is

$$\operatorname{sp}_\pi(x) := \{\varphi \in \Sigma(\mathcal{A});\ \operatorname{ann}_\pi(x) \subseteq \ker \varphi\} = \operatorname{h}(\operatorname{ann}_\pi(x))$$

(see [10], §4.12). If a commutative Banach algebra $\mathcal{A}$ is semisimple and regular, then $\operatorname{sp}_\lambda(a) = \operatorname{supp} \widehat{a}$, for every $a \in \mathcal{A}$. Hence, in this case, $\operatorname{sp}_\lambda(a) = \emptyset$ if and only if $a = 0$ (cf. [10], Proposition 4.12.4).

It is not hard to see that local Arveson spectra have the following properties. Obviously, they are hk-closed subsets of $\Sigma(\mathcal{A})$. If $\mathcal{A}$ is unital (and therefore

$\pi(1) = I$), then $\mathrm{sp}_\pi(x) = \emptyset$ if and only if $x = 0$. For $a \in \mathcal{A}$ and $x, x_1, x_2 \in \mathscr{X}$, we have

$$\mathrm{hk}(\omega(a) \cap \mathrm{sp}_\pi(x)) \subseteq \mathrm{sp}_\pi(a \cdot x) \subseteq \mathrm{sp}_\lambda(a) \cap \mathrm{sp}_\pi(x).$$

and

$$\mathrm{sp}_\pi(x_1 + x_2) \subseteq \mathrm{sp}_\pi(x_1) \cup \mathrm{sp}_\pi(x_2).$$

At the end of this section let us recall two definitions from the local spectral theory. Let $\mathscr{X}$ be a Banach space and $T \in B(\mathscr{X})$. If for every open covering $\{U_1, U_2\}$ of $\mathbb{C}$ there exists a pair of closed T-invariant linear subspaces $\mathscr{Y}_1$ and $\mathscr{Y}_2$ of $\mathscr{X}$ such that $\mathscr{X} = \mathscr{Y}_1 + \mathscr{Y}_2$, and $\sigma(T|\mathscr{Y}_k) \subseteq U_k$ $(k = 1, 2)$, then T is said to be *decomposable* (in the sense of Foiaş). A decomposable operator T is *super-decomposable* if there exists an operator $S \in B(\mathscr{X})$ commuting with T such that the subspaces $\mathscr{Y}_1$ and $\mathscr{Y}_2$ in the above definition of decomposability are of the form $\mathscr{Y}_1 = \overline{\mathrm{im}\, S}$ and $\mathscr{Y}_2 = \overline{\mathrm{im}\,(I - S)}$. By [10, Proposition 1.4.3], an operator $T \in B(\mathscr{X})$ is super-decomposable if and only if, for every open covering $\{U_1, U_2\}$ of $\mathbb{C}$, there exist T-invariant closed linear subspaces $\mathscr{Y}_1$ and $\mathscr{Y}_2$ of $\mathscr{X}$ and operators T_1, $T_2 \in B(\mathscr{X})$ commuting with T such that $T_1 + T_2 = I$, $T_k\mathscr{X} \subseteq \mathscr{Y}_k$, and $\sigma(T|\mathscr{Y}_k) \subseteq U_k$ $(k = 1, 2)$.

3. **Some basic properties of** $\mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$

Some assertions about $\mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$ are easily seen. For instance, if $\mathcal{A} \cdot \mathscr{X}$, the linear span of all products $a \cdot x$ $(a \in \mathcal{A}, x \in \mathscr{X})$, is finite dimensional, then $\mathrm{Dec}_{\mathcal{A}}(\mathscr{X}) = \mathcal{A}$. Another simple observation is that

$$\mathrm{Dec}_{\mathcal{A}}(\mathscr{X}) = \mathrm{Dec}_{\mathcal{A}}(\mathscr{X}^*),$$

whenever $\mathcal{A}$ is a commutative Banach algebra, $\mathscr{X}$ is a left Banach $\mathcal{A}$-module, and $\mathscr{X}^*$ is the dual module of $\mathscr{X}$. Namely, by [10, Theorem 2.5.19], an operator T on a Banach space $\mathscr{X}$ is decomposable if and only if its adjoint operator T^* is decomposable on $\mathscr{X}^*$.

Proposition 3.1. *Let $\mathcal{A}$ be a commutative Banach algebra and $\mathscr{X}$ be a left Banach $\mathcal{A}$-module. Then* $\mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$ *is a closed subset of $\mathcal{A}$.*

Proof. Let $\{a_n\}_{n=1}^\infty \subseteq \mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$ be a convergent sequence with the limit point $a \in \mathcal{A}$. Denote by T_k $(k \in \mathbb{N})$ the multiplication operator induced by a_k on $\mathscr{X}$ and let T be the multiplication operator that corresponds to a. Then $\{T_n\}_{n=1}^\infty$ is a Cauchy sequence in $B(\mathscr{X})$ with the limit point T. Note that T commutes with each of the operators T_n. By [10, Theorem 3.4.10], T is decomposable, which means $a \in \mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$. □

The algebraic structure of $\mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$ is a harder issue. The problem is closely connected with a long-standing open problem whether sums and products of commuting decomposable operators on Banach spaces are decomposable (see [10, 6.1.4]). Namely, let $\mathscr{X}$ be a complex Banach space and S, $T \in B(\mathscr{X})$

be commuting decomposable operators. Denote by $\mathcal{A}$ an arbitrary closed commutative subalgebra of $B(\mathscr{X})$ which contains S and T. For the multiplication wich is given by

$$A \cdot x := Ax \qquad (A \in \mathcal{A},\ x \in \mathscr{X})$$

the space $\mathscr{X}$ becomes a left Banach $\mathcal{A}$-module. It is obvious that the multiplication operator which is induced by $A \in \mathcal{A}$ on $\mathscr{X}$ is A itself. Thus, by the assumption, $S,\ T \in \mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$. It is clear now that the sum $S+T$ (respectively, the product ST) is in $\mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$ if and only if $S+T$ (respectively, ST) is a decomposable operator on $\mathscr{X}$.

For $a \in \mathcal{A}$, let $\mathcal{A}[a]$ denote the closed subalgebra of $\mathcal{A}$ which is generated by a. If $\mathcal{A}$ is a commutative Banach algebra and $\mathscr{X}$ is a left Banach $\mathcal{A}$-module, then $\mathcal{A}[a] \subseteq \mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$, for any $a \in \mathrm{Dec}_{\mathcal{A}}(X)$. Namely, for an arbitrary complex polynomial p, the multiplication operator induced by $p(a)$ on $\mathscr{X}$ is $p(T_a)$, where T_a is a multiplication operator induced by a on $\mathscr{X}$. By [6, Corollary 2.1.11], $p(T_a)$ is decomposable, which means that $p(a) \in \mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$. Now, by Proposition 3.1, we conclude that $\mathcal{A}[a] \subseteq \mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$.

For a unital commutative Banach algebra $\mathcal{A}$, it is clear that the spectrum $\sigma(T_a)$ of a multiplication operator induced by $a \in \mathcal{A}$ on a left Banach $\mathcal{A}$-module $\mathscr{X}$ is included in the spectrum $\sigma(a)$. The same is true if $\mathcal{A}$ is without a unit. To see this, let $\mathcal{A}$ be a non-unital commutative Banach algebra and let $\mathcal{A}_1$ be its standard unitisation ([10], p. 335). Then each left Banach $\mathcal{A}$-module $\mathscr{X}$ is also a left Banach $\mathcal{A}_1$-module through the multiplication

$$(a+\lambda)\cdot x = a \cdot x + \lambda x \qquad (a+\lambda \in \mathcal{A}_1,\ x \in \mathscr{X}).$$

A number $\lambda \in \mathbb{C}$ is in the spectrum of $a \in \mathcal{A}$ if and only if $a-\lambda$ is invertible in $\mathcal{A}_1$, see [9, §1.2].

Proposition 3.2. *Let $\mathcal{A}$ be a unital commutative Banach algebra. If, for $a \in \mathcal{A}$, the spectrum $\sigma(a)$ is totally disconnected, then $a \in \mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$, for every left Banach $\mathcal{A}$-module $\mathscr{X}$. In particular, the radical of $\mathcal{A}$ is a subset of $\mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$ and each algebraic element $a \in \mathcal{A}$ is in $\mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$.*

Proof. Let $\mathscr{X}$ be an arbitrary left Banach $\mathcal{A}$-module. If $a \in \mathcal{A}$ has a totally disconnected spectrum, then, because of $\sigma(T_a) \subseteq \sigma(a)$, the spectrum of the multiplication operator T_a, which is induced by a on $\mathscr{X}$, is totally disconnected. Since, by [10, Proposition 1.4.5], every operator with totally disconnected spectrum is super-decomposable, we conclude that $a \in \mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$. ☐

4. **Spectrally separable algebras**

If $\mathcal{A}$ is a semisimple regular commutative Banach algebra $\mathcal{A}$, then

$$\mathrm{Dec}_{\mathcal{A}}(\mathcal{A}) = \mathcal{A}, \tag{4.1}$$

by Theorem 1.1. Does there exist a non-semisimple commutative Banach algebra $\mathcal{A}$ which satisfies (4.1)? In this section we will give an affirmative answer to this question.

A unital commutative Banach algebra $\mathcal{A}$ is said to be *spectrally separable* if for any two distinct characters $\varphi, \psi \in \Sigma(\mathcal{A})$ there exist elements $a, b \in \mathcal{A}$ such that

$$ab = 0 \qquad \text{and} \qquad \varphi(a)\psi(b) \neq 0. \tag{4.2}$$

Spectrally separable algebras have been introduced and studied by Baskakov, see [3] and references therein.

Every spectrally separable algebra is regular. Indeed, let φ and ψ be two distinct characters on $\mathcal{A}$ and let $a, b \in \mathcal{A}$ be elements which fulfill (4.2). Then $\omega(a)$ and $\omega(b)$ are disjoint hk-open neighbourhoods of φ and ψ, respectively, which means that the hk-topology is Hausdorff.

On the other hand, it is not known whether every regular unital commutative Banach algebra is spectrally separable. In the particular case of semisimple algebras both notions coincide. Namely, let $\mathcal{A}$ be semisimple and regular. Then, for two distinct characters φ and ψ on $\mathcal{A}$, there exist, by regularity, elements $a, b \in \mathcal{A}$ such that $\widehat{a}(\varphi) = 1$, $\psi \notin \operatorname{supp} \widehat{a}$, $\widehat{b}(\psi) = 1$, and $\varphi \notin \operatorname{supp} \widehat{b}$. It follows $\widehat{ab} = \widehat{a}\widehat{b} = 0$ and consequently, by semisimplicity, $ab = 0$.

Besides unital semisimple regular commutative Banach algebras there exist also non-semisimple spectrally separable algebras. For instance, if T is a bounded linear operator on a Banach space $\mathscr{X}$ and the spectrum of T is totally disconnected, then the closed subalgebra of $B(\mathscr{X})$ which is generated by T and I is spectrally separable and, in general, non-semisimple (see [5, Example 3.1]). For more examples see [4].

If $\mathcal{A}$ is a spectrally separable algebra and $\{U_1, \ldots, U_n\}$ is an open covering of $\Sigma(\mathcal{A})$, then there exist elements $a_1, \ldots, a_n$ in $\mathcal{A}$ such that $a_1 + \cdots + a_n = 1$ and $\operatorname{sp}_\lambda(a_k) \subset U_k$ for all $k = 1, \ldots, n$. Relying on this result we can characterize spectrally separable algebras through the local spectral theory as follows.

Theorem 4.1. *Let $\mathcal{A}$ be a unital commutative Banach algebra. If there exists a subset $\mathcal{A}_0 \subseteq \mathcal{A}$ such that the Gelfand transforms of elements in $\mathcal{A}_0$ separate the points of $\Sigma(\mathcal{A})$ and for each $a \in \mathcal{A}_0$ the corresponding multiplication operator on $\mathcal{A}$ is decomposable, then $\mathcal{A}$ is spectrally separable.*

On the other hand, if $\mathcal{A}$ is spectrally separable and $\mathscr{X}$ is a left Banach $\mathcal{A}$-module, then, for every $a \in \mathcal{A}$, the multiplication operator which is induced by a on $\mathscr{X}$ is super-decomposable.

For a proof of Theorem 4.1 see [4]. An immediate corollary of the theorem is the following assertion.

Corollary 4.2. *Let $\mathcal{A}$ be a unital commutative Banach algebra. If $\operatorname{Dec}_{\mathcal{A}}(\mathcal{A}) = \mathcal{A}$, then $\mathcal{A}$ is spectrally separable. On the other hand, if $\mathcal{A}$ is spectrally separable, then $\operatorname{Dec}_{\mathcal{A}}(\mathscr{X}) = \mathcal{A}$, for every left Banach $\mathcal{A}$-module $\mathscr{X}$.*

Every commutative Banach algebra $\mathcal{A}$ (with or without identity) contains a maximum closed regular subalgebra $\mathrm{Reg}(\mathcal{A})$. For a unital semisimple algebra, this was observed by Albrecht [1]. In the general case it was proven by Neumann [11] and, independently, by Inoue and Takahasi [8]. Now we shall show that every unital commutative Banach algebra contains a maximal spectrally separable subalgebra.

Theorem 4.3. *Let $\mathcal{A}$ be a unital commutative Banach algebra. There exists a greatest spectrally separable subalgebra of $\mathcal{A}$, denoted by* $\mathrm{Sep}(\mathcal{A})$, *such that: (i)* $\mathrm{Sep}(\mathcal{A})$ *is closed and contains* 1, *the unit of $\mathcal{A}$, and (ii)* $\mathrm{Sep}(\mathcal{A})$ *is a full subalgebra of $\mathcal{A}$.*

Proof. Let $\mathcal{F}$ be the family of all spectrally separable subalgebras of $\mathcal{A}$. This family is not empty because it contains $\mathbb{C}1$, the onedimensional subalgebra of $\mathcal{A}$ which is generated by 1. Denote by $\mathcal{S}$ the closed subalgebra of $\mathcal{A}$ which is generated by the union $\mathcal{D} = \cup_{\mathcal{C}\in\mathcal{F}}\mathcal{C}$.

Let φ and ψ be two distinct characters on $\mathcal{S}$. If $\varphi(a) = \psi(a)$ for all $a \in \mathcal{D}$, then φ and ψ would be equal on $\mathcal{S}$. Since this is not the case, it follows that the Gelfand transforms of elements in $\mathcal{D}$ separate the points of $\Sigma(\mathcal{S})$.

Choose $a \in \mathcal{D}$ and consider the corresponding multiplication operator L_a on $\mathcal{S}$. Since a is in $\mathcal{D}$ there is an algebra $\mathcal{C}_a$ in the family $\mathcal{F}$ such that $a \in \mathcal{C}_a$. The algebra $\mathcal{S}$ is a Banach left $\mathcal{C}_a$-module. Hence L_a is decomposable, by Theorem 4.1. By the same theorem, we may conclude that $\mathcal{S}$ is spectrally separable. It is clear, by the construction, that $\mathcal{S}$ is the greatest algebra in $\mathcal{F}$ and that it satisfies (i).

In order to show (ii), assume that $\mathcal{S}$ is not a full subalgebra of $\mathcal{A}$. Then there exists an element $a \in \mathcal{S}$, which is invertible in $\mathcal{A}$, and $a^{-1} \notin \mathcal{S}$. Denote by $\mathcal{T}$ the closed subalgebra of $\mathcal{A}$ that is generated by $\mathcal{S} \cup \{a^{-1}\}$. Of course, Gelfand transforms of the elements in $\mathcal{S}\cup\{a^{-1}\}$ separate the points of $\Sigma(\mathcal{T})$. Since $\mathcal{T}$ is a left Banach module over $\mathcal{S}$, all multiplication operators $L_b : \mathcal{T} \to \mathcal{T}$ $(b \in \mathcal{S})$ are decomposable, by Theorem 4.1. In particular, $L_a : \mathcal{T} \to \mathcal{T}$ is decomposable. The operator L_a is invertible and its inverse is $L_{a^{-1}}$. Since, by [6, Proposition 2.1.12], $L_{a^{-1}}$ is decomposable we may use Theorem 4.1 again and conclude that $\mathcal{T}$ is spectrally separable. However this contradicts the assumption that $\mathcal{S}$ is the greatest spectrally separable subalgebra of $\mathcal{A}$. □

Following the proof of [10, Proposition 4.3.7] it can be shown that $\mathrm{Sep}(\mathcal{A})$ is spectrally closed, however we will not include that proof here.

If $\mathcal{A}$ is unital and semisimple, then $\mathrm{Reg}(\mathcal{A}) = \mathrm{Sep}(\mathcal{A})$. Since we do not know if every unital regular algebra is spectrally separable, we also do not know if the inclusion $\mathrm{Reg}(\mathcal{A}) \supseteq \mathrm{Sep}(\mathcal{A})$ is actually an equality, for any unital commutative Banach algebra $\mathcal{A}$. However, there is a great similarity between these two algebras. For instance, we have the following result, which should be compared with [10, Proposition 4.4.16].

Proposition 4.4. *Let $\mathcal{A}$ be a unital commutative Banach algebra, and consider the transfinite sequence of closed subalgebras of $\mathcal{A}$ given by $\mathcal{A}_0 := \mathcal{A}$, $\mathcal{A}_{\alpha+1} := \mathrm{Dec}_{\mathcal{A}_\alpha}(\mathcal{A}_\alpha)$ for each ordinal α, and $\mathcal{A}_\alpha := \cap\{\mathcal{A}_\beta;\ \beta < \alpha\}$ for each limit ordinal α. Then the sequence $\{\mathcal{A}_\alpha\}_\alpha$ is eventually constant, and its eventual constant value is* $\mathrm{Sep}(\mathcal{A})$.

Proof. It is clear that the transfinite sequence of algebras is decreasing. Since there exists an ordinal number that is greater than the number of all elements in $\mathcal{A}$, the sequence $\{\mathcal{A}_\alpha\}_\alpha$ has to be eventually constant. Let $\mathcal{S}$ be this constant value. By the second part of Theorem 4.1 it is obvious that $\mathrm{Sep}(\mathcal{A}) \subseteq \mathrm{Dec}(\mathcal{A}_\alpha)$ for each ordinal α. Thus $\mathrm{Sep}(\mathcal{A}) \subseteq \mathcal{S}$. On the other hand, it follows from $\mathcal{S} = \mathrm{Dec}_{\mathcal{S}}(\mathcal{S})$ that $\mathcal{S}$ is spectrally separable. □

In general, it is hard to determine $\mathrm{Sep}(\mathcal{A})$, for a given unital commutative Banach algebra $\mathcal{A}$. For instance, let G be a non-discrete locally compact abelian group and $M(G)$ be the measure algebra. Then $M(G)$ is a unital semisimple commutative Banach algebra that is not regular. Since $M(G)$ is semisimple we have $\mathrm{Sep}(M(G)) = \mathrm{Reg}(M(G))$ however it is not known (even for $G = \mathbb{R}$ and $G = \mathbb{T}$) which measures are in this subalgebra (for the details see [10], Section 4.4, especially Example 4.3.11).

If $\mathcal{A}$ is a unital commutative Banach algebra and $\mathscr{X}$ is a left Banach $\mathcal{A}$-module, then $\mathscr{X}$ is also a left Banach $\mathrm{Sep}(\mathcal{A})$-module. Thus, a simple application of Theorem 4.1 gives the following result.

Proposition 4.5. *Let $\mathcal{A}$ be a unital commutative Banach algebra. Then*

$$\mathrm{Sep}(\mathcal{A}) \subseteq \mathrm{Dec}_{\mathcal{A}}(\mathscr{X}),$$

for every left Banach $\mathcal{A}$-module $\mathscr{X}$.

When $\mathcal{A}$ is considered as a left Banach module over itself through the usual multiplication, we have $\mathrm{Sep}(\mathcal{A}) \subseteq \mathrm{Dec}_{\mathcal{A}}(\mathcal{A})$. It is not known, in general, whether $\mathrm{Sep}(\mathcal{A}) = \mathrm{Dec}_{\mathcal{A}}(\mathcal{A})$. Even in the case of semisimple algebras, when $\mathrm{Sep}(\mathcal{A}) = \mathrm{Reg}(\mathcal{A})$, it is an open question whether $\mathrm{Reg}(\mathcal{A}) = \mathrm{Dec}_{\mathcal{A}}(\mathcal{A})$ (see Section 6.2 in [10] for related open questions). The class of semisimple non-regular commutative Banach algebras for which the equality $\mathrm{Reg}(\mathcal{A}) = \mathrm{Dec}_{\mathcal{A}}(\mathcal{A})$ is confirmed is not large and some examples of algebras in this class may be found in Sections 4.4 and 4.5 of [10].

In the end of this section we prove an assertion by using spectrally separable algebras.

Proposition 4.6. *Let $\mathcal{A}$ be a unital commutative Banach algebra and $\mathscr{X}$ be a left Banach $\mathcal{A}$-module with the corresponding representation π. If the Arveson spectrum of π is totally disconnected, then* $\mathrm{Dec}_{\mathcal{A}}(\mathscr{X}) = \mathcal{A}$.

Proof. Denote by $\mathcal{B}$ the quotient algebra $\mathcal{A}/\mathrm{ann}_\pi(\mathscr{X})$. This is a commutative Banach algebra with the character space $\Sigma(\mathcal{B})$ which can be identified with

$h(\mathrm{ann}_\pi(\mathscr{X})) = \mathrm{sp}(\pi) \subseteq \Sigma(\mathcal{A})$ ([9, Lemma 2.2.15]). The multiplication

$$(a + \mathrm{ann}_\pi(\mathscr{X})) \cdot x := a \cdot x \qquad (a + \mathrm{ann}_\pi(\mathscr{X}) \in \mathcal{B},\ x \in \mathscr{X})$$

is well defined and $\mathscr{X}$ is through it a left Banach $\mathcal{B}$-module. It is clear that $a \in \mathcal{A}$ and $a + \mathrm{ann}_\pi(\mathscr{X}) \in \mathcal{B}$ induce the same multiplication operator T_a on $\mathscr{X}$. Since the character space of $\mathcal{B}$ is totally disconnected, the algebra is spectrally separable, by [4, Example 2.1]. Since, by Theorem 4.1, each element in $\mathcal{B}$ induces a decomposable multiplication operator on $\mathscr{X}$, we conclude that T_a is decomposable. □

5. Elements with continuous Gelfand transform

For a semisimple commutative Banach algebra $\mathcal{A}$, Neumann ([11,12]) characterized the Apostol algebra of $\mathcal{A}$ as follows.

Theorem 5.1. *Let $\mathcal{A}$ be a semisimple commutative Banach algebra. If $a \in \mathrm{Dec}_{\mathcal{A}}(\mathcal{A})$, then the Gelfand transform $\widehat{a}$ is hk-continuous.*

On the other hand, if the Gelfand transform of $a \in \mathcal{A}$ is hk-continuous, then $a \in \mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$, for every left Banach $\mathcal{A}$-module $\mathscr{X}$.

It is an immediate consequence of Theorem 5.1 that an element $a \in \mathcal{A}$ is in $\mathrm{Dec}_{\mathcal{A}}(\mathcal{A})$ if and only if its Gelfand transform is hk-continuous and that $\mathrm{Dec}_{\mathcal{A}}(\mathcal{A}) \subseteq \mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$, for every left Banach $\mathcal{A}$-module $\mathscr{X}$. Since $\mathrm{Reg}(\mathcal{A}) \subseteq \mathrm{Dec}_{\mathcal{A}}(\mathcal{A})$ the Gelfand transform $\widehat{a}$ is hk-continuous, for every $a \in \mathrm{Reg}(\mathcal{A})$. On the other hand, we do not know if there exists a semisimple commutative Banach algebra $\mathcal{A}$ and an element $a \in \mathcal{A}$ whose Gelfand transform is hk-continuous but $a \notin \mathrm{Reg}(\mathcal{A})$. Hence, for semisimple commutative Banach algebras, this question is equivalent to the problem whether $\mathrm{Reg}(\mathcal{A}) = \mathrm{Dec}_{\mathcal{A}}(\mathcal{A})$.

We do not know whether Theorem 5.1 holds for non-semisimple algebras. However if the hk-topology is replaced by a weaker one, a variant of Theorem 5.1 can be proven for unital commutative Banach algebras that are not necessarily semisimple. The mentioned topology on $\Sigma(\mathcal{A})$ is defined by the help of the largest spectrally separable subalgebra as follows.

Let $\mathcal{A}$ be a unital commutative Banach algebra. For $\varphi \in \Sigma(\mathcal{A})$ let $r(\varphi)$ denote the restriction of φ to $\mathrm{Sep}(\mathcal{A})$. It is obvious that r maps $\Sigma(\mathcal{A})$ into $\Sigma(\mathrm{Sep}(\mathcal{A}))$, i.e., $r(\varphi)$ is a character on $\mathrm{Sep}(\mathcal{A})$. Since $\mathrm{Sep}(\mathcal{A})$ is regular the map r is surjective (see [9]), however it does not need to be injective.

Define a family τ of subsets of $\Sigma(\mathcal{A})$ in the following way. A subset $U \subseteq \Sigma(\mathcal{A})$ is in τ if and only if it is hk-open and $r^{-1}(r(\varphi)) \subseteq U$ for each $\varphi \in U$. It is not hard to see that τ is indeed a topology on $\Sigma(\mathcal{A})$ and that it is weaker than the hk-topology. This is the reason why we shall call it *shk-topology* (sub-hull-kernel topology).

It is obvious that $r^{-1}(r(\varphi)) \subseteq F$ for each $\varphi \in F$ whenever F is a shk-closed subset of $\Sigma(\mathcal{A})$. Hence, singletons are not shk-closed, in general. In fact, the shk-closure of $\{\varphi\}$ is $r^{-1}(r(\varphi))$. Namely, let ψ be in the hk-closure of $r^{-1}(r(\varphi))$.

Then $r(\psi)$ annihilates the kernel of $r(\varphi)$ which means that $r(\psi) = r(\varphi)$ and consequently $\psi \in r^{-1}(r(\varphi))$.

The shk-closure of a subset $S \subseteq \Sigma(\mathcal{A})$ will be denoted by $\mathrm{shk}(S)$.

In the sequel let $\mu : \mathrm{Sep}(\mathcal{A}) \to B(\mathrm{Sep}(\mathcal{A}))$ be the left regular representation of $\mathrm{Sep}(\mathcal{A})$.

Proposition 5.2. *Let $\mathcal{A}$ be a unital commutative Banach algebra. If $\Sigma(\mathcal{A})$ is endowed with the shk-topology, then $r : \Sigma(\mathcal{A}) \to \Sigma(\mathrm{Sep}(\mathcal{A}))$ is continuous, closed, and open.*

Proof. Note that $\Sigma(\mathrm{Sep}(\mathcal{A}))$ is endowed with Gelfand topology which coincides with hk-topology because of the regularity of $\mathrm{Sep}(\mathcal{A})$.

We shall start with the continuity of r. Let $F \subseteq \Sigma(\mathrm{Sep}(\mathcal{A}))$ be a closed set. Since it is obvious that $r^{-1}(r(\varphi)) \subseteq r^{-1}(F)$ whenever $\varphi \in r^{-1}(F)$, we have to show that $\mathrm{hk}(r^{-1}(F)) \subseteq r^{-1}(F)$, i.e. $r^{-1}(F)$ is hk-closed.

Assume that $\psi \in \mathrm{hk}(r^{-1}(F))$. If $a \in \mathrm{Sep}(\mathcal{A})$ is in $\mathrm{k}_S(F)$, where k_S stands for a kernel in $\mathrm{Sep}(\mathcal{A})$, then $r(\varphi)(a) = 0$ for each $\varphi \in r^{-1}(F)$. It follows that $\varphi(a) = 0$, which gives $a \in \mathrm{k}(r^{-1}(F))$. Now, since $\psi \in \mathrm{hk}(r^{-1}(F))$ we have $\psi(a) = 0$ and therefore $r(\psi) \in \mathrm{h}_S\mathrm{k}_S(F) = F$. It follows $\psi \in r^{-1}(F)$. By h_S we have denoted a hull in $\Sigma(\mathrm{Sep}(\mathcal{A}))$.

Now we are going to prove that r maps shk-closed sets into closed sets. We have to show that $\mathrm{h}_S\mathrm{k}_S(r(F)) \subseteq r(F)$, for a shk-closed subset $F \subseteq \Sigma(\mathcal{A})$. Assume that $r(\varphi) \notin r(F)$. Then, of course, $r^{-1}(r(\varphi)) \cap F = \emptyset$. It follows, by [4, Corollary 2.9], that for each $\psi \in F$ there exists $b_\psi \in \mathrm{Sep}(\mathcal{A})$ such that $\widehat{b}_\psi(r(\psi)) = 1$ and $r(\psi) \notin \mathrm{sp}_\mu(b_\psi)$. Let

$$E_\psi = \{\eta \in \Sigma(\mathrm{Sep}(\mathcal{A}));\ |\widehat{b}_\psi(\eta)| \geq 1/3\} \quad \text{and} \quad W_\psi = \{\eta \in \Sigma(\mathrm{Sep}(\mathcal{A}));\ |\widehat{b}_\psi(\eta)| > 2/3\}.$$

Of course, E_ψ is a closed subset and W_ψ is an open subset of $\Sigma(\mathrm{Sep}(\mathcal{A}))$. By [13, Theorem 3.6.15], there exists $c_\psi \in \mathrm{Sep}(\mathcal{A})$ such that $\eta(b_\psi c_\psi) = 1$ for all $\eta \in E_\psi$. Let $d_\psi = b_\psi c_\psi$. Then $r(\varphi) \notin \mathrm{sp}_\mu(d_\psi)$. Since r is continuous $r^{-1}(E_\psi)$ is a shk-closed subset and $r^{-1}(W_\psi)$ is a shk-open subset of $\Sigma(\mathcal{A})$. The family $\{r^{-1}(W_\psi);\ \psi \in F\}$ is a shk-open covering of F, which is shk-closed and therefore shk-compact. Thus, there exist $\psi_1, \ldots, \psi_n \in F$ such that

$$F \subseteq r^{-1}(W_{\psi_1}) \cup \ldots \cup r^{-1}(W_{\psi_n}) \subseteq r^{-1}(E_{\psi_1}) \cup \ldots \cup r^{-1}(E_{\psi_n}).$$

It is easily seen that

$$r(r^{-1}(E_{\psi_1}) \cup \ldots \cup r^{-1}(E_{\psi_n})) = E_{\psi_1} \cup \ldots \cup E_{\psi_n},$$

therefore

$$r(F) \subseteq E_{\psi_1} \cup \ldots \cup E_{\psi_n}.$$

Let

$$e_1 := d_{\psi_1}, \quad e_2 := (1 - d_{\psi_1})d_{\psi_2}, \quad \ldots, \quad e_n := (1 - d_{\psi_1}) \cdots (1 - d_{\psi_{n-1}})d_{\psi_n}.$$

Then $e_k \in \mathrm{Sep}(\mathcal{A})$ and $r(\varphi) \notin \mathrm{sp}_\mu(e_k)$ for all $k = 1, \dots, n$. Inductively we get

$$e_1 + \cdots e_k = 1 - (1 - d_{\psi_1}) \cdots (1 - d_{\psi_k}) \qquad (k = 1, \dots, n).$$

Set $e = e_1 + \cdots + e_n$ and $a = 1 - e$. Then $r(\varphi) \notin \mathrm{sp}_\mu(e)$ and consequently $\widehat{a}(r(\varphi)) = 1$. For each $\eta \in r(F)$, there exists E_{ψ_k} that contains η. It follows that $\widehat{d}_{\psi_k}(\eta) = 1$, which gives $\widehat{e}(\eta) = 1$ and consequently $\widehat{a}(\eta) = 0$. We have proven that $a \in \mathrm{k}_S(r(F))$. Since $\widehat{a}(r(\varphi)) = 1$ we conclude $r(\varphi) \notin \mathrm{h}_S\mathrm{k}_S(r(F))$.

Let now $U \subseteq \Sigma(\mathcal{A})$ be shk-open. Then $F := \Sigma(\mathcal{A}) \setminus U$ is shk-closed and consequently $r(F)$ is a closed subset of $\Sigma(\mathrm{Sep}(\mathcal{A}))$. Since $r(U) = \Sigma(\mathrm{Sep}(\mathcal{A})) \setminus r(F)$ we conclude that $r(U)$ is open. □

Now it is easy to deduce the following assertion.

Proposition 5.3. *Let $\mathcal{A}$ be a unital commutative Banach algebra and $\{U_1, \dots, U_n\}$ be an shk-open covering of $\Sigma(\mathcal{A})$. Then there exist $a_1, \dots, a_n \in \mathrm{Sep}(\mathcal{A})$ such that $a_1 + \cdots + a_n = 1$ and $\mathrm{sp}_\lambda(a_k) \subseteq U_k$, for $k = 1, \dots, n$. In particular, if $F \subseteq \Sigma(\mathcal{A})$ is shk-closed and $\varphi \in \Sigma(\mathcal{A})$ is not in F, then there exists $a \in \mathrm{Sep}(\mathcal{A})$ such that $\mathrm{sp}_\lambda(a) \cap F = \emptyset$ and $\widehat{a}(\varphi) = 1$.*

Proof. Since $\{U_1, \dots, U_n\}$ is an shk-open covering of $\Sigma(\mathcal{A})$ the sets $V_k = r(U_k)$ $(k = 1, \dots, n)$ form an open covering of $\Sigma(\mathrm{Sep}(\mathcal{A}))$. By [4, Theorem 2.10], there exist $a_1, \dots, a_n \in \mathrm{Sep}(\mathcal{A})$ such that $a_1 + \cdots + a_n = 1$ and $\mathrm{sp}_\mu(a_k) \subseteq V_k$ $(k = 1, \dots, n)$. Since $r^{-1}(V_k) \subseteq U_k$ we have $r^{-1}(\mathrm{sp}_\mu(a_k)) \subseteq r^{-1}(V_k) \subseteq U_k$. Let us show that $\mathrm{sp}_\lambda(a_k) \subseteq r^{-1}(\mathrm{sp}_\mu(a_k))$. It is obvious that $\mathrm{ann}_\mu(a_k) \subseteq \mathrm{ann}_\lambda(a_k)$. Thus, if $\varphi \in \mathrm{sp}_\lambda(a_k)$, then $\mathrm{ann}_\lambda(a_k) \subseteq \ker\varphi$ gives $\mathrm{ann}_\mu(a_k) \subseteq \ker r(\varphi)$, which means $r(\varphi) \in \mathrm{sp}_\mu(a_k)$ and consequently $\varphi \in r^{-1}(\mathrm{sp}_\mu(a_k))$. □

Proposition 5.4. *Let $\mathcal{A}$ be a unital commutative Banach algebra, $\mathscr{X}$ be a left Banach $\mathcal{A}$-module, and $F \subseteq \Sigma(\mathcal{A})$ be an shk-closed set.*

(i) The set $\mathscr{X}(F) = \{x \in \mathscr{X};\ \mathrm{sp}_\pi(x) \subseteq F\}$ is a submodule of $\mathscr{X}$.

(ii) If $\mathscr{X} = \mathcal{A}$, then $\mathcal{A}(F)$ is a closed ideal of $\mathcal{A}$ and

$$hk(F^c) \subseteq h(\mathcal{A}(F)) \subseteq \mathrm{shk}(F^c).$$

Proof. (i) It is easily seen that $\mathscr{X}(F)$ is a submodule. Let $\{x_n\}_{n=1}^\infty \subset \mathscr{X}(F)$ be a convergent sequence with the limit point $x \in \mathscr{X}$. If $\varphi \in \Sigma(\mathcal{A})$ is not in F, then, by Proposition 5.3, there exists $a \in \mathrm{Sep}(\mathcal{A})$ such that $\widehat{a}(\varphi) = 1$ and $\mathrm{sp}_\lambda(a) \cap F = \emptyset$. It follows that $a \cdot x_n = 0$, for all $n = 1, 2, \dots$. Since $a \cdot x = \lim_{n\to\infty} a \cdot x_n = 0$ and $\varphi \in \omega(a)$ we have $\varphi \notin \mathrm{sp}_\pi(x)$.

(ii) Since $\mathrm{sp}_\lambda(a) \cap F^c = \emptyset$, for each $a \in \mathcal{A}(F)$, the ideal $\mathcal{A}(F)$ is included in the kernel $\mathrm{k}(F^c)$ and consequently the first inclusion follows. On the other hand, if $\varphi \in \Sigma(\mathcal{A})$ is not in $\mathrm{shk}(F^c)$, then there exists, by Proposition 5.3, $a \in \mathrm{Sep}(\mathcal{A})$ such that $\widehat{a}(\varphi) = 1$ and $\mathrm{sp}_\lambda(a) \cap \mathrm{shk}(F^c) = \emptyset$. Thus $a \in \mathcal{A}(F)$ and therefore $\varphi \notin h(\mathcal{A}(F))$. □

Now we can prove the analog of Theorem 5.1 for unital commutative Banach algebras which are not necessarily semisimple.

Theorem 5.5. *Let $\mathcal{A}$ be a unital commutative Banach algebra. If $a \in \mathrm{Dec}_{\mathcal{A}}(\mathcal{A})$ and $\widehat{a}$ is a constant function on $r^{-1}(r(\varphi))$, for each $\varphi \in \Sigma(\mathcal{A})$, then $\widehat{a}$ is shk-continuous.*

On the other hand, if $\widehat{a}$ is shk-continuous, then $a \in \mathrm{Dec}_{\mathcal{A}}(\mathscr{X})$, for every left Banach $\mathcal{A}$-module $\mathscr{X}$.

Proof. The first part of the theorem is easy to prove. Namely, by [10, Proposition 4.4.4], $\widehat{a}$ is hk-continuous. Thus, for an open subset $\mathcal{U} \subseteq \mathbb{C}$, the set $\widehat{a}^{-1}(\mathcal{U})$ is hk-open. Since $\widehat{a}$ is a constant function on $r^{-1}(r(\varphi))$, for all $\varphi \in \Sigma(\mathcal{A})$, we conclude that $\widehat{a}^{-1}(\mathcal{U})$ is shk-open.

Let $\mathscr{X}$ be an arbitrary left Banach $\mathcal{A}$-module and let T_a be the multiplication operator induced by a on $\mathscr{X}$. We shall prove that T_a is super-decomposable. Let $\{\mathcal{U}_1, \mathcal{U}_2\}$ be an open covering of $\mathbb{C}$. Since $\widehat{a}$ is shk-continuous the sets $U_k = \widehat{a}^{-1}(\mathcal{U}_k)$ $(k = 1, 2)$ form an shk-open covering of $\Sigma(\mathcal{A})$. Using Proposition 5.3 we get $a_1, a_2 \in \mathrm{Sep}(\mathcal{A})$ such that $a_1 + a_2 = 1$ and $\mathrm{sp}_\lambda(a_k) \subseteq U_k$ $(k = 1, 2)$. Let T_k be the multiplication operator induced by a_k on X $(k = 1, 2)$. Then, of course, $T_1 + T_2 = I$, and $T_k T_a = T_a T_k$ $(k = 1, 2)$. By Proposition 5.4, $\mathscr{Y}_k := \mathscr{X}(\mathrm{shk}(U_k))$ $(k = 1, 2)$ are closed $\mathcal{A}$-submodules in X, which means that $T_k \mathscr{X} \subseteq \mathscr{Y}_k$. It remains to prove that $\sigma(T_a|\mathscr{Y}_k) \subseteq \overline{\mathcal{U}}_k$ $(k = 1, 2)$.

Let z be a complex number in $\overline{\mathcal{U}}_k^c$. Then there is an open neighbourhood $\mathcal{W}$ of $\overline{\mathcal{U}}_k$ such that $z \notin \overline{\mathcal{W}}$. The set $\mathcal{F} = \mathcal{W}^c$ is closed and therefore $F = \widehat{a}^{-1}(\mathcal{F})$ is shk-closed. By Proposition 5.4, $\mathcal{A}(F)$ is a closed ideal in $\mathcal{A}$. Of course, $F \cap \widehat{a}^{-1}(\overline{\mathcal{U}}_k) = \emptyset$. Also, $F^c = \widehat{a}^{-1}(\mathcal{W})$ gives $\mathrm{shk}(F^c) \subseteq \widehat{a}^{-1}(\overline{\mathcal{W}})$. The character space of the quotient algebra $\mathcal{A}/\mathcal{A}(F)$ may be identified by $\mathrm{h}(\mathcal{A}(F))$ (see [9, Lemma 2.2.15]). Using Proposition 5.4 (i) we conclude that $\Sigma(\mathcal{A}/\mathcal{A}(F)) = \mathrm{h}(\mathcal{A}(F)) \subseteq \widehat{a}^{-1}(\overline{\mathcal{W}})$.

Consider the element $a - z + \mathcal{A}(F) \in \mathcal{A}/\mathcal{A}(F)$. Since $\psi(a - z + \mathcal{A}(F)) = \widehat{a}(\psi) - z$, for every $\psi \in \Sigma(\mathcal{A}/\mathcal{A}(F)) = \mathrm{h}(\mathcal{A}(F))$, and since z is not in $\overline{\mathcal{W}}$, the spectrum $\sigma(a - z + \mathcal{A}(F))$ does not contain 0. Thus $a - z + \mathcal{A}(F)$ is invertible in $\mathcal{A}/\mathcal{A}(F)$, which means that there exists b in $\mathcal{A}$ such that

$$(b + \mathcal{A}(F))(a - z + \mathcal{A}(F)) = 1 + \mathcal{A}(F).$$

Let $d \in \mathcal{A}(F)$ be such that $b(a - z) = 1 + d$. Denote by T_b the multiplication operator induced by b on $\mathscr{X}$. Then

$$T_b|\mathscr{Y}_k(T_a|\mathscr{Y}_k - z)x = b(a - z) \cdot x = x + d \cdot x,$$

for each $x \in \mathscr{Y}_k$. However $d \cdot x = 0$ because of

$$\mathrm{sp}_\pi(d \cdot x) \subseteq \mathrm{sp}_\lambda(d) \cap \mathrm{sp}_\pi(x) \subseteq F \cap \widehat{a}^{-1}(\overline{\mathcal{U}}_k) = \emptyset.$$

Thus $T_b|\mathscr{Y}_k(T_a|\mathscr{Y}_k - z)x = x$, $x \in \mathscr{Y}_k$, or, equivalently, $z \notin \sigma(T_a|\mathscr{Y}_k)$.

It is obvious that $\widehat{a}$ is a constant function on $r^{-1}(r(\varphi))$ for each $\varphi \in \Sigma(\mathcal{A})$. $\square$

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