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Author(s):

A. Taghavi

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ON A FUNCTIONAL EQUATION FOR SYMMETRIC LINEAR OPERATORS ON C^* ALGEBRAS

A. TAGHAVI

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Abstract. Let A be a C^* algebra, $T : A \to A$ be a linear map which satisfies the functional equation $T(x)T(y) = T^2(xy), \quad T(x^*) = T(x)^*$. We prove that under each of the following conditions, T must be the trivial map $T(x) = \lambda x$ for some $\lambda \in \mathbb{R}$:
i) A is a simple C^*-algebra. ii) A is unital with trivial center and has a faithful trace such that each zero-trace element lies in the closure of the span of commutator elements. iii) $A = B(H)$ where H is a separable Hilbert space.

For a given field F, we consider a similar functional equation $T(x)T(y) = T^2(xy), T(x^{tr}) = T(x)^{tr}$, where T is a linear map on $M_n(F)$ and "tr" is the transpose operator. We prove that this functional equation has trivial solution for all $n \in \mathbb{N}$ if and only if F is a formally real field.
Keywords: Functional Equations, C^* algebras, formally real field.
MSC(2010): Primary: 39B42; Secondary: 46L05.

1. Introduction

Motivated by the classical operator of differentiation, we shall consider the functional equation

$$\begin{cases} T(x)T(y) = T^2(xy) \\ T(x^*) = T(x)^* \end{cases} \tag{1.1}$$

where T is a linear operator on a C^* algebra A. We give some sufficient conditions under which the equation (1.1) has only trivial solution $T(x) = \lambda x$ for some $\lambda \in \mathbb{R}$. On the other hand, in the particular case $A = M_n(\mathbb{C})$, this equation gives us the following functional equation on $M_n(\mathbb{R})$ or more generally

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on $M_n(F)$ where F is an arbitrary field:

$$\begin{cases} T(x)T(y) = T^2(xy) \\ T(x^{tr}) = T(x)^{tr} \end{cases} \tag{1.2}$$

where "tr" is the transpose operator. We observe that for all $n \in \mathbb{N}$, this functional equation has only trivial scalar solution

$$T(x) = \lambda x, \text{ for some } \lambda \in F$$

if and only if F is a formally real field, that is $\sum_{i=1}^{n} f_i^2 = 0,\ f_i \in F,\ n \in \mathbb{N}$ implies that $f_i = 0,\ \forall i = 1, 2, \ldots, n$.
First we explain that how we construct our main functional equation (1.1) from the differentiation operator:

Recall that a complex coalgebra is a complex vector space C with linear maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \mathbb{C}$ such that

$$(Id \otimes \Delta) \circ \Delta = (\Delta \otimes Id) \circ \Delta$$
$$(Id \otimes \varepsilon) \circ \Delta = Id = (\varepsilon \otimes Id) \circ \Delta.$$

Let $C = (\mathbb{C}[x], \Delta, \varepsilon)$ be the coalgebra of complex polynomials with the divided power structure $\Delta(x^n) = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} \otimes x^i$, $\varepsilon(1) = 1$, $\varepsilon(x^n) = 0$ for $n > 0$. The formulation of this structure comes from both the expansion $(x+y)^n = \sum_{k=1}^{n} \binom{n}{k} x^k y^{n-k}$ and formula $B_n(x+y) = \sum_{k=1}^{n} \binom{n}{k} B_k(x) B_{n-k}(y)$ where B_k's are Bell polynomials introduced by E.T. Bell in [1]. Assume that T is the operator of differentiation on $\mathbb{C}[x]$. Then T satisfies

$$(T \otimes T) \circ \Delta = \Delta \circ T^2. \tag{1.3}$$

So we have the following commutative diagram;

$$\begin{array}{ccc} C \otimes C & \xleftarrow{\ \Delta\ } & C \\ {\scriptstyle T\otimes T}\big\uparrow & & \big\uparrow{\scriptstyle T^2} \\ C \otimes C & \xleftarrow{\ \Delta\ } & C \end{array}$$

Now assume that A is a (not necessarily unital) algebra with multiplication $m : A \otimes A \to A$ and T is a linear map on A. By reversing the direction of arrows in the above diagram and replacing the coproduct Δ of C by the product m of A, we find the following commutative diagram:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\ m\ } & A \\ {\scriptstyle T\otimes T}\big\downarrow & & \big\downarrow{\scriptstyle T^2} \\ A \otimes A & \xrightarrow{\ m\ } & A \end{array}$$

This means that T is a linear map on the algebra A which satisfies:

$$T(x)T(y) = T^2(xy). \tag{1.4}$$

In fact, motivated by the classical operator of differentiation, we construct (1.3) as a coalgebraic functional equation on an arbitrary coalgebra. This equation naturally gives us the equation (1.4), as an algebraic functional equation for linear maps on a complex algebra A. If we wish to consider (1.4) on a C^* algebra A, it is natural to add the symmetric condition $T(x^*) = (T(x))^*$.

In this paper we are mainly interested in the functional equation

$$\begin{aligned} T(x)T(y) &= T^2(xy) \\ T(x^*) &= (T(x))^* \end{aligned} \tag{1.5}$$

where T is a (not necessarily continuous) linear map on a C^* algebra A and $T^2 = T \circ T$. An operator which satisfies (1.5) is called a *partial multiplier*. We observe that a partial multiplier is automatically continuous. Despite of its pure algebraic nature, we will see that for certain C^* algebras, this functional equation has a geometric interpretation in terms of inner product preserving maps, see Proposition 2.

Our reason that we choose the name "partial multiplier" for such operators is that an injective partial multiplier on an algebra A can be considered as an element of multiplier algebra of A, see (d) Proposition 1. Another reason for this name is that a partial multiplier T on a unial C^* algebra A is equal to multiplication by $T(1)$, provided that we restrict T to $T(A)$.

Obviously for every $\lambda \in \mathbb{R}$, the trivial linear map $T(x) = \lambda x$ is a partial multiplier. In this paper we are interested in conditions on a C^* algebra A, under which every partial multiplier is necessarily a trivial map.

Theorem 1.1. Every partial multiplier on a C^* algebra A is trivial if A satisfies any one of the following conditions:

(I) A is a simple C^*-algebra.

(II) A is unital, with trivial center, and has a faithful trace such that each zero-trace element lies in the closure of the span of commutator elements.

(III) $A = B(H)$ where H is a separable Hilbert space.

Our next Theorem, is a characterization of all formally real fields F, in terms of the functional equation (1.2) on the matrix algebra $M_n(F)$:

Theorem 1.2. A field F is a formally real field if and only if for each $n \in \mathbb{N}$ the equation (1.2) has only trivial solution $T(x) = \lambda x$ for some $\lambda \in F$.

2. **Preliminaries**

In this section we give some definitions and notations. For a C^* algebra A, a positive linear map on A is a linear map T such that $T(x) \geq 0$ for $x \geq 0$. A faithful (positive) trace on A is a bounded linear map $tr : A \to \mathbb{C}$ such that $tr(xy) = tr(yx)$ and $tr(x) > 0$ for $x > 0$ ($tr(x) \geq 0$ for $x \geq 0$). A zero trace element is an element $x \in A$ with $tr(x) = 0$. Elements of the form $xy - yx$ are called commutators. For a C^* algebra A with a faithful trace tr we define an inner product $< . >_{tr}$ on A with $< a, b >_{tr} = tr(ab^*)$.

The dual and bidual of A is denoted by A^* and A^{**}, respectively. A bounded linear map T on A induces natural linear maps T^* and T^{**} on A^* and A^{**}, respectively. The space A^{**} is a C^* algebra with the Arens product and a natural involution. The Arens product is defined in three stages as follows: (for more information on Arens product see [3])

- For $f \in A^*$ and $x \in A$ define $< f,\ x > \in A^*$ with $< f,\ x > (y) = f(xy)$ for $y \in A$.
- For $F \in A^{**}$ and $g \in A^*$ define $[F,\ g] \in A^*$ with $[F,\ g](x) = F(< g,\ x >)$ for $x \in A$.
- For $F, G \in A^{**}$ the Arens product $F.G \in A^{**}$ is defined with $F.G(f) = F([G,\ f])$ for $f \in A^*$.

The involution on A^{**} is defined as follows: $F^*(\phi) = \overline{F(\phi^*)}$, $\phi^*(a) = \overline{\phi(a^*)}$, where $F \in A^{**}$, $\phi \in A^*$, $a \in A$. Note that the natural involution is available when A is a C^* algebra but not in general when A is just a Banach $*$ algebra. Then A^{**} is a unital C^* algebra which contains A as a C^* subalgebra, via the natural imbedding of A into A^{**}. The multiplier algebra of A, denoted by $\mathcal{M}(A)$, is the idealizor of A in A^{**}, that is the algebra $\{z \in A^{**} \mid zA \subseteq A \ \&\ Az \subseteq A\}$.

A pair $(L,\ R)$ of linear maps on A is called a double centralizer if $R(x)y = xL(y)$ for $x, y \in A$. The space of double centralizers on A is a C^* algebra with natural operations and is isomorphic to the multiplier algebra $\mathcal{M}(A)$. In fact for every double centralizer $(L,\ R)$ on A there is a unique element $a \in \mathcal{M}(A)$ such that $L(x) = ax$, $R(x) = xa$ for $x \in A$. For the algebra of double centralizers of a C^* algebras see [7].

3. **Partial multipliers**

Let A be a C^* algebra. A linear map $T : A \to A$ is called a partial multiplier if T satisfies (1.5). Some algebraic properties of partial multipliers are as follows:

Proposition 3.1. Let T be a partial multiplier on a C^* algebra A. Then

(a) T is a bounded operator, and T^{**} is a partial multiplier on A^{**}.

(b) $\ker(T)$ is a closed two sided ideal in A.

(c) $\prod_{i=1}^{n} T(x_i) = T^n(\prod_{i=1}^{n} x_i)$ where $x_i \in A$ for $i = 1, 2, \ldots, n$.

(d) If T is an injective operator then $(T,\ T)$ is a double centralizer on A.

Proof. To prove (a) assume that T is a partial multiplier on A. Then $T^2(xx^*) = T(x)(T(x))^*$ so T^2 is a positive map. Since T^2 is a positive map on a C^* algebra, it is a bounded operator, see '[4, page 260]'. This implies that T is a bounded operator too because

$$\| T(x) \|^2 = \| T(x)(T(x))^* \| = \| T^2(xx^*) \| \leq \| T^2 \| \| xx^* \| = \| T^2 \| \| x \|^2 .$$

We omit the proof of the last part of (a) since it is a repetition of the proof of Theorem 6.1 in [3]

Now we prove (b). Assume that $T(x) = 0$. Then for each $y \in A$

$$T(xy)(T(xy))^* = T(xy)T(y^*x^*) = T^2(xyy^*x^*) = T(x)T(yy^*x^*) = 0.$$

This shows that $\ker T$ is a right ideal in A. On the other hand $\ker T$ is a $*$-subspace of A, since T is a symmetric operator. This shows that $\ker T$ is a two sided ideal in A.

To prove the remaining parts of the Proposition, without loss of generality, we may assume that A is a unital algebra. Otherwise we consider the extension T^{**} of T on A as a linear operator on A^{**}. So in the remaining part of the proof, in the non-unital case, T is implicitly used for T^{**}. This implicit usage of T for T^{**} is legal since the restriction of T^{**} to A is equal to T.

To prove (c) we first note that $T(1)^k T(x) = T^{k+1}(x)$ for all $k \in \mathbb{N}$. Now we prove (c) by induction on n. Assume that the statement is true for all $k \leq n-1$. Then

$$\prod_{i=1}^{n} T(x_i) = T^{n-1}(\prod_{i=1}^{n-1} x_i)T(x_n) =$$
$$T(1)^{n-2}T(\prod_{i=1}^{n-1} x_i)T(x_n) = T(1)^{n-2}T^2(\prod_{i=1}^{n} x_i) =$$
$$T(1)^{n-2}T(T(\prod_{i=1}^{n} x_i)) = T^{n-1}(T(\prod_{i=1}^{n} x_i) = T^n(\prod_{i=1}^{n} x_i).$$

To prove (d) we note that for all $x, y \in A$

$$T(x)T^2(y) = T^2(x)T(y) \tag{3.1}$$

since each side of the equality is equal to $T(x)T(1)T(y)$. In the non-unital case, this $T(1)$ is replaced by $T^{**}(1)$. Now (3.1) implies that

$$T^2(xT(y)) = T^2(T(x)y). \tag{3.2}$$

Since T is injective we conclude that $xT(y) = T(x)y$. Then (T,T) is a double centralizer on A. □

In the next result we give a geometric interpretation for partial multipliers. First we need the following lemma which is proved in [2, Theorem 1]:

Lemma 3.2. Assume that T is a linear map on a complex inner product space V which preserves orthogonality. Then there is a real number k such that $< T(x), T(y) >= k < x, y >$ for all $x, y \in V$.

In the following result $< x, y >_{tr}= tr(xy^*)$, is the inner product induced from a faithful trace.

Proposition 3.3. Assume that a C^* algebra A has a faithful trace such that every zero trace element lies in the closure of span of commutator elements. Let T be a partial multiplier on A. Then there is a $\lambda \in \mathbb{C}$ such that $< T(x), T(y) >_{tr}= \lambda < x, y >_{tr}$, for all $x, y \in A$

Proof. By the above Lemma, it is sufficient to prove that T preserves the orthogonality with respect to the inner product $< ., . >_{tr}$. Note that for every $a, b \in A$, $tr(T^2(ab - ba)) = tr(T(a)T(b) - T(b)T(a)) = 0$. Hence the functional $tr \circ T^2$ vanishes on the closure of the span of commutator elements. Next suppose that $tr(xy^*) = 0$ for some $x, y \in A$. Then $tr(T^2(xy^*)) = 0$. Hence $tr(T(x)T(y)^*) = 0$. So T preserves the orthogonality with respect to $< ., . >_{tr}$. This completes the proof of the Proposition.

□

Proof of Theorem 1.1. To prove (I), assume that T is a non-zero partial multiplier on a simple C^* algebra A. By proposition 1 (b), the kernel of T is an ideal in A so $\ker T = \{0\}$, that is T is injective. By (d), (T,T) is a double centralizer on A. This means that there is an element $z \in \mathcal{M}(A)$ such that $zx = xz$ for all $x \in A$. By the following argument we conclude that z is a multiple of the identity element of $\mathcal{M}(A)$. (Communicated to us by Professor J. Rosenberg).
Every C^* algebra has an irreducible representation on a Hilbert space H, see [4, Corollary I.9.11]. Since A is simple this representation is injective. So A is an irreducible subalgebra of $B(H)$. From an equivalent definition of the multiplier algebra which is mentioned in [8, Proposition 2.2.11], we have that $\mathcal{M}(A)$ is the idealizer of A in $B(H)$. Moreover irreducibility of A in $B(H)$ implies that the centralizer of A in $B(H)$ reduces to one dimensional scalars $\mathbb{C}.1$,[4, Lemma I.9.1]. This obviously shows that z is a multiple of the identity. Then the partial multiplier T is in the form $T(x) = \lambda x$ for some $\lambda \in \mathbb{C}$. Since T is a

symmetric operator, λ is a real number. This completes the proof of (I).

Assume that T is a partial multiplier on a C^* algebra A which satisfies the hypothesis of (II). Then T is injective by Proposition 2 and $\mathcal{M}(A) = A$ since A is unital. Then, similar to the above situation, (T, T) is a double centralizer for A. Since $\mathcal{M}(A) = A$ we have $T(x) = zx = xz$ for a central element $z \in A$. Since A has trivial center we conclude that the symmetric linear map T is in the form $T(x) = \lambda x$ for some $\lambda \in \mathbb{R}$. This proves (II).

The same argument as above also shows that an injective partial multiplier on $B(H)$ is a trivial map. Then to prove (III) we assume that $\ker T$ is non trivial, then we will obtain a contradiction.
Since $K(H)$, the space of compact operators on H, is the unique closed two sided ideal in $B(H)$, we may assume that $\ker T = K(H)$. Then T induces the quotient operators $\widetilde{T} : B(H)/K(H) \to B(H)$and $\widehat{T} : B(H)/K(H) \to B(H)/K(H)$. The Calkin algebra $B(H)/K(H)$ is a simple algebra and $\widehat{T}$ is a partial multiplier on $B(H)/K(H)$. Then (I) implies that $\widehat{T}(a) \equiv \lambda a$ for some $\lambda \in \mathbb{R}$. If $\lambda = 0$ then $T(x)$ is a compact operator, for all $x \in B(H)$. Thus $T^2 = 0$ since $\ker T = K(H)$. So $T(x)T(x)^* = T^2(xx^*) = 0$, then $T(x) = 0$ for all $x \in B(H)$. Now assume that $\lambda \neq 0$. After a rescaling $T := T/\lambda$ we can assume that $\widehat{T}$ is the identity operator. This means that $T(x) - x \in K(H) = \ker T$ for all $x \in B(H)$. Then $T^2 = T$ so T is a C^* morphism on $B(H)$, since T satisfies (1.5). Let $\pi : B(H) \to B(H)/K(H)$ be the canonical map. We have $\pi \circ \widetilde{T} = Id$, the identity operator on the Calkin algebra. This is a contradiction by each of the following arguments:

- It is well known that the Calkin algebra can not be embedded in $B(H)$, see [5, page 41].
- The equality $\pi \circ \widetilde{T} = Id$ implies that the following short exact sequence is splitting:

$$0 \to K(H) \to B(H) \to B(H)/K(H) \to 0.$$

On the other hand a splitting short exact sequence of C^* algebras, gives us a splitting exact sequence of their K-theory, see [8, Corollary 8.2.2]. In particular for $i = 0, 1$ we would obtain that

$$K_i(B(H)) \approx K_i(K(H)) \bigoplus K_i((B(H)/K(H)).$$

This is a contradiction, see the catalogue of K-groups in [8, pages 123].

This completes the proof of Theorem 1.1. $\square$

To prove Theorem 1.2 we need the following lemma:

Lemma 3.4. Assume that T is a linear operator on $M_n(F)$ which satisfies (1.2) then, for all $x \in \ker(T)$ and for all $y \in M_n(F)$, xy and yx belong to $\ker(T^2)$. Moreover, for every $k \in \mathbb{N}$, T^k satisfies (1.2).

Proof. The last part of the Lemma is obvious. We prove the first part. Assume that $T(x) = 0$. Then for each $y \in M_n(F)$ we have $T^2(xy) = T(x)T(y) = 0$. Moreover $(T^2(yx))^{tr} = (T^2(yx)^{tr}) = T(x^{tr})T(y^{tr}) = 0$, hence $T^2(yx) = 0$. □

Proof of Theorem 1.2. Assume that F is a formally real field and T is a linear map on $M_n(F)$ which satisfies (1.2). We prove that if $T \neq 0$ then T is an injective operator. On the other hand T satisfies (3.2) hence injectivity of T implies $xT(y) = T(x)y, \quad \forall x, y \in M_n(F)$. Then $T(x) = xT(Id) = T(Id)x$. So $T(Id)$, being a central element in $M_n(F)$, is an scalar element. This would complete the proof of one side of the Theorem. Now assume that $T \neq 0$ and is not injective. Since $\ker(T^i)_{i\in\mathbb{N}}$ is an increasing sequence of sub vector spaces of $M_n(F)$, there exists $k \in \mathbb{N}$ such that $\ker(T^k) = \ker(T^{2k})$. So the above Lemma implies that $\ker T^k$ is an ideal in $M_n(F)$. This shows that $\ker(T^k) = M_n(F)$, since $M_n(F)$ is a simple algebra. Then T is a nilpotent operator. Note that (c) of Proposition 2 is applicable here. Then for all $x \in M_n(F)$, $T(x)$ is a nilpotent matrix. So the image of $T : M_n(F) \to M_n(F)$ is a vector subspace of $M_n(F)$ which consists of nilpotent matrices together with zero, moreover it is closed under transpose operator. This implies that the image of T is the zero space, that is T is identically zero, a contradiction to $T \neq 0$. The reason is that for a formally real field F, the zero matrix is the only nilpotent symmetric or anti symmetric matrix. So the above contradiction completes the proof of one side of the Theorem. Now we prove the converse: Assume that F is not a formally real field. Then there are $b_1, b_2, \ldots, b_n$ in F with $\sum_{i=1}^n b_i^2 = 0$. Put $B = (b_i b_j)_{n\times n}$. Then $B = B^{tr} \neq 0$, $B^2 = 0$. Now define $T : M_n(F) \to M_n(F)$ with $T(A) = trace(A)B$. T is a non-scalar operator which satisfies (1.2). □

4. **Remarks**

Remark 4.1. There is a wide class of C^* algebras which satisfy part(II) of Theorem 1. Apart from the matrix algebra $M_n(\mathbb{C})$, for every non Abelian free group F, the reduced C^* algebra $C^*_{red}(F)$, being a unital C^* algebra with trivial center, possesses a unique faithful trace with the property that each zero trace element is a limit of commutator elements. This was communicated to us by Professor A. Valette.

Remark 4.2. The following example shows that the assumption "faithful trace" in Theorem 1, part (II) cannot be weakened to "positive trace":
Let $\mathcal{K}$ be the algebra of compact operators on an infinite dimensional separable Hilbert space. Assume that $A = \mathcal{K} \bigoplus \mathbb{C}$ is the unitization of $\mathcal{K}$. Obviously A is a unital algebra with trivial center. Then define $tr : A \to \mathbb{C}$ with $tr(x,\ \lambda) = \lambda$. This is a positive but not faithful trace on A. Every zero trace element is in

the form $(T,\ 0)$ which is a sum of three commutator elements, because every compact operator on an infinite dimensional separable Hilbert space is a sum of three commutators, see [6]. The operator $T(x,\ \lambda) = (0,\ \lambda)$ is a nontrivial partial multiplier on A. This shows that we cannot replace the assumption "faithful trace" in (II) by the weaker assumption "positive trace".

Remark 4.3. As a consequence of part (II) of the main Theorem we conclude that the existence of a nontrivial idempotent C^* morphism on a unital C^* algebra A with trivial center, is necessary for A to have a faithful trace with the property that each zero trace element is a sum of commutator elements.

Remark 4.4. We observed in Proposition 1 that every partial multiplier on a C^* algebra is automatically continuous. The following example shows that if we remove the symmetric condition $T(x^*) = (T(x))^*$ from the definition of partial multiplier, we may loose the automatic continuity: Let H be an infinite dimensional Hilbert space and $A = B(H \bigoplus H)$. Then each element of A is in the form $\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ where X, Y, Z and W are elements of $B(H)$. Assume that ϕ is an unbounded functional on $B(H)$.
Then $T(\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}) = \begin{pmatrix} 0 & \phi(X)Id \\ 0 & 0 \end{pmatrix}$ is an unbounded operator on A which satisfies $T(x)T(y) = T^2(xy)$.

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(Ali Taghavi) Faculty of Mathematics and Computer Science, Damghan University, Damghan, Iran.
E-mail address: taghavi@du.ac.ir