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SOME COMMUTATIVITY THEOREMS FOR $*$-PRIME RINGS WITH (σ, τ)-DERIVATION

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Abstract. Let R be a $*$-prime ring with center $Z(R)$, d a non-zero (σ,τ)-derivation of R with associated automorphisms σ and τ of R, such that σ, τ and d commute with $'*'$. Suppose that U is an ideal of R such that $U^* = U$, and $C_{\sigma,\tau} = \{c \in R \mid c\sigma(x) = \tau(x)c$ for all $x \in R\}$. In the present paper, it is shown that if characteristic of R is different from two and $[d(U), d(U)]_{\sigma,\tau} = \{0\}$, then R is commutative. Commutativity of R has also been established in case if $[d(R), d(R)]_{\sigma,\tau} \subseteq C_{\sigma,\tau}$.
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MSC(2012): Primary: 16W10; Secondary: 16N60, 16U80.

1. Introduction

Throughout, R will denote an associative ring with center $Z(R)$. An additive mapping $d : R \to R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation. Recall that R is said to be prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$. A ring R is said to be 2-torsion free, if $2x = 0$ implies $x = 0$.

For any two endomorphisms σ and τ of R, we call an additive mapping $d : R \to R$ a (σ,τ)-derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ for all $x, y \in R$. Of course, a $(1, 1)$-derivation is a derivation on R, where 1 is the identity mapping on R. It is also to remark that there exist (σ,τ)-derivations which are not derivations. For example, let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Z} \right\}$ be the ring of all 2×2 matrices over $\mathbb{Z}$, the ring of integers. Define $d, \sigma, \tau : R \to R$ such that $d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $\sigma\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $\tau\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} =$

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$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. It can be easily seen that σ and τ are automorphisms of R, and d is a (σ,τ)-derivation which is not a derivation of R. We set $C_{\sigma,\tau} = \{x \in R \mid x\sigma(y) = \tau(y)x$ for all $y \in R\}$ and $[x,y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$. In particular $C_{1,1} = Z(R)$, is the center of R, and $[x,y]_{1,1} = [x,y] = xy - yx$, is the usual Lie product.

An additive mapping $x \mapsto x^*$ on a ring R is called an involution if $(x^*)^* = x$ and $(xy)^* = y^*x^*$ hold for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or $*$-ring. A ring R equipped with an involution $'*'$ is said to be $*$-prime if $aRb = aRb^* = \{0\}$ (or, equivalently $aRb = a^*Rb = \{0\}$) implies $a = 0$ or $b = 0$. It is important to note that, a prime ring is $*$-prime, but the converse is in general not true. An example due to Shulaing [8] justifies this fact. If R° denotes the opposite ring of a prime ring R, then $S = R \times R^\circ$ equipped with the exchange involution $*_{ex}$ defined by $*_{ex}(x,y) = (y,x)$ is $*_{ex}$-prime, but not a prime ring because of the fact that $(1,0)S(0,1) = 0$. In all that follows, $Sa_*(R)$ will denote the set of symmetric and skew symmetric elements of R, i.e., $Sa_*(R) = \{x \in R | x^* = \pm x\}$. An ideal U of R is said to be a $*$-ideal of R if $U^* = U$. It can also be noted that an ideal of a ring R may not be $*$-ideal of R. As an example, let $R = \mathbb{Z} \times \mathbb{Z}$, and consider the involution $'*'$ on R such that $(a,b)^* = (b,a)$ for all $(a,b) \in R$. The subset $U = \mathbb{Z} \times \{0\}$ of R is an ideal of R but it is not a $*$-ideal of R, because $U^* = \{0\} \times \mathbb{Z} \neq U$.

Recently many authors have studied commutativity of prime and semiprime rings with involution admitting suitably constrained derivations. A lot of work have been done by L. Okhtite and co-authors on rings with involution (see for reference [11, 12, 13], where further references can be found).

In [10], Lee and Lee proved that if a prime ring of characteristic different from 2 admits a derivation d such that $[d(R), d(R)] \subseteq Z(R)$, then R is commutative. On the other hand in [7] for $a \in R$, Herstein proved that if $[a, d(R)] = \{0\}$, then $a \in Z(R)$. Further in the year 1992, Aydin together with Kaya [4] extended the theorems mentioned above by replacing derivation by (σ,τ)-derivation and in some of those, R by a nonzero ideal of R. In this note, we investigate the commutativity of $*$-prime ring R equipped with an involution $'*'$ admitting a (σ,τ)-derivation d satisfying $[d(U), d(U)]_{\sigma,\tau} = \{0\}$ and $[d(R), d(R)]_{\sigma,\tau} \subseteq C_{\sigma,\tau}$, where U is a nonzero $*$-ideal of R.

2. **The results**

In the remaining part of the paper, R will represent a $*$-prime ring which admits a nonzero (σ,τ)-derivation d with automorphisms σ and τ such that $'*'$ commutes with d, σ and τ. We shall use the following relations frequently without specific mention:

$$[xy, z]_{\sigma,\tau} = x[y,z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x,z]_{\sigma,\tau}y,$$

$$[x, yz]_{\sigma,\tau} = \tau(y)[x, z]_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z),$$

and

$$[x, [y, z]]_{\sigma,\tau} + [[x, z]_{\sigma,\tau}, y]_{\sigma,\tau} - [[x, y]_{\sigma,\tau}, z]_{\sigma,\tau} = 0.$$

Remark 2.1. We find that if R is a $*$-prime ring of characteristic different from 2, then R is 2-torsion free. In fact, if $2x = 0$ for all $x \in R$, then $xr(2s) = 0$ for all $r, s \in R$. But since *char* $R \neq 2$, there exists a nonzero $l \in R$ such that $2l \neq 0$ and hence by the above $xR(2l) = \{0\}$. This also gives that $xR(2l)^* = \{0\}$ and $*$-primeness of R yields that $x = 0$, i.e., R is 2-torsion free.

The main result of the present paper states as follows:

Theorem 2.2. *Let R be a $*$- prime ring with characteristic different from two and σ, τ be automorphisms of R, and U a $*$-ideal of R. If R admits a non-zero (σ, τ)-derivation $d : R \to R$ such that $[d(U), d(U)]_{\sigma,\tau} = \{0\}$, then R is commutative.*

We facilitate our discussion with the following lemmas which are required for developing the proof of our main result.

Since every $*$-prime ring is semiprime and every $*$-right ideal is right ideal, hence Lemmas 1.1.4 and 1.1.5 of [5] can be rewritten in case of $*$-prime ring as follows:

Lemma 2.3. *Suppose that R is a $*$-prime ring and that $a \in R$ is such that $a(ax - xa) = 0$ for all $x \in R$. Then $a \in Z(R)$.*

Lemma 2.4. *Let R be a $*$-prime ring and U a non-zero $*$-right ideal of R. Then $Z(U) \subseteq Z(R)$.*

Corollary 2.5. *Let R be a $*$-prime ring and U a non-zero $*$-right ideal of R. If U is commutative then R is commutative.*

Proof. Since U is commutative, by the Lemma 2.4, we have $U = Z(U) \subseteq Z(R)$. If for any $x, y \in R$, $a \in U$ we have $ax \in U$ then $ax \in Z(R)$, and hence $(ax)y = y(ax) = ayx$. This further yields $U(xy - yx) = \{0\}$. Since U is a non-zero $*$-right ideal of R, we have $UR(xy - yx) = \{0\} = U^*R(xy - yx)$. Also, since $U \neq \{0\}$ is a right ideal, $*$-primeness of R gives $xy - yx = 0$, for all $x, y \in R$. Hence R is commutative. □

Lemma 2.6. *Let R be a $*$-prime ring and U a non-zero $*$-right ideal of R. Suppose that $a \in R$ centralizes U. Then $a \in Z(R)$.*

Proof. Since a centralizes U, for all $u \in U$ and $x \in R$, $aux = uxa$. But $au = ua$, therefore $uax = uxa$, i.e., $u[a, x] = 0$. On replacing u by uy for any $y \in R$, we get $uR[a, x] = \{0\}$ for all $u \in U$, $x \in R$. Also, since U is $*$-right ideal, we get $u^*R[a, x] = \{0\}$. Again since $U \neq \{0\}$, $*$-primeness of R yields that $[a, x] = 0$ for all $x \in R$. Therefore, $a \in Z(R)$. □

Lemma 2.7. *Let R be a $*$- prime ring with characteristic different from two and suppose that $a \in R$ commutes with all its commutators $ax - xa$ for all $x \in R$. Then $a \in Z(R)$.*

Proof. Define $d : R \to R$ by $d(x) = ax - xa$ for all $x \in R$. By hypothesis we arrive at

$$d^2(x) = 0 \text{ for all } x \in R. \tag{2.1}$$

Also; $d^2(xy) = d^2(x)y + 2d(x)d(y) + xd^2(y)$. By (2.1) and using torsion restriction on R, we get $d(x)d(y) = 0$ for all $x, y \in R$. On replacing y by yz for any $z \in R$, we obtain $d(x)Rd(y) = \{0\}$, also $d(x)^*Rd(y) = \{0\}$ for all $x, y \in R$. Using $*$-primeness of R yields that $d(x) = 0$ for all $x \in R$. Recalling that $d(x) = ax - xa$, we obtain $a \in Z(R)$. □

Lemma 2.8. *Let R be a $*$-prime ring. Suppose that $ab, a^*b, b \in C_{\sigma,\tau}$ for all $a, b \in R$. Then either $a \in Z(R)$ or $b = 0$.*

Proof. Since $ab \in C_{\sigma,\tau}$, $ab\sigma(x) = \tau(x)ab$ for all $x \in R$. Also since $b \in C_{\sigma,\tau}$, i.e., $b\sigma(x) = \tau(x)b$ for all $x \in R$, we have $a(b\sigma(x)) = \tau(x)ab$, or $a(\tau(x)b) = (\tau(x)a)b$, i.e., $[a, \tau(x)]b = 0$. On replacing x by xy for any $y \in R$, we get

$$[a, \tau(x)]Rb = \{0\} \text{ for all } x \in R.$$

Similarly, since $a^*b \in C_{\sigma,\tau}$ we have

$$[a^*, \tau(x)]Rb = \{0\} \text{ for all } x \in R.$$

On replacing x by x^* in the above relation, we find that

$$[a, \tau(x)]^*Rb = \{0\} \text{ for all } x \in R.$$

Therefore, on using $*$-primeness of R, we find that either $[a, \tau(x)] = 0$ or $b = 0$ for all $x \in R$. Hence, we conclude that $a \in Z(R)$ or $b = 0$. □

Corollary 2.9. *Let R be a $*$-prime ring. Suppose that $ab = 0 = a^*b$, $b \in C_{\sigma,\tau}$ for all $a, b \in R$. Then either $a = 0$ or $b = 0$.*

Proof. Since $b \in C_{\sigma,\tau}$, $b\sigma(x) = \tau(x)b$. Left multiplying by a and a^* and on using $ab = 0$ and $a^*b = 0$, we obtain $ab\sigma(x) = a\tau(x)b = 0$, for all $x \in R$, i.e., $aRb = \{0\}$ and $a^*b\sigma(x) = a^*\tau(x)b = 0$, for all $x \in R$, i.e., $a^*Rb = \{0\}$ respectively. Hence, $*$-primeness of R yields either $a = 0$ or $b = 0$. □

Lemma 2.10. *Let R be a $*$-prime ring and U a $*$-right ideal of R. If $d(U) = \{0\}$, then $d = 0$.*

Proof. For all $u \in U$ and $x \in R$, $0 = d(ux) = d(u)\sigma(x) + \tau(u)d(x) = \tau(u)d(x)$. On replacing x by xy for any $y \in R$, we get $\tau(u)d(x)\sigma(y) + \tau(u)\tau(x)d(y) = 0$, or, $\tau(u)\tau(x)d(y) = 0$, i.e., $\tau(u)Rd(y) = \{0\}$ for all $u \in U$ and $y \in R$. Also since U is a $*$-right ideal, we get $\tau(u)^*Rd(y) = \{0\}$. Also, $*$-primeness of R yields that $\tau(u) = 0$ for all $u \in U$ or $d = 0$. Since $U \neq \{0\}$ we get $d = 0$. □

Lemma 2.11. *Let R be a $*$-prime ring, U a non-zero $*$-ideal of R and $a \in R$. If $ad(U) = \{0\}$ (or, $d(U)a = \{0\}$), then $a = 0$ or $d = 0$.*

Proof. For $u \in U$, $x \in R$, $0 = ad(ux) = ad(u)\sigma(x) + a\tau(u)d(x)$. By assumption, we have $a\tau(u)d(x) = 0$, for all $x \in R$. On replacing u by uy for any $y \in R$, we obtain $a\tau(u)Rd(x) = \{0\}$ for all $u \in U$, $x \in R$. Also, $a\tau(u)Rd(x)^* = \{0\}$. Since R is $*$-prime, we find that either $a\tau(u) = 0$ or $d(x) = 0$. If $a\tau(u) = 0$ for all $u \in U$ or $\tau^{-1}(a)u = 0$, or $\tau^{-1}(a)U = \{0\}$. Now since U is $*$-ideal, we can write $\tau^{-1}(a)U^* = \{0\}$. This implies that $\tau^{-1}(a)RU = \{0\} = \tau^{-1}(a)RU^*$. By the $*$-primeness of R, we obtain $\tau^{-1}(a) = 0$, since $U \neq \{0\}$. In conclusion, we get either $a = 0$ or $d = 0$. Similarly, $d(U)a = \{0\}$ implies $a = 0$ or $d = 0$. $\square$

Lemma 2.12. *Let d be a non-zero (σ, τ)-derivation of $*$-prime ring R and U a $*$-right ideal of R. If $d(U) \subseteq Z(R)$, then R is commutative.*

Proof. Since $d(U) \subseteq Z(R)$, we have $[d(U), R] = \{0\}$. For $u, v \in U$ and $x \in R$,

$$[x, d(uv)] = [x, d(u)\sigma(v) + \tau(u)d(v)] = d(u)[x, \sigma(v)] + d(v)[x, \tau(u)] = 0. \tag{2.2}$$

Replacing x by $x\sigma(v)$, $v \in U$ in (2.2), we have

$$\begin{aligned} 0 &= d(u)[x\sigma(v), \sigma(v)] + d(v)[x\sigma(v), \tau(u)] \\ &= d(u)[x, \sigma(v)]\sigma(v) + d(v)(x[\sigma(v), \tau(u)] + [x, \tau(u)]\sigma(v)). \end{aligned}$$

By using (2.2), we get

$$d(v)R[\sigma(v), \tau(u)] = \{0\}, \text{ for all } u, v \in U. \tag{2.3}$$

Let $v \in U \cap Sa_*(R)$. From (2.3), it follows that

$$d(v)^*R[\sigma(v), \tau(u)] = \{0\}, \text{ for all } u \in U. \tag{2.4}$$

By (2.3) and (2.4), the $*$-primeness of R yields that $d(v) = 0$ or $[\sigma(v), \tau(u)] = 0$ for any $v \in U \cap Sa_*(R)$ and for all $u \in U$. Let $w \in U$, since $w - w^* \in U \cap Sa_*(R)$, then

$$d(w - w^*) = 0 \text{ or } [\sigma(w - w^*), \tau(u)] = 0.$$

Assume that $d(w - w^*) = 0$. Then $d(w) = d(w^*)$. Replacing v by w^* in (2.3) and since U is $*$-right ideal, we get $d(w^*)R[\sigma(w^*), \tau(u)] = \{0\}$ for all $u \in U$. Consequently,

$$d(w)R[\sigma(w), \tau(u)]^* = \{0\}, \text{ for all } u, w \in U. \tag{2.5}$$

Also by (2.3), we get $d(w)R[\sigma(w), \tau(u)] = \{0\}$, the $*$-primeness of R together with (2.5) assures that $d(w) = 0$ or $[\sigma(w), \tau(u)] = 0$, for all $u \in U$. Now suppose that $[\sigma(v), \tau(u)] = 0$, for all $v \in U \cap Sa_*(R)$ and $u \in U$. We have $[\sigma(w - w^*), \tau(u)] = 0$, for all $u \in U$, or $[\sigma(w), \tau(u)] = [\sigma(w^*), \tau(u)]$. Replacing v by w^* in (2.3), we get $d(w^*)R[\sigma(w^*), \tau(u)] = \{0\}$ for all $u \in U$. Consequently,

$$d(w^*)R[\sigma(w),\tau(u)]^* = \{0\}, \text{ for all } u \in U. \tag{2.6}$$

Since $d(w)R[\sigma(w),\tau(u)] = \{0\}$, by (2.3), the $*$-primeness of R together with (2.6) assures that $d(w) = 0$ or $[\sigma(w),\tau(u)] = 0$, for all $u \in U$. In conclusion, for all $u \in U$ we have

$$\text{either } d(w) = 0 \text{ or } [\sigma(w),\tau(u)] = 0.$$

Now, define

$$K = \{w \in U \mid d(w) = 0\} \text{ and } L = \{w \in U \mid [\sigma(w),\tau(u)] = 0 \text{ for all } u \in U\}.$$

Then $U = K \cup L$. Since $d \neq 0$, we have $d(U) \neq \{0\}$ by Lemma 2.10, therefore, $U \neq K$. By Brauer's trick, we have

$$[\sigma(w),\tau(u)] = 0 \text{ for all } u, w \in U. \tag{2.7}$$

Replacing w by $w\sigma^{-1}(\tau(v))$, $u \in U$, in (2.7) and using (2.7), we get $\sigma(w)\tau([v,u]) = 0$, for all $u, v, w \in U$. On replacing w by wx for any $x \in R$, we get $\sigma(w)R\tau([v,u]) = \{0\}$, for all $u, v, w \in U$. Also, since U is $*$-right ideal, we get $\sigma(w)^*R\tau([v,u]) = \{0\}$, for all $u, v, w \in U$. Since R is $*$-prime, we find that $\sigma(w) = 0$ or $\tau[v,u] = 0$ for all $u, v, w \in U$. Since $U \neq \{0\}$, we have U is commutative. In view of Corollary 2.5, we obtain the commutativity of R. □

Using the same technique as in Lemma 4 of [4], we get the following lemma.

Lemma 2.13. *Let R be a $*$- prime ring with characteristic different from two, $d_1 : R \to R$ be a (σ,τ)-derivation and $d_2 : R \to R$ be a derivation. If $d_1d_2(R) = \{0\}$, then $d_1 = 0$ or $d_2 = 0$.*

Proof. Let us assume that $d_1 \neq 0$. Then for all $x, y \in R$,

$$0 = d_1d_2(xy) = d_1(d_2(x)y + xd_2(y)) = \tau(d_2(x))d_1(y) + d_1(x)\sigma(d_2(y)).$$

That is,

$$\tau(d_2(x))d_1(y) = -d_1(x)\sigma(d_2(y)) \text{ for all } x, y \in R. \tag{2.8}$$

If we replace x by $d_2(x)$ in (2.8), we have $\tau(d_2^2(x))d_1(y) = 0$. This further reduces to $\tau(d_2^2(x)) = 0$ for all $x \in R$, in view of Lemma 2.11. Therefore

$$d_2^2(x) = 0 \text{ for all } x \in R. \tag{2.9}$$

Replacing x by $xd_2(z)$, $z \in R$, in (2.8) and using (2.8) and (2.9), we get

$$\begin{aligned}
0 &= \tau(d_2(xd_2(z)))d_1(y) + d_1(xd_2(z))\sigma(d_2(y))\\
&= \tau(d_2(x))\tau(d_2(z))d_1(y) + d_1(x)\sigma(d_2(z))\sigma(d_2(y))\\
&= -\tau(d_2(x))d_1(z)\sigma(d_2(y)) + d_1(x)\sigma(d_2(z))\sigma(d_2(y))\\
&= d_1(x)\sigma(d_2(z))\sigma(d_2(y)) + d_1(x)\sigma(d_2(z))\sigma(d_2(y)).
\end{aligned}$$

So we obtain,

$$2d_1(x)\sigma(d_2(z))\sigma(d_2(y)) = 0 \text{ for all } x, y, z \in R.$$

Since characteristic of R is different from 2. Then by Lemma 2.11, we have

$$d_2(z)d_2(y) = 0 \text{ for all } x, y \in R. \tag{2.10}$$

Again applying Lemma 2.11 to (2.10), we get $d_2 = 0$. □

We are now well equipped to prove our main theorem:

Proof of Theorem 2.2. First we will show that if any $a \in Sa_*(R)$ satisfies $[d(U), a]_{\sigma,\tau} = \{0\}$, then $a \in Z(R)$.

$$\begin{aligned} 0 &= [d(uv), a]_{\sigma,\tau} \\ &= [d(u)\sigma(v) + \tau(u)d(v), a]_{\sigma,\tau} \\ &= d(u)\sigma(v)\sigma(a) + \tau(u)d(v)\sigma(a) - \tau(a)d(u)\sigma(v) - \tau(a)\tau(u)d(a). \end{aligned}$$

By hypothesis, $d(u)\sigma(a) = \tau(a)d(u)$ for all $u \in U$. We have

$$d(u)\sigma([v, a]) + \tau([u, a])d(v) = 0 \text{ for all } u, v \in U. \tag{2.11}$$

Replace v by va in (2.11) and use (2.11) to get

$$\begin{aligned} 0 &= d(u)\sigma([v, a])\sigma(a) + \tau([u, a])(d(v)\sigma(a) + \tau(v)d(a)) \\ &= \{d(u)\sigma([v, a]) + \tau([u, a])d(v)\}\sigma(a) + \tau([u, a])\tau(v)d(a). \end{aligned}$$

We have $\tau([u, a])\tau(v)d(a) = 0$, for all $u, v \in U$. Replacing v by vx for any $x \in R$, we find that $\tau([u, a])\tau(v)Rd(a) = \{0\}$, for all $u, v \in U$. Since $a \in Sa_*(R)$, the above expression can be rewritten as $\tau([u, a])\tau(v)Rd(a)^* = \{0\}$, for all $u, v \in U$. On using $*$-primeness of R, we obtain for all $u, v \in U$

$$\tau([u, a])\tau(v) = 0 \text{ or } d(a) = 0. \tag{2.12}$$

Let us suppose that $d(a) = 0$, then for all $u \in U$, $d([u, a]) = [d(u), a]_{\sigma,\tau} - [d(a), u]_{\sigma,\tau} = 0$. That is

$$d([U, a]) = \{0\}. \tag{2.13}$$

On replacing v by vw, $w \in U$, in (2.11), we get

$$\begin{aligned} 0 &= d(u)\sigma([vw, a]) + \tau([u, a])d(vw) \\ &= d(u)\sigma(v)\sigma([w, a]) + d(u)\sigma([v, a])\sigma(w) + \tau([u, a])d(v)\sigma(w) \\ &\quad + \tau([u, a])\tau(v)d(w) \\ &= d(u)\sigma(v)\sigma([w, a]) + \tau([u, a])\tau(v)d(w) \\ &\quad + \{d(u)\sigma([v, a]) + \tau([u, a])d(v)\}\sigma(w). \end{aligned}$$

By using (2.11), we have

$$d(u)\sigma(v)\sigma([w, a]) + \tau([u, a])\tau(v)d(w) = 0 \text{ for all } u, v, w \in U. \tag{2.14}$$

Replacing w by $[w, a]$ in (2.14) and using (2.13), we get

$$d(u)\sigma(v)\sigma([[w, a], a]) = 0 \text{ for all } u, v, w \in U.$$

Replacing v by xv for any $x \in R$ in the above relation, we find that $d(u)R\sigma(v)\sigma([[w, a], a]) = \{0\}$ for all $u, v, w \in U$. Also since U is $*$-ideal, we may obtain $d(u)^*R\sigma(v)\sigma([[w,a],a]) = \{0\}$ for all $u, v, w \in U$. Using ∗-primeness of R, we get

$$d(U) = \{0\} \text{ or } \sigma(v)\sigma([[w,a],a]) = 0 \text{ for all } u, v, w \in U.$$

But $d(U) \neq \{0\}$, therefore, $\sigma(v)\sigma([[w,a],a]) = 0$ for all $u, v, w \in U$. Replacing v by vx, and using U is ∗-ideal, we obtain

$$\sigma(U)R\sigma([[w,a],a]) = \{0\} \text{ and } \sigma(U)^*R\sigma([[w,a],a]) = \{0\} \text{ for all } w \in U.$$

Since R is ∗-prime and $\sigma(U) \neq \{0\}$ is ∗-ideal of R,

$$\sigma([[U,a],a]) = \{0\}.$$

In other words, if we define $I_a(x) = [x,a]$ an inner derivation determined by a then we have $I_a^2(U) = \{0\}$. By Lemma 2.13, $I_a = \{0\}$, i.e., $[a,U] = \{0\}$, and so by Lemma 2.6, $a \in Z(R)$. In view of (2.12) let us now suppose that $\tau([u,a])\tau(v) = 0$ for all $u, v \in U$. On replacing v by xv for any $x \in R$, the above equation reduces to $\tau([u,a])R\tau(v) = \{0\}$, for all $u, v \in U$. Also, U being a ∗-ideal, we get $\tau([u,a])R\tau(v)^* = \{0\}$. Using the ∗-primeness of R yields either $\tau([U,a]) = \{0\}$ or $\tau(U) = \{0\}$. Since $\tau(U) = \{0\}$ is not possible, it reduces to $\tau([U,a]) = \{0\}$ and so $[U,a] = \{0\}$. In view of Lemma 2.6, we find that $a \in Z(R)$. Hence by our hypothesis we obtain that $d(U) \subseteq Z(R)$. So by Lemma 2.12, R is commutative. □

Theorem 2.14. *Let R be a ∗- prime ring with characteristic different from two and σ, τ be automorphisms of R. If R admits a non-zero (σ,τ)-derivation $d : R \to R$ such that $[d(R), d(R)]_{\sigma,\tau} \subseteq C_{\sigma,\tau}$, then R is commutative.*

Proof. First we will show that for any $a \in Sa_*(R)$ satisfying $[d(R), a]_{\sigma,\tau} \subseteq C_{\sigma,\tau}$, we have $a \in Z(R)$. Suppose on contrary that $a \notin Z(R)$. Using the hypothesis we have $[d(a^2), a]_{\sigma,\tau} \in C_{\sigma,\tau}$

$$\begin{aligned}
[d(a^2), a]_{\sigma,\tau} &= [d(a)\sigma(a) + \tau(a)d(a), a]_{\sigma,\tau} \\
&= d(a)\sigma(a)\sigma(a) - \tau(a)\tau(a)d(a) \\
&= [d(a), a^2]_{\sigma,\tau} = \tau(a)[d(a), a]_{\sigma,\tau} + [d(a), a]_{\sigma,\tau}\sigma(a) \\
&= 2\tau(a)[d(a), a]_{\sigma,\tau}.
\end{aligned}$$

Since *char* $R \neq 2$, we have $\tau(a)[d(a), a]_{\sigma,\tau} \in C_{\sigma,\tau}$. Since $a \in Sa_*(R)$, we also have $\tau(a)^*[d(a), a]_{\sigma,\tau} \in C_{\sigma,\tau}$. In view of the hypothesis and Lemma 2.8, we get either $\tau(a) \in Z(R)$ or $[d(a), a]_{\sigma,\tau} = 0$. Since by our assumption $a \notin Z(R)$, we have

$$[d(a), a]_{\sigma,\tau} = 0. \tag{2.15}$$

On the other hand, since $[d(R), a]_{\sigma,\tau} \subseteq C_{\sigma,\tau}$, for any $x \in R$,
$[d([a,x]), a]_{\sigma,\tau} \in C_{\sigma,\tau}$. Therefore

$$[d([a,x]), a]_{\sigma,\tau} = [[d(a), x]_{\sigma,\tau}, a]_{\sigma,\tau} - [[d(x), a]_{\sigma,\tau}, a]_{\sigma,\tau}.$$

We obtain

$$[[d(a), x]_{\sigma,\tau}, a]_{\sigma,\tau} \in C_{\sigma,\tau} \text{ for all } x \in R. \tag{2.16}$$

Replacing x by ax in (2.16)

$$\begin{aligned}
[[d(a), ax]_{\sigma,\tau}, a]_{\sigma,\tau} &= [\tau(a)[d(a), x]_{\sigma,\tau} + [d(a), a]_{\sigma,\tau}\sigma(x), a]_{\sigma,\tau} \\
&= [\tau(a)[d(a), x]_{\sigma,\tau}, a]_{\sigma,\tau} \\
&= \tau(a)[[d(a), x]_{\sigma,\tau}, a]_{\sigma,\tau} + [\tau(a), \tau(a)][d(a), x]_{\sigma,\tau}.
\end{aligned}$$

We get $\tau(a)[[d(a), x]_{\sigma,\tau}, a]_{\sigma,\tau} \in C_{\sigma,\tau}$ for all $x \in R$. Since $a \in Sa_*(R)$, we have $\tau(a)^*[[d(a), x]_{\sigma,\tau}, a]_{\sigma,\tau} \in C_{\sigma,\tau}$ for all $x \in R$. In view of (2.16), together with above two relations and Lemma 2.8, we obtain $\tau(a) \in Z(R)$ or $[[d(a), x]_{\sigma,\tau}, a]_{\sigma,\tau} = 0$. Since $a \notin Z(R)$, we have

$$[[d(a), x]_{\sigma,\tau}, a]_{\sigma,\tau} = 0 \text{ for all } x \in R. \tag{2.17}$$

Now, applying the relation

$$[x, [y, z]]_{\sigma,\tau} + [[x, z]_{\sigma,\tau}, y]_{\sigma,\tau} - [[x, y]_{\sigma,\tau}, z]_{\sigma,\tau} = 0$$

to (2.17) and using (2.15), we obtain

$$[d(a), [a, x]]_{\sigma,\tau} = 0 \text{ for all } x \in R. \tag{2.18}$$

In other words, if we define $I_a(x) = [a, x]$ an inner derivation determined by a and $I_{d(a)}(x) = [d(a), x]_{\sigma,\tau}$, a (σ, τ)-derivation determined by $d(a)$, in view of (2.18), we find that $I_{d(a)}I_a(x) = 0$, for all $x \in R$. By Lemma 2.13, either $I_{d(a)} = 0$ or $I_a = 0$. That is, $d(a) \in C_{\sigma,\tau}$ or $a \in Z(R)$. Since $a \notin Z(R)$, this gives us

$$d(a) \in C_{\sigma,\tau}.$$

On the other hand, since $[d(R), a]_{\sigma,\tau} \subseteq C_{\sigma,\tau}$. For $x \in R$, $[d(ax), a]_{\sigma,\tau} \in C_{\sigma,\tau}$. Then

$$\begin{aligned}
[d(ax), a]_{\sigma,\tau} &= [d(a)\sigma(x) + \tau(a)d(x), a]_{\sigma,\tau} \\
&= d(a)\sigma(x)\sigma(a) + \tau(a)d(x)\sigma(a) - \tau(a)d(a)\sigma(x) \\
&\quad -\tau(a)\tau(a)d(x).
\end{aligned}$$

Now since we have $d(a) \in C_{\sigma,\tau}$, the above equation reduces to

$$[d(ax), a]_{\sigma,\tau} = d(a)\sigma(ax) + \tau(a)d(x)\sigma(a) - d(a)\sigma(ax) - \tau(a)\tau(a)d(x),$$

or,

$$d(a)\sigma([x, a]) + \tau(a)[d(x), a]_{\sigma,\tau} \in C_{\sigma,\tau} \text{ for all } x \in R. \tag{2.19}$$

Commuting (2.19) with a and using $d(a)$, $[d(x), a]_{\sigma,\tau} \in C_{\sigma,\tau}$, we get

$$\begin{aligned} 0 &= [d(a)\sigma([x,a]) + \tau(a)[d(x),a]_{\sigma,\tau}, a]_{\sigma,\tau} \\ &= d(a)\sigma([x,a])\sigma(a) + \tau(a)[d(x),a]_{\sigma,\tau}\sigma(a) - \tau(a)d(a)\sigma([x,a]) \\ &\quad -\tau(a)\tau(a)[d(x),a]_{\sigma,\tau} \\ &= d(a)\sigma([x,a]a) + \tau(a)[d(x),a]_{\sigma,\tau}\sigma(a) - d(a)\sigma(a[x,a]) \\ &\quad -\tau(a)[d(x),a]_{\sigma,\tau}\sigma(a) \\ &= d(a)\sigma([[x,a],a]). \end{aligned}$$

Also since $a \in Sa_*(R)$, we have $d(a)\sigma([[x,a],a])^* = 0$. Therefore, by Corollary 2.9, $d(a) = 0$ or $[a,[a,x]] = 0$ for all $x \in R$. If $[a,[a,x]] = 0$, for all $x \in R$, we have by Lemma 2.7, $a \in Z(R)$, a contradiction. Therefore, $d(a) = 0$. Now (2.19) can be rewritten as $\tau(a)[d(x),a]_{\sigma,\tau} \in C_{\sigma,\tau}$, for all $x \in R$. Also $\tau(a)^*[d(x),a]_{\sigma,\tau} \in C_{\sigma,\tau}$, for all $x \in R$. But $[d(x),a]_{\sigma,\tau} \in C_{\sigma,\tau}$ yields by Lemma 2.8 either $\tau(a) \in Z(R)$ or $[d(x),a]_{\sigma,\tau} = 0$ for all $x \in R$. Now in application of Theorem 2.2, we obtain $a \in Z(R)$. This contradicts our assumption. Hence, $a \in Z(R)$. By our hypothesis we have $d(R) \subseteq Z(R)$, and hence R is commutative by Lemma 2.12. □

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