SOME COMMUTATIVITY THEOREMS FOR \( * \)-PRIME RINGS WITH \((\sigma, \tau)\)-DERIVATION

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Abstract. Let \( R \) be a \( * \)-prime ring with center \( Z(R) \), \( d \) a non-zero \((\sigma, \tau)\)-derivation of \( R \) with associated automorphisms \( \sigma \) and \( \tau \) of \( R \), such that \( \sigma, \tau \) and \( d \) commute with each other. Suppose that \( U \) is an ideal of \( R \) such that \( U^* = U \) and \( C_{\sigma, \tau} = \{ c \in R \mid \sigma(x) = \tau(x)c \text{ for all } x \in R \} \). In the present paper, it is shown that if characteristic of \( R \) is different from two and \( [d(U), d(U)]_{\sigma, \tau} = \{0\} \), then \( R \) is commutative. Commutativity of \( R \) has also been established in case if \( [d(R), d(R)]_{\sigma, \tau} \subseteq C_{\sigma, \tau} \).

Keywords: Prime-rings, derivations, ideal, involution map.


1. Introduction

Throughout, \( R \) will denote an associative ring with center \( Z(R) \). An additive mapping \( d : R \to R \) is said to be a derivation if \( d(xy) = d(x)y + xd(y) \) holds for all \( x, y \in R \). For a fixed \( a \in R \), the mapping \( I_a : R \to R \) given by \( I_a(x) = [a, x] \) is a derivation which is said to be an inner derivation. Recall that \( R \) is said to be prime if \( aRb = \{0\} \) implies \( a = 0 \) or \( b = 0 \). A ring \( R \) is said to be 2-torsion free, if \( 2x = 0 \) implies \( x = 0 \).

For any two endomorphisms \( \sigma \) and \( \tau \) of \( R \), we call an additive mapping \( d : R \to R \) a \((\sigma, \tau)\)-derivation if \( d(xy) = d(x)\sigma(y) + \tau(x)d(y) \) for all \( x, y \in R \). Of course, a \((1, 1)\)-derivation is a derivation on \( R \), where 1 is the identity mapping on \( R \). It is also to remark that there exist \((\sigma, \tau)\)-derivations which are not derivations. For example, let \( R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\} \) be the ring of all \( 2 \times 2 \) matrices over \( \mathbb{Z} \), the ring of integers. Define \( d, \sigma, \tau : R \to R \) such that \( d \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \left( \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right), \sigma \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \left( \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right) \) and \( \tau \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \left( \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right) \).
An additive mapping $x \mapsto x^*$ on a ring $R$ is called an involution if $(x^*)^* = x$ and $(xy)^* = y^*x^*$ hold for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or *-ring. A ring $R$ equipped with an involution $^*$ is said to be *-prime if $aRb = a^*Rb^* = \{0\}$ (or, equivalently $aRb = a^*Rb = \{0\}$) implies $a = 0$ or $b = 0$. It is important to note that, a prime ring is *-prime, but the converse is in general not true. An example due to Shulaing [8] justifies this fact. If $R^o$ denotes the opposite ring of a prime ring $R$, then $S = R \times R^o$ equipped with the exchange involution $^\text{ex}$ defined by $^\text{ex}(x, y) = (y, x)$ is *-ex- prime, but not a prime ring because of the fact that $(1, 0)S(0, 1) = 0$. In all that follows, $S_+(R)$ will denote the set of symmetric and skew symmetric elements of $R$, i.e., $S_+(R) = \{x \in R \mid x^* = \pm x\}$. An ideal $U$ of $R$ is said to be a *-ideal of $R$ if $U^* = U$. It can also be noted that an ideal of a ring $R$ may not be *-ideal of $R$. As an example, let $R = \mathbb{Z} \times \mathbb{Z}$, and consider the involution $^s$ on $R$ such that $(a, b)^* = (b, a)$ for all $(a, b) \in R$. The subset $U = \mathbb{Z} \times \{0\}$ of $R$ is an ideal of $R$ but it is not a *-ideal of $R$, because $U^* = \{0\} \times \mathbb{Z} \neq U$.

Recently many authors have studied commutativity of prime and semiprime rings with involution admitting suitably constrained derivations. A lot of work have been done by L. Okhtite and co-authors on rings with involution (see for reference [11, 12, 13], where further references can be found). In [10], Lee and Lee proved that if a prime ring of characteristic different from 2 admits a derivation $d$ such that $[d(R), d(R)] \subseteq Z(R)$, then $R$ is commutative. On the other hand in [7] for $a \in R$, Herstein proved that if $[a, d(R)] = \{0\}$, then $a \in Z(R)$. Further in the year 1992, Aydin together with Kaya [4] extended the theorems mentioned above by replacing derivation by $(\sigma, \tau)$-derivation and in some of those, $R$ by a nonzero ideal of $R$. In this note, we investigate the commutativity of *-prime ring $R$ equipped with an involution $^s$ admitting a $(\sigma, \tau)$-derivation $d$ satisfying $[d(U), d(U)]_{\sigma, \tau} = \{0\}$ and $[d(R), d(R)]_{\sigma, \tau} \subseteq C_{\sigma, \tau}$, where $U$ is a nonzero *-ideal of $R$.

2. The results

In the remaining part of the paper, $R$ will represent a *-prime ring which admits a nonzero $(\sigma, \tau)$-derivation $d$ with automorphisms $\sigma$ and $\tau$ such that $^s$ commutes with $d, \sigma$ and $\tau$. We shall use the following relations frequently without specific mention:

$$[xy, z]_{\sigma, \tau} = x[y, z]_{\sigma, \tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma, \tau}y,$$
We find that if 

\[ \alpha^2 \]

Let \( ax \)

Suppose that \( \alpha = 2.4 \)

Since \( ax \)

Let \( x; y \)

If for any \( x; y \)

\[ \tau(y)(x, z) = x \sigma_{\tau}(z), \]

and

\[ [x, y]_{\sigma, \tau} + [[x, z]_{\sigma, \tau}, y]_{\sigma, \tau} - [[x, y]_{\sigma, \tau}, z]_{\sigma, \tau} = 0. \]

**Remark 2.1.** We find that if \( R \) is a \(*\)-prime ring of characteristic different from 2, then \( R \) is 2-torsion free. In fact, if \( 2x = 0 \) for all \( x \in R \), then \( xR(2s) = 0 \) for all \( r, s \in R \). But since \( \text{char } R \neq 2 \), there exists a nonzero \( l \in R \) such that \( 2l \neq 0 \) and hence by the above \( xR(2l) = \{0\} \). This also gives that \( xR(2l)^* = \{0\} \) and \(*\)-primeness of \( R \) yields that \( x = 0 \), i.e., \( R \) is 2-torsion free.

The main result of the present paper states as follows:

**Theorem 2.2.** Let \( R \) be a \(*\)-prime ring with characteristic different from two and \( \sigma, \tau \) be automorphisms of \( R \), and \( U \) a \(*\)-ideal of \( R \). If \( R \) admits a non-zero \((\sigma, \tau)\)-derivation \( d : R \to R \) such that \([d(U), d(U)]_{\sigma, \tau} = \{0\}\), then \( R \) is commutative.

We facilitate our discussion with the following lemmas which are required for developing the proof of our main result.

Since every \(*\)-prime ring is semiprime and every \(*\)-right ideal is right ideal, hence Lemmas 1.1.4 and 1.1.5 of [5] can be rewritten in case of \(*\)-prime ring as follows:

**Lemma 2.3.** Suppose that \( R \) is a \(*\)-prime ring and that \( a \in R \) is such that \( a(ax - xa) = 0 \) for all \( x \in R \). Then \( a \in Z(R) \).

**Lemma 2.4.** Let \( R \) be a \(*\)-prime ring and \( U \) a non-zero \(*\)-right ideal of \( R \). Then \( Z(U) \subseteq Z(R) \).

**Corollary 2.5.** Let \( R \) be a \(*\)-prime ring and \( U \) a non-zero \(*\)-right ideal of \( R \). If \( U \) is commutative then \( R \) is commutative.

**Proof.** Since \( U \) is commutative, by the Lemma 2.4, we have \( U = Z(U) \subseteq Z(R) \).

If for any \( x, y \in R \), \( a \in U \) we have \( ax \in U \) then \( ax \in Z(R) \), and hence \( (ax)y = y(ax) = ayx \). This further yields \( UR(xy - yx) = \{0\} \). Since \( U \) is a non-zero \(*\)-right ideal of \( R \), we have \( UR(xy - yx) = \{0\} = U \sigma R(xy - yx) \).

Also, since \( U \neq \{0\} \) is a right ideal, \(*\)-primeness of \( R \) gives \( xy - yx = 0 \), for all \( x, y \in R \). Hence \( R \) is commutative. \( \square \)

**Lemma 2.6.** Let \( R \) be a \(*\)-prime ring and \( U \) a non-zero \(*\)-right ideal of \( R \). Suppose that \( a \in R \) centralizes \( U \). Then \( a \in Z(R) \).

**Proof.** Since \( a \) centralizes \( U \), for all \( u \in U \) and \( x \in R \), \( aux = uxa \). But \( au = ua \), therefore \( uax = uxa \), i.e., \( u[a, x] = 0 \). On replacing \( u \) by \( uy \) for any \( y \in R \), we get \( uR[a, x] = \{0\} \) for all \( u \in U \), \( x \in R \). Also, since \( U \) is \(*\)-right ideal, we get \( u^*R[a, x] = \{0\} \). Again since \( U \neq \{0\} \), \(*\)-primeness of \( R \) yields that \( [a, x] = 0 \) for all \( x \in R \). Therefore, \( a \in Z(R) \). \( \square \)
Lemma 2.7. Let $R$ be a $*$-prime ring with characteristic different from two and suppose that $a \in R$ commutes with all its commutators $ax - xa$ for all $x \in R$. Then $a \in Z(R)$.

Proof. Define $d : R \to R$ by $d(x) = ax - xa$ for all $x \in R$. By hypothesis we arrive at

\begin{equation}
(2.1) \quad d^2(x) = 0 \quad \text{for all} \quad x \in R.
\end{equation}

Also, $d^2(xy) = d^2(x)y + 2d(x)d(y) + xd^2(y)$. By (2.1) and using torsion restriction on $R$, we get $d(x)d(y) = 0$ for all $x, y \in R$. On replacing $y$ by $y^2$ for any $z \in R$, we obtain $d(x)Rd(y) = \{0\}$, also $d(x)^Rd(y) = \{0\}$ for all $x, y \in R$. Using $*$-primeness of $R$ yields that $d(x) = 0$ for all $x \in R$. Recalling that $d(x) = ax - xa$, we obtain $a \in Z(R)$. □

Lemma 2.8. Let $R$ be a $*$-prime ring. Suppose that $ab, a^*b, b \in C_{\sigma, \tau}$ for all $a, b \in R$. Then either $a \in Z(R)$ or $b \in Z(R)$.

Proof. Since $ab \in C_{\sigma, \tau}$, $\sigma(x) = \tau(x)ab$ for all $x \in R$. Also since $b \in C_{\sigma, \tau}$, i.e., $b\sigma(x) = \tau(x)b$ for all $x \in R$, we have $a(b\sigma(x)) = \tau(x)ab$, or $a(\tau(x)b) = (\tau(x)a)b$, i.e., $[a, \tau(x)]b = 0$. On replacing $x$ by $xy$ for any $y \in R$, we get

\[ [a, \tau(x)]Rb = \{0\} \quad \text{for all} \quad x \in R. \]

Similarly, since $a^*b \in C_{\sigma, \tau}$ we have

\[ [a^*, \tau(x)]Rb = \{0\} \quad \text{for all} \quad x \in R. \]

On replacing $x$ by $x^*$ in the above relation, we find that

\[ [a, \tau(x)]^*Rb = \{0\} \quad \text{for all} \quad x \in R. \]

Therefore, on using $*$-primeness of $R$, we find that either $[a, \tau(x)] = 0$ or $b = 0$ for all $x \in R$. Hence, we conclude that $a \in Z(R)$ or $b = 0$. □

Corollary 2.9. Let $R$ be a $*$-prime ring. Suppose that $ab = 0 = a^*b$, $b \in C_{\sigma, \tau}$ for all $a, b \in R$. Then either $a = 0$ or $b = 0$.

Proof. Since $b \in C_{\sigma, \tau}$, $\sigma(x) = \tau(x)b$. Left multiplying by $a$ and $a^*$ and on using $ab = 0$ and $a^*b = 0$, we obtain $a\sigma(x) = a\tau(x)b = 0$, for all $x \in R$, i.e., $aRb = \{0\}$ and $a^*b\sigma(x) = a^*\tau(x)b = 0$, for all $x \in R$, i.e., $a^*Rb = \{0\}$ respectively. Hence, $*$-primeness of $R$ yields either $a = 0$ or $b = 0$. □

Lemma 2.10. Let $R$ be a $*$-prime ring and $U$ a $*$-right ideal of $R$. If $d(U) = \{0\}$, then $d = 0$.

Proof. For all $u \in U$ and $x \in R$, $0 = d(ux) = d(u)\sigma(x) + \tau(u)d(x) = \tau(u)d(x)$. On replacing $x$ by $xy$ for any $y \in R$, we get $\tau(u)d(x)\sigma(y) + \tau(u)\tau(x)d(y) = 0$, or, $\tau(u)\tau(x)d(y) = 0$, i.e., $\tau(u)Rd(y) = \{0\}$ for all $u \in U$ and $y \in R$. Also since $U$ is a $*$-right ideal, we get $\tau(u)^*Rd(y) = \{0\}$. Also, $*$-primeness of $R$ yields that $\tau(u) = 0$ for all $u \in U$ or $d = 0$. Since $U \neq \{0\}$ we get $d = 0$. □
Lemma 2.11. Let $R$ be a $*$-prime ring, $U$ a non-zero $*$-ideal of $R$ and $a \in R$. If $ad(U) = \{0\}$ (or, $d(U)a = \{0\}$), then $a = 0$ or $d = 0$.

Proof. For $u \in U$, $x \in R$, $0 = ad(ux) = ad(u)\sigma(x) + \alpha(u)d(x)$. By assumption, we have $\sigma(u)d(x) = 0$, for all $x \in R$. On replacing $u$ by $uy$ for any $y \in R$, we obtain $\sigma(u)Rd(x) = \{0\}$ for all $u \in U$, $x \in R$. Also, $\sigma(u)Rd(x)^* = \{0\}$. Since $R$ is $*$-prime, we find that either $\sigma(u) = 0$ or $d(x) = 0$. If $\sigma(u) = 0$ for all $u \in U$ or $\tau^{-1}(a)u = 0$, or $\tau^{-1}(a)U = \{0\}$. Now since $U$ is $*$-ideal, we can write $\tau^{-1}(a)U^* = \{0\}$. This implies that $\tau^{-1}(a)RU = \{0\} = \tau^{-1}(a)RU^*$. By the $*$-primeness of $R$, we obtain $\tau^{-1}(a) = 0$, since $U \neq \{0\}$. In conclusion, we get either $a = 0$ or $d = 0$. □

Lemma 2.12. Let $d$ be a non-zero $(\sigma, \tau)$-derivation of $*$-prime ring $R$ and $U$ a $*$-right ideal of $R$. If $d(U) \subseteq Z(R)$, then $R$ is commutative.

Proof. Since $d(U) \subseteq Z(R)$, we have $[d(U), R] = \{0\}$. For $u, v \in U$ and $x \in R$,

\[(2.2) \quad [x, d(uv)] = [x, d(u)]\sigma(v) + [x, \sigma(v)]d(u)] = d(u)[x, \sigma(v)] + d(v)[x, \tau(u)] = 0.\]

Replacing $x$ by $x\sigma(v)$, $v \in U$ in (2.2), we have

\[
0 = d(u)[x\sigma(v), \sigma(v)] + d(v)[x\sigma(v), \tau(u)] - d(u)[x, \sigma(v)] + d(v)[x\sigma(v), \tau(u)] + [x, \tau(u)]\sigma(v)).
\]

By using (2.2), we get

\[(2.3) \quad d(v)R[\sigma(v), \tau(u)] = \{0\}, \text{ for all } u, v \in U.\]

Let $v \in U \cap Sa_*(R)$. From (2.3), it follows that

\[(2.4) \quad d(v)^*R[\sigma(v), \tau(u)] = \{0\}, \text{ for all } u \in U.\]

By (2.3) and (2.4), the $*$-primeness of $R$ yields that $d(v) = 0$ or $[\sigma(v), \tau(u)] = 0$ for any $v \in U \cap Sa_*(R)$ and for all $u \in U$. Let $w \in U$, since $w - w^* \in U \cap Sa_*(R)$, then

\[
d(w - w^*) = 0 \text{ or } [\sigma(w - w^*), \tau(u)] = 0.
\]

Assume that $d(w - w^*) = 0$. Then $d(w) = d(w^*)$. Replacing $v$ by $w^*$ in (2.3) and since $U$ is $*$-right ideal, we get $d(w^*)R[\sigma(w^*), \tau(u)] = \{0\}$ for all $u \in U$. Consequently,

\[(2.5) \quad d(w)R[\sigma(w), \tau(u)] = \{0\}, \text{ for all } u, w \in U.\]

Also by (2.3), we get $d(w)R[\sigma(w), \tau(u)] = \{0\}$, the $*$-primeness of $R$ together with (2.5) assures that $d(w) = 0$ or $[\sigma(w), \tau(u)] = 0$, for all $u \in U$. Now suppose that $[\sigma(v), \tau(u)] = 0$, for all $v \in U \cap Sa_*(R)$ and $u \in U$. We have $[\sigma(w - w^*), \tau(u)] = 0$, for all $u \in U$, or $[\sigma(w), \tau(u)] = [\sigma(w^*), \tau(u)]$. Replacing $v$ by $w^*$ in (2.3), we get $d(w^*)R[\sigma(w^*), \tau(u)] = \{0\}$ for all $u \in U$. Consequently,
(2.6) \[ d(w^*)R[\sigma(w), \tau(u)]^* = \{0\}, \text{ for all } u \in U. \]

Since \( d(w)R[\sigma(w), \tau(u)] = \{0\} \), by (2.3), the \(*\)-primeness of \( R \) together with (2.6) assures that \( d(w) = 0 \) or \([\sigma(w), \tau(u)] = 0\), for all \( u \in U \). In conclusion, for all \( u \in U \) we have

\[ \text{either } d(w) = 0 \text{ or } [\sigma(w), \tau(u)] = 0. \]

Now, define

\[ K = \{ w \in U \mid d(w) = 0 \} \text{ and } L = \{ w \in U \mid [\sigma(w), \tau(u)] = 0 \text{ for all } u \in U \}. \]

Then \( U = K \cup L \). Since \( d \neq 0 \), we have \( d(U) \neq \{0\} \) by Lemma 2.10, therefore, \( U \neq K \). By Brauer’s trick, we have

\[ (2.7) \quad [\sigma(w), \tau(u)] = 0 \text{ for all } u, w \in U. \]

Replacing \( w \) by \( w\sigma^{-1}(\tau(v)) \), \( u \in U \), in (2.7) and using (2.7), we get

\[ \sigma(w)\tau([v, u]) = 0, \text{ for all } u, v, w \in U. \]

Replacing \( w \) by \( wx \) for any \( x \in R \), we get \( \sigma(w)R\tau([v, u]) = \{0\} \), for all \( u, v, w \in U \). Also, since \( U \) is \(*\)-right ideal, we get \( \sigma(w)^*R\tau([v, u]) = \{0\} \), for all \( u, v, w \in U \). Since \( R \) is \(*\)-prime, we find that \( \sigma(w) = 0 \) or \( \tau[v, u] = 0 \) for all \( u, v, w \in U \). Since \( U \neq \{0\} \), we have \( U \) is commutative. In view of Corollary 2.5, we obtain the commutativity of \( R \). \[ \square \]

Using the same technique as in Lemma 4 of [4], we get the following lemma.

**Lemma 2.13.** Let \( R \) be a \(*\)-prime ring with characteristic different from two, \( d_1 : R \to R \) be a \((\sigma, \tau)\)-derivation and \( d_2 : R \to R \) be a derivation. If \( d_1d_2(R) = \{0\} \), then \( d_1 = 0 \) or \( d_2 = 0 \).

**Proof.** Let us assume that \( d_1 \neq 0 \). Then for all \( x, y \in R \),

\[ 0 = d_1d_2(xy) = d_1(d_2(x)y + xd_2(y)) = \tau(d_2(x))d_1(y) + d_1(x)\sigma(d_2(y)). \]

That is,

\[ (2.8) \quad \tau(d_2(x))d_1(y) = -d_1(x)\sigma(d_2(y)) \text{ for all } x, y \in R. \]

If we replace \( x \) by \( d_2(x) \) in (2.8), we have \( \tau(d_2^2(x))d_1(y) = 0 \). This further reduces to \( \tau(d_2^2(x)) = 0 \) for all \( x \in R \), in view of Lemma 2.11. Therefore

\[ (2.9) \quad d_2^2(x) = 0 \text{ for all } x \in R. \]

Replacing \( x \) by \( xd_2(z), z \in R \), in (2.8) and using (2.8) and (2.9), we get

\[ 0 = \tau(d_2(xd_2(z)))d_1(y) + d_1(xd_2(z))\sigma(d_2(y)) \]
\[ = \tau(d_2(x))\tau(d_2(z))d_1(y) + d_1(x)\sigma(d_2(z))\sigma(d_2(y)) \]
\[ = -\tau(d_2(x))d_1(z)\sigma(d_2(y)) + d_1(x)\sigma(d_2(z))\sigma(d_2(y)) \]
\[ = d_1(x)\sigma(d_2(z))\sigma(d_2(y)) + d_1(x)\sigma(d_2(z))\sigma(d_2(y)). \]

So we obtain,

\[ 2d_1(x)\sigma(d_2(z))\sigma(d_2(y)) = 0 \text{ for all } x, y, z \in R. \]
Since characteristic of $R$ is different from 2. Then by Lemma 2.11, we have
\[ d_2(z)d_2(y) = 0 \text{ for all } x, y \in R. \]
Again applying Lemma 2.11 to (2.10), we get $d_2 = 0$. \hfill \Box

We are now well equipped to prove our main theorem:

Proof of Theorem 2.2. First we will show that if any $a \in S_{a_\ast}(R)$ satisfies $[d(U), a]_{\sigma, \tau} = \{0\}$, then $a \in Z(R)$.

\[ 0 = [d(uv), a]_{\sigma, \tau} = \cdots = d(u)\sigma(v)\sigma(a) + \tau(u)d(v)\sigma(a) - \tau(a)d(u)\sigma(v) - \tau(a)\tau(u)d(a). \]

By hypothesis, $d(u)\sigma(a) = \tau(u)d(u)$ for all $u \in U$. We have
\[ d(u)\sigma([v, a]) + \tau([u, a])d(v) = 0 \text{ for all } u, v \in U. \]

Replace $v$ by $va$ in (2.11) and use (2.11) to get
\[ \cdots = d(u)\sigma([v, a])\sigma(a) + \tau([u, a])d(v)\sigma(a) + \tau(v)d(a). \]

We have $\tau([u, a])\tau(v)d(a) = 0$, for all $u, v \in U$. Replacing $v$ by $vx$ for any $x \in R$, we find that $\tau([u, a])\tau(v)Rd(a) = \{0\}$, for all $u, v \in U$. Since $a \in S_{a_\ast}(R)$, the above expression can be rewritten as $\tau([u, a])\tau(v)Rd(a)^* = \{0\}$, for all $u, v \in U$. On using *-primeness of $R$, we obtain for all $u, v \in U$
\[ \tau([u, a])\tau(v) = 0 \text{ or } d(a) = 0. \]

Let us suppose that $d(a) = 0$, then for all $u \in U$, $d([u, a]) = [d(u), a]_{\sigma, \tau} = [d(a), u]_{\sigma, \tau} = 0$. That is
\[ \cdots = d([u, a]) = \{0\}. \]

On replacing $v$ by $vw$, $w \in U$, in (2.11), we get
\[ \cdots = d(u)\sigma([vw, a]) + \tau([u, a])d(vw) \]
\[ \cdots = d(u)\sigma(v)\sigma([w, a]) + d(u)\sigma([v, a])\sigma(w) + \tau([u, a])d(v)\sigma(w) \]
\[ \cdots = d(u)\sigma(v)\sigma([w, a]) + \tau([u, a])\tau(v)d(w) \]
\[ \cdots = d(u)\sigma([v, a]) + \tau([u, a])\tau(v)d(w) \]
\[ \cdots = d(u)\sigma([v, a]) + \tau([u, a])\tau(v)d(w) \]
\[ \cdots = d(u)\sigma([v, a]) + \tau([u, a])\tau(v)d(w). \]

By using (2.11), we have
\[ d(u)\sigma(v)\sigma([w, a]) + \tau([u, a])\tau(v)d(w) = 0 \text{ for all } u, v, w \in U. \]

Replacing $w$ by $[w, a]$ in (2.14) and using (2.13), we get
\[ d(u)\sigma(v)\sigma([w, a]) = 0 \text{ for all } u, v, w \in U. \]

Replacing $v$ by $xw$ for any $x \in R$ in the above relation, we find that $d(u)R\sigma(v)\sigma([w, a]) = \{0\}$ for all $u, v, w \in U$. Also since $U$ is *-ideal, we may
obtain $d(u)^* R \sigma(v) \sigma([w, a], a) = \{0\}$ for all $u, v, w \in U$. Using $*$-primeness of $R$, we get

$$d(U) = \{0\} \text{ or } \sigma(v) \sigma([w, a], a) = 0 \text{ for all } u, v, w \in U.$$  

But $d(U) \neq \{0\}$, therefore, $\sigma(v) \sigma([w, a], a) = 0$ for all $u, v, w \in U$. Replacing $v$ by $vx$, and using $U$ is $*$-ideal, we obtain

$$\sigma(U) R \sigma([w, a], a) = \{0\} \text{ and } \sigma(U)^* R \sigma([w, a], a) = \{0\} \text{ for all } w \in U.$$  

Since $R$ is $*$-prime and $\sigma(U) \neq \{0\}$ is $*$-ideal of $R$,

$$\sigma([U, a], a) = \{0\}.$$  

In other words, if we define $I_a(x) = [x, a]$ an inner derivation determined by $a$ then we have $I_a^2(U) = \{0\}$. By Lemma 2.13, $I_a = \{0\}$, i.e., $[a, U] = \{0\}$, and so by Lemma 2.6, $a \in Z(R)$. In view of (2.12) let us now suppose that $\tau([u, a]) \tau(v) = 0$ for all $u, v \in U$. On replacing $v$ by $uv$ for any $x \in R$, the above equation reduces to $\tau([u, a]) R \tau(v) = \{0\}$, for all $u, v \in U$. Also, $U$ being a $*$-ideal, we get $\tau([u, a]) R \tau(v)^* = \{0\}$. Using the $*$-primeness of $R$ yields either $\tau([U, a]) = \{0\}$ or $\tau(U) = \{0\}$. Since $\tau(U) = \{0\}$ is not possible, it reduces to $\tau([U, a]) = \{0\}$ and so $[U, a] = \{0\}$. In view of Lemma 2.6, we find that $a \in Z(R)$. Hence by our hypothesis we obtain that $d(U) \subseteq Z(R)$. So by Lemma 2.12, $R$ is commutative. \hfill $\Box$

**Theorem 2.14.** Let $R$ be a $*$-prime ring with characteristic different from two and $\sigma, \tau$ be automorphisms of $R$. If $R$ admits a non-zero $(\sigma, \tau)$-derivation $d : R \rightarrow R$ such that $[d(R), d(R)]_{\sigma, \tau} \subseteq C_{\sigma, \tau}$, then $R$ is commutative.

**Proof.** First we will show that for any $a \in S_{\sigma}(R)$ satisfying $[d(R), a]_{\sigma, \tau} \subseteq C_{\sigma, \tau}$, we have $a \in Z(R)$. Suppose on contrary that $a \notin Z(R)$. Using the hypothesis we have $[d(a^2), a]_{\sigma, \tau} \in C_{\sigma, \tau}$

$$[d(a^2), a]_{\sigma, \tau} = [d(a) \sigma(a) + \tau(a) d(a), a]_{\sigma, \tau} = d(a) \sigma(a) \sigma(a) - \tau(a) \tau(a) d(a) = [d(a), a^2]_{\sigma, \tau} = \tau(a) [d(a), a]_{\sigma, \tau} + [d(a), a]_{\sigma, \tau} \sigma(a) = 2 \tau(a) [d(a), a]_{\sigma, \tau}.$$  

Since $\text{char } R \neq 2$, we have $\tau(a) [d(a), a]_{\sigma, \tau} \in C_{\sigma, \tau}$. Since $a \in S_{\sigma}(R)$, we also have $\tau(a)^* [d(a), a]_{\sigma, \tau} \in C_{\sigma, \tau}$. In view of the hypothesis and Lemma 2.8, we get either $\tau(a) \in Z(R)$ or $[d(a), a]_{\sigma, \tau} = 0$. Since by our assumption $a \notin Z(R)$, we have

$$[d(a), a]_{\sigma, \tau} = 0.$$  

On the other hand, since $[d(R), a]_{\sigma, \tau} \subseteq C_{\sigma, \tau}$, for any $x \in R$, $[d([a, x]), a]_{\sigma, \tau} \in C_{\sigma, \tau}$. Therefore

$$[d([a, x]), a]_{\sigma, \tau} = [[d(a), x]_{\sigma, \tau}, a]_{\sigma, \tau} - [[d(x), a]_{\sigma, \tau}, a]_{\sigma, \tau}.$$
We obtain
(2.16)  \[ [d(a), x]_{\sigma, \tau}, a]_{\sigma, \tau} \in C_{\sigma, \tau} \text{ for all } x \in R. \]

Replacing \( x \) by \( ax \) in (2.16)
\[
[[d(a), ax]_{\sigma, \tau}, a]_{\sigma, \tau} = [\tau(a)[d(a), x]_{\sigma, \tau} + [d(a), a]_{\sigma, \tau}\sigma(x), a]_{\sigma, \tau}
= [\tau(a)[d(a), x]_{\sigma, \tau}, a]_{\sigma, \tau}
= \tau(a)[[d(a), x]_{\sigma, \tau}, a]_{\sigma, \tau} + [\tau(a), \tau(a)][d(a), x]_{\sigma, \tau}.
\]

We get \( \tau(a)[[d(a), x]_{\sigma, \tau}, a]_{\sigma, \tau} \in C_{\sigma, \tau} \) for all \( x \in R \). Since \( a \in Sa_{\sigma}(R) \), we have \( \tau(a)^*[[[d(a), x]_{\sigma, \tau}, a]_{\sigma, \tau} \in C_{\sigma, \tau} \) for all \( x \in R \). In view of (2.16), together with above two relations and Lemma 2.8, we obtain \( \tau(a) \in Z(R) \) or \( [[d(a), x]_{\sigma, \tau}, a]_{\sigma, \tau} = 0 \). Since \( a \notin Z(R) \), we have
(2.17)  \[ [[d(a), x]_{\sigma, \tau}, a]_{\sigma, \tau} = 0 \text{ for all } x \in R. \]

Now, applying the relation
\[ [x, [y, z]]_{\sigma, \tau} + [[x, z]_{\sigma, \tau}, y]_{\sigma, \tau} - [[x, y]_{\sigma, \tau}, z]_{\sigma, \tau} = 0 \]
to (2.17) and using (2.15), we obtain
(2.18)  \[ [d(a), [a, x]]_{\sigma, \tau} = 0 \text{ for all } x \in R. \]

In other words, if we define \( I_a(x) = [a, x] \) an inner derivation determined by \( a \) and \( I_{d(a)}(x) = [d(a), x]_{\sigma, \tau}, \) a \( (\sigma, \tau) \)-derivation determined by \( d(a) \), in view of (2.18), we find that \( I_{d(a)}I_a(x) = 0 \), for all \( x \in R \). By Lemma 2.13, either \( I_{d(a)} = 0 \) or \( I_a = 0 \). That is, \( d(a) \in C_{\sigma, \tau} \) or \( a \in Z(R) \). Since \( a \notin Z(R) \), this gives us
\[ d(a) \in C_{\sigma, \tau}. \]

On the other hand, since \( [d(R), a]_{\sigma, \tau} \subseteq C_{\sigma, \tau} \). For \( x \in R \), \([d(ax), a]_{\sigma, \tau} \subseteq C_{\sigma, \tau} \). Then
\[
[d(ax), a]_{\sigma, \tau} = [d(a)\sigma(x) + \tau(a)d(x), a]_{\sigma, \tau}
= d(a)\sigma(x)\sigma(a) + \tau(a)d(x)\sigma(a) - \tau(a)d(a)\sigma(x)
- \tau(a)\tau(a)d(x).
\]

Now since we have \( d(a) \in C_{\sigma, \tau} \), the above equation reduces to
\[ [d(ax), a]_{\sigma, \tau} = d(a)\sigma(ax) + \tau(a)d(x)\sigma(a) - d(a)\sigma(ax) - \tau(a)\tau(a)d(x), \]
or,
(2.19)  \[ d(a)\sigma([x, a]) + \tau(a)[d(x), a]_{\sigma, \tau} \in C_{\sigma, \tau} \text{ for all } x \in R. \]
Commuting (2.19) with \(a\) and using \(d(a), [d(x), a]_{\sigma,\tau} \in C_{\sigma,\tau}\), we get
\[
0 = [d(a)\sigma([x, a]) + \tau(a)[d(x), a]_{\sigma,\tau}, a]_{\sigma,\tau}
= d(a)\sigma([x, a])\sigma(a) + \tau(a)[d(x), a]_{\sigma,\tau}\sigma(a) - \tau(a)d(a)\sigma([x, a])
- \tau(a)d(x), a]_{\sigma,\tau}\sigma(a)
= d(a)\sigma([x, a])\sigma(a).
\]
Also since \(a \in S_{\sigma,\tau}(R)\), we have \(d(a)\sigma([x, a], a])^* = 0\). Therefore, by Corollary 2.9, \(d(a) = 0\) or \(a, [a, x] = 0\) for all \(x \in R\). If \([a, [a, x]] = 0\), for all \(x \in R\), we have by Lemma 2.7, \(a \in Z(R)\), a contradiction. Therefore, \(d(a) = 0\). Now (2.19) can be rewritten as \(\tau(a)[d(x), a]_{\sigma,\tau} \in C_{\sigma,\tau}\), for all \(x \in R\). Also \(\tau(a)^*[d(x), a]_{\sigma,\tau} \in C_{\sigma,\tau}\) yields by Lemma 2.8 either \(\tau(a) \in Z(R)\) or \([d(x), a]_{\sigma,\tau} = 0\) for all \(x \in R\). Now in application of Theorem 2.2, we obtain \(a \in Z(R)\). This contradicts our assumption. Hence, \(a \in Z(R)\). By our hypothesis we have \(d(R) \subseteq Z(R)\), and hence \(R\) is commutative by Lemma 2.12. \(\square\)

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