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$\mathcal{X}$-INJECTIVE AND $\mathcal{X}$-PROJECTIVE COMPLEXES

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Abstract. Let $\mathcal{X}$ be a class of R-modules. In this paper, we investigate $\mathcal{X}$-injective (projective) and DG-$\mathcal{X}$-injective (projective) complexes which are generalizations of injective (projective) and DG-injective (projective) complexes. We prove that some known results can be extended to the class of $\mathcal{X}$-injective (projective) and DG-$\mathcal{X}$-injective (projective) complexes for this general settings.
Keywords: Injective (Projective) complex, precover, preenvelope.
MSC(2010): Primary: 18G35.

1. Introduction

Throughout this paper, R denotes an associative ring with identity and all modules are unitary. Let $\mathcal{X}$ be a class of R-modules. An R-module E is called $\mathcal{X}$-injective (see [6]), if $Ext^1(B/A, E) = 0$ for every module $B/A \in \mathcal{X}$ or equivalently if E is injective with respect to every exact sequence $0 \to A \to B \to B/A \to 0$ where $B/A \in \mathcal{X}$. Dually we can define an $\mathcal{X}$-projective module. In Section 2, we define and characterize $\mathcal{X}$-injective, $\mathcal{X}$-projective, DG-$\mathcal{X}$-injective and DG-$\mathcal{X}$-projective complexes which are generalizations of injective, projective, DG-injective and DG-projective complexes, respectively (see [1] and [2]). By [2] we know that (ε,DG-injective) is a cotorsion pair. We denote the class of all $\mathcal{X}$-complexes, that is, exact complexes with kernel in $\mathcal{X}$, by $\varepsilon_{\mathcal{X}}$, (in [5] the same class is denoted by $\widetilde{\mathcal{X}}$). We prove that if $\mathcal{X}$ is extension closed, then ${\varepsilon_{\mathcal{X}}}^{\perp}({}^{\perp}\varepsilon_{\mathcal{X}})$ = DG-$\mathcal{X}$-injective (projective) which is proved in [5] when $(\mathcal{X}, \mathcal{X}^{\perp})$ is a cotorsion pair.

In the last section, we investigate when a complex has an exact C($\mathcal{X}$-projective (injective))-precover (preenvelope). We know that an injective (projective) complex is exact, thus we give some conditions that an $\mathcal{X}$-injective (projective) complex is exact and in particular in $\varepsilon_{\mathcal{X}-injective(projective)}$. We prove that if $\mathcal{X}$-injective (projective)$\subseteq \mathcal{X}$ and ($\mathcal{X}$, $\mathcal{X}$-injective) (($\mathcal{X}$-projective,$\mathcal{X}$)) is a

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complete cotorsion pair, then every complex has a monic (epic) $C(\mathcal{X}$-injective (projective))-preenvelope (precover) in $\varepsilon_{\mathcal{X}-injective}$ ($\varepsilon_{\mathcal{X}-projective}$) and hence $C(\mathcal{X}$-injective (projective)) and $\varepsilon_{\mathcal{X}-injective(projective)}$ complexes are identical.

Since every complex has an injective and projective resolution, we can compute the right derived functors $Ext^i(X,Y)$ of $Hom(-,-)$ where $Hom(X,Y)$ is the set of all chain maps from X to Y.
Moreover $\mathcal{H}om(X,Y)$ is the complex defined by $\mathcal{H}om(X,Y)_n = \prod_{p+q=n}(X_{-p}, Y_q)$.
(See for more details and the other definitions [1, 2, 3, 7]).

2. DG-$\mathcal{X}$-injective and DG-$\mathcal{X}$-projective complexes

We begin with the following generalized definitions.

Definition 2.1. Let $\mathcal{X}$ be a class of R-modules. A complex $\mathcal{C} : \ldots \longrightarrow C^{n-1} \longrightarrow C^n \longrightarrow C^{n+1} \longrightarrow \ldots$ is called an $\mathcal{X}^*$-(cochain) complex, if $C^i \in \mathcal{X}$ for all $i \in \mathbb{Z}$. A complex $\mathcal{C} : \ldots \longrightarrow C_{n+1} \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \ldots$ is called an $\mathcal{X}^*$-(chain) complex, if $C_i \in \mathcal{X}$ for all $i \in \mathbb{Z}$. The class of all $\mathcal{X}^*$-complexes is denoted by $C(\mathcal{X}^*)$.

Definition 2.2. A complex $\mathcal{C}$ is called an $\mathcal{X}$-injective complex, if $Ext^1(Y/X, C) = 0$ for every complex $Y/X \in C(\mathcal{X}^*)$. Equivalently, a complex $\mathcal{C}$ is an $\mathcal{X}$-injective complex if for any exact sequence $0 \to X \to Y \to Y/X \to 0$ with a complex $Y/X \in C(\mathcal{X}^*)$, the sequence $Hom(Y,C) \to Hom(X,C) \to 0$ is exact.

Dually we can define an $\mathcal{X}$-projective complex. A complex $\mathcal{C}$ is called an $\mathcal{X}$-projective complex, if $Ext^1(C,X) = 0$ for every complex $X \in \mathcal{C}(\mathcal{X}^*)$, or equivalently a complex $\mathcal{C}$ is an $\mathcal{X}$-projective complex if for any exact sequence $0 \to X \to A \to B \to 0$ with a complex $X \in \mathcal{C}(\mathcal{X}^*)$, the sequence $Hom(C,A) \to Hom(C,B) \to 0$ is exact. We denote the class of all $\mathcal{X}$-injective (projective) complexes by C($\mathcal{X}$-injective (projective)).

Definition 2.3. Let ε be the class of exact complexes. Then we can define $\varepsilon_{\mathcal{X}}$ as the class of exact complexes with kernels in $\mathcal{X}$.

Example 2.4. If P is an $\mathcal{X}$-projective ($\mathcal{X}$-injective) module, then $\overline{P} : \ldots \longrightarrow 0 \longrightarrow P \longrightarrow P \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots$ is an $\mathcal{X}$-projective ($\mathcal{X}$-injective) complex. Moreover any direct sum (product) of $\mathcal{X}$-projective ($\mathcal{X}$-injective) complexes is again an $\mathcal{X}$-projective ($\mathcal{X}$-injective) complex. Since C($\mathcal{X}$-injective (projective)) is closed under extensions, every bounded exact complex $Y : \ldots 0 \to Y^0 \to \ldots \to Y^n \to 0 \ldots$ with kernels an $\mathcal{X}$-injective (projective) module is in C($\mathcal{X}$-injective (projective)).

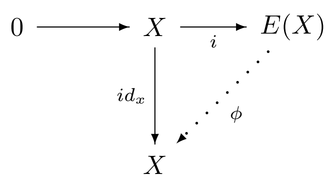
Since every right (left) bounded exact complex with kernels $\mathcal{X}$-injecti- ve (projective) module is an inverse (direct) limit of bounded exact complexes with kernels $\mathcal{X}$-injective (projective) module, then every left (right) bounded exact complex with kernels $\mathcal{X}$-injective (projective) module is in C($\mathcal{X}$-injective (projective)).

Moreover if $\mathcal{X}$-injective $\subseteq \mathcal{X}$, then every $\varepsilon_{\mathcal{X}-injective}$ complex is a direct sum of $\mathcal{X}$-injective complexes, which is the same as injective complexes. Similarly, if $\mathcal{X}$-projective $\subseteq \mathcal{X}$, then every $\varepsilon_{\mathcal{X}-projective}$ complexes is a direct sum of $\mathcal{X}$-projective complexes. Thus, $\varepsilon_{\mathcal{X}-injective(projective)} \subseteq C(\mathcal{X}$-injective (projective)).

Notice that if P is an $\mathcal{X}$-injective ($\mathcal{X}$-projective) module and P is not in the class $\mathcal{X}$, then $\overline{P}$ is an $\mathcal{X}$-injective complex, but not an $\mathcal{X}^*$-complex. So an $\mathcal{X}$-injective (projective) complex may not be an $\mathcal{X}^*$-complex.

Lemma 2.5. *Let X be an $\mathcal{X}$-injective complex such that $\frac{E(X)}{X} \in C(\mathcal{X}^*)$ (or $\frac{Y}{X} \in C(\mathcal{X}^*)$) where $E(X)$ is an injective envelope of X. Then $X = E(X)$ and so it is an injective complex (X is a direct summand of Y).*

Proof. We know that every complex has an injective envelope, so X has an injective envelope $E(X)$. Then $E(X)$ is an injective complex, and so it is exact. We have the following commutative diagram:

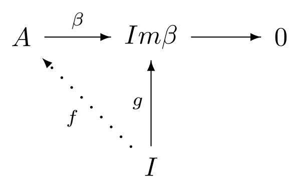


such that $\phi i = id_x$. Therefore X is a direct summand of $E(X)$. So X is an injective complex and hence it is exact. Similarly, if $\frac{Y}{X} \in C(\mathcal{X}^*)$, then we can prove that X is a direct summand of Y. □

Definition 2.6. A complex I is called DG-$\mathcal{X}$-injective, if each I^n is $\mathcal{X}$-injective and $\mathcal{H}om(E, I)$ is exact for all $E \in \varepsilon_{\mathcal{X}}$. A complex I is called a DG-$\mathcal{X}$-projective, if each I^n is $\mathcal{X}$-projective and $\mathcal{H}om(I, E)$ is exact for all $E \in \varepsilon_{\mathcal{X}}$.

Lemma 2.7. *Let $A \xrightarrow{\beta} B \xrightarrow{\theta} C$ be an exact sequence of modules (complexes) where $Ker\beta \in \mathcal{X}$ ($C(\mathcal{X}^*)$). Then for all $\mathcal{X}$-projective modules (complexes) I, $Hom(I, A) \longrightarrow Hom(I, B) \longrightarrow Hom(I, C)$ is exact.*

Proof. By the exact sequence $0 \longrightarrow Ker\theta \xrightarrow{i} B \xrightarrow{\theta} C$, $0 \longrightarrow Hom(I, Ker\theta) \longrightarrow Hom(I, B) \longrightarrow Hom(I, C)$ is exact. We have the following commutative diagram:



such that $\beta f = g$. Since I is an $\mathcal{X}$-projective module (complex) and $Ker\beta \in \mathcal{X}$ $(C(\mathcal{X}^*)), Hom(I, A) \longrightarrow Hom(I, B) \longrightarrow Hom(I, C)$ is exact. □

Dually we can give the following lemma:

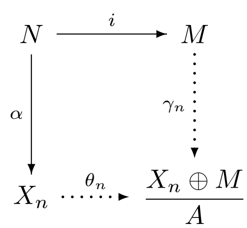
Lemma 2.8. *Let* $A \xrightarrow{\beta} B \xrightarrow{\theta} C$ *be an exact sequence of modules (complexes) where* $\frac{C}{Im\theta} \in \mathcal{X}(C(\mathcal{X}^*))$. *Then for all* $\mathcal{X}$*-injective modules (complexes) I,* $Hom(C, I) \longrightarrow Hom(B, I) \longrightarrow Hom(A, I)$ *is exact.*

Example 2.9. Let $I = ... \longrightarrow 0 \longrightarrow I^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow ...$ where I^0 is an $\mathcal{X}$-injective ($\mathcal{X}$-projective) module. Then I is a DG-$\mathcal{X}$-injective (DG-$\mathcal{X}$-projective) complex.

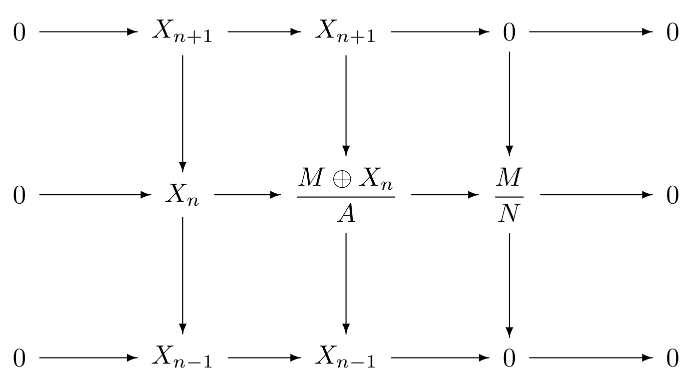
Proof. Let $E : ... \longrightarrow E^{-1} \xrightarrow{d^{-1}} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \xrightarrow{d^2} E^3 \longrightarrow ...$ be exact and $Kerd^n \in \mathcal{X}$, then $\mathcal{H}om(E, I) \cong ...Hom(E^2, I^0) \longrightarrow Hom(E^1, I^0) \longrightarrow Hom(E^0, I^0)....$ By Lemma 2.8, $\mathcal{H}om(E, I)$ is exact. □

Lemma 2.10. *If a complex* $X : \ldots \longrightarrow X_{n+1} \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \ldots$ *is an* $\mathcal{X}$*-injective (*$\mathcal{X}$*-projective) complex, then for all* $n \in \mathbb{Z}$ X_n *is an* $\mathcal{X}$*-injective (*$\mathcal{X}$*-projective) module.*

Proof. Let $0 \longrightarrow N \xrightarrow{i} M$ be exact such that $\frac{M}{N} \in \mathcal{X}$ and $\alpha : N \to X_n$ be linear. Form the pushout:



where $A = \{(\alpha(n), -i(n)) : n \in N\}$. By the following diagram:



we have the exact sequence $0 \longrightarrow X \longrightarrow T \longrightarrow S \longrightarrow 0$ where $T : ... \longrightarrow X_{n+2} \longrightarrow X_{n+1} \longrightarrow \frac{M \oplus X_n}{A} \longrightarrow X_{n-1}...$ and $S : ... \longrightarrow 0 \longrightarrow 0 \longrightarrow \frac{M}{N} \longrightarrow 0....$ Since X is an $\mathcal{X} - injective$ complex, $Ext^1(S, X) = 0$, and so $0 \to Hom(S, X) \to Hom(T, X) \to Hom(X, X) \to Ext^1(S, X) = 0$. Therefore there exists $\beta_n : T_n = \frac{M \oplus X_n}{A} \longrightarrow X_n$ such that $\beta_n \theta_n = 1$. So

$$\beta^n \theta^n(\alpha(n)) = \alpha(n)$$

$$\beta^n((\alpha(n), 0) + A) = \alpha(n)$$

$$\beta^n((0, i) + A) = \alpha(n)$$

$$\beta^n \gamma_n i(n) = \alpha(n)$$

and hence $\beta^n \gamma_n i = \alpha$. So X_n is an $\mathcal{X}$-injective module. □

The following example shows that if $X : ... \to X_{n+1} \to X_n \to X_{n-1} \to ...$ is a complex such that X_n are $\mathcal{X}$-injective ($\mathcal{X}$-projective) modules for all $n \in \mathbb{Z}$, then X does not need to be an $\mathcal{X}$-injective ($\mathcal{X}$-projective) complex.

Example 2.11. Let $R \in \mathcal{X}$ be an $\mathcal{X}$-injective module and $f : R \to R \oplus R$ be a morphism such that $f(a) = (0, a)$ and $g : R \oplus R \to R$ be a morphism such that $g(a, b) = a$. Then $gf = 0$ where $g \neq 0$. Consider the following diagrams:

$$\begin{array}{ccccccc}
...0 & \longrightarrow & R & \xrightarrow{f} & R \oplus R & \longrightarrow & 0... \\
\downarrow & & \downarrow f & & \downarrow g & & \downarrow \\
...0 & \longrightarrow & 0 & \longrightarrow & R & \longrightarrow & 0...
\end{array}$$

$$\begin{array}{ccccccc}
...0 & \longrightarrow & R & \xrightarrow{f} & R \oplus R & \longrightarrow & 0... \\
\downarrow & & \downarrow f & & \downarrow 1 & & \downarrow \\
...0 & \longrightarrow & R \oplus R & \xrightarrow{1} & R \oplus R & \longrightarrow & 0...
\end{array}$$

Then we have the diagram:

$$\begin{array}{ccccccc}
...0 & \longrightarrow & R & \xrightarrow{f} & R \oplus R & \longrightarrow & 0... \\
\downarrow & & \downarrow f & & \downarrow 1 & & \downarrow \\
...0 & \longrightarrow & R \oplus R & \xrightarrow{1} & R \oplus R & \longrightarrow & 0... \\
\downarrow & & \downarrow & & \downarrow g & & \downarrow \\
...0 & \longrightarrow & 0 & \longrightarrow & R & \longrightarrow & 0...
\end{array}$$

such that $g1 = 0$. But this is impossible. So $\underline{R}$ cannot be an $\mathcal{X}$-injective complex. Dually, we can give an example for $\mathcal{X}$-projectivity.

Remark 2.12. There exists a module which is both in $\mathcal{X}$ and an $\mathcal{X}$-injective module. Let $\mathcal{X}$ be a class of injective modules and R be an injective module, then R is both in $\mathcal{X}$ and an $\mathcal{X}$-injective module. Moreover let M be a flat cotorsion module (see Theorem 5.3.28 in [3] for the existence of such a module) and $\mathcal{X}$ be a class of flat modules, then M is both in $\mathcal{X}$ and an $\mathcal{X}$-injective module.

Lemma 2.13. *If $I \in \varepsilon_{\mathcal{X}}^{\perp}$, then each I^n is an $\mathcal{X}$-injective module for each $n \in \mathbb{Z}$.*

Proof. Let $S \subseteq M$ be a submodule of a module M with $\frac{M}{S} \in \mathcal{X}$ and $\alpha : S \longrightarrow I_n$ be linear. Form the pushout:

$$\begin{array}{ccc} S & \xrightarrow{\ i\ } & M \\ {\scriptstyle\alpha}\downarrow & & \vdots\ {\scriptstyle i_1} \\ I^n & \xrightarrow{\ i_2\ } & \dfrac{I^n \oplus M}{A} = I^n \oplus_S M \end{array}$$

where $A = \{(\alpha(s), -s) : s \in S\}$. Thus i_2 is one-to-one the same as i. Then $\overline{I} : ... \longrightarrow I^{n-1} \longrightarrow I^n \oplus_S M \longrightarrow I^{n+1} \longrightarrow I^{n+2} \longrightarrow ...$ is a complex.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^{n-1} & \longrightarrow & I^{n-1} & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^n & \longrightarrow & I^n \oplus_S M & \longrightarrow & \dfrac{M}{S} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^{n+1} & \longrightarrow & I^{n+1} & \longrightarrow & \dfrac{M}{S} & \longrightarrow & 0 \end{array}$$

Therefore, we have an exact sequence $0 \longrightarrow I \longrightarrow \overline{I} \longrightarrow E \longrightarrow 0$ where $E : ... \longrightarrow \frac{M}{S} \longrightarrow \frac{M}{S} \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow ...$ and so we have an exact sequence $0 \longrightarrow Hom(E, I) \longrightarrow Hom(\overline{I}, I) \longrightarrow Hom(I, I) \longrightarrow Ext^1(E, I) = 0$ since $I \in \varepsilon_{\mathcal{X}}^{\perp}$. This implies that we can find $\overline{f} : \overline{I} \longrightarrow I$ with $\overline{f}f = 1$. Therefore, there exists a function $\overline{f}^n : I^n \oplus_S M \longrightarrow I^n$ with $\overline{f}^n f^n = 1$. So,

$$\overline{f}^n f^n(\alpha(s)) = \alpha(s)$$
$$\overline{f}^n((\alpha(s), 0) + A) = \alpha(s)$$
$$\overline{f}^n((0, s) + A) = \alpha(s)$$
$$\overline{f}^n i_1 i(s) = \alpha(s)$$

and hence $\overline{f}_n i_1 i = \alpha$ and thus each $I^n \in \mathcal{X}$-injective.

□

Lemma 2.14. *Let $f : X \longrightarrow Y$ be a morphism of complexes. Then the exact sequence $0 \longrightarrow Y \longrightarrow M(f) \longrightarrow X[1] \longrightarrow 0$ associated with the mapping cone $M(f)$ splits if and only if f is homotopic to 0.*

Proof. The proof follows from [2]. □

Lemma 2.15. *Let X and I be complexes. If $Ext^1(X, I[n]) = 0$ for all $n \in \mathbb{Z}$, then $\mathcal{H}om(X, I)$ is exact.*

Proof. Since $Ext^1(X, I[n]) = 0$, if $f : X[-1] \to I[n]$ is a morphism, then $0 \to I[n] \to M(f) \to X \to 0$ splits.

By Lemma 2.14, $f : X[-1] \to I[n]$ is homotopic to zero for all n. So $f^1 : X \to I[n+1]$ is homotopic to zero for all $n \in \mathbb{Z}$. Thus $\mathcal{H}om(X, I)$ is exact. □

In [5] the following proposition is proved in the case when $(\mathcal{X}, \mathcal{X}^\perp)$ is a cotorsion pair.

Proposition 2.16. *Let $\mathcal{X}$ be extension closed. Then ${\varepsilon_\mathcal{X}}^\perp ({}^\perp\varepsilon_\mathcal{X})$ = DG-$\mathcal{X}$-injective (projective).*

Proof. By Lemma 2.13 and Lemma 2.15 we have that ${\varepsilon_\mathcal{X}}^\perp ({}^\perp\varepsilon_\mathcal{X}) \subseteq DG - \mathcal{X}$-injective (projective). Let $I \in DG - \mathcal{X}$-injective. Therefore $\mathcal{H}om(X, I)$ is exact for all $X \in \varepsilon_\mathcal{X}$ and so for all n, $f : X \to I[n]$ is homotopic to zero. By Lemma 2.14 $A : 0 \to I[n] \to M(f) \to X[1] \to 0$ is split exact. We know that any exact complex $B : 0 \to I[n] \to Y \to X[1] \to 0$ splits at module level since the $I[n]^m$ are $\mathcal{X}$-injective modules and $X^m \in \mathcal{X}$. Therefore the exact sequences A and B are isomorphic. It is known that $Ext^1(C, A) = 0$ if and only if every short exact sequence $0 \to A \to B \to C \to 0$ splits. This implies that $Ext^1(X, I[n]) = 0$ and thus the converse inclusion is proved. □

If we use Proposition 2.16, then we can give the following example since $\mathcal{X}$ and ${\varepsilon_\mathcal{X}}^\perp ({}^\perp\varepsilon_\mathcal{X})$ are extension closed and every right(left) bounded complex is a direct (inverse) transfinite limit of bounded complexes.

Example 2.17. Let $\mathcal{X}$ be extension closed. Then every $\mathcal{X}$-projective (injective) complex is DG-$\mathcal{X}$-projective (injective). Every right (left) bounded complex I where I_i is an $\mathcal{X}$-projective (injective) module is a DG-$\mathcal{X}$-projective (injective) complex. Moreover $\varepsilon_{\mathcal{X}-injective(projective)} \subseteq$ DG-$\mathcal{X}$-injective (projective) since the direct (inverse) limit of DG-$\mathcal{X}$-injective complexes is also an inverse (direct) transfinite limit of bounded $\varepsilon_{\mathcal{X}-injective(projective)}$ complexes.

$\varepsilon_\mathcal{X}$ and DG-$\mathcal{X}$-injective cannot be a cotorsion pair if $\mathcal{X}$ is extension closed. We have the following theorem:

Theorem 2.18. *Let $\mathcal{X}$ be extension closed and we have enough $\mathcal{X}$-object. Then (DG-$\mathcal{Y}$-projective,$\varepsilon_{\mathcal{Y}}$) is cotorsion pair where $\mathcal{Y} = \mathcal{X}$-injective.*

Proof. It follows from the proof of Proposition 3.6 in [5] and Proposition 2.16. □

3. C($\mathcal{X}$-projective)-precovers and C($\mathcal{X}$-injective)-preenvelopes

In this section we prove that if a complex has a C($\mathcal{X}$-projective)-precover or C($\mathcal{X}$-injective)-preenvelope in C($\mathcal{X}^*$), then such precovers or preenvelopes are homotopic. Moreover we investigate when a complex has an exact C($\mathcal{X}$-projective (injective))- precover (preenvelope) and we give some conditions when an $\mathcal{X}$-projective (injective) complex is exact and in particular in $\varepsilon_{\mathcal{X}-proje-ctive(injective)}$.

Lemma 3.1. *i) Let $f : X \longrightarrow Y$ be a chain morphism, let X be an $\mathcal{X}^*$ complex and let Y be an $\mathcal{X}-injective$ complex. Then f is homotopic to zero. Moreover if a complex has a C($\mathcal{X}$-injective)-preenvelope in $C(\mathcal{X}^*)$, then such preenvelopes are homotopic.*
ii) Let $f : X \longrightarrow Y$ be a chain homomorphism such that Y is an $\mathcal{X}^*$ complex and X is an $\mathcal{X}$-projective complex. Then f is homotopic to zero. Moreover if a complex has a C($\mathcal{X}$-projective)-precover in $C(\mathcal{X}^*)$, then such precovers are homotopic.

Proof. i) Let $id : X \longrightarrow X$, then we have the following exact sequence:

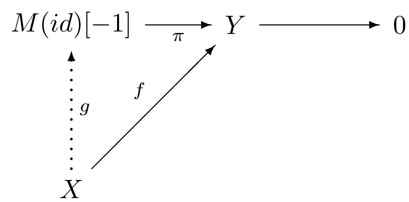
$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i} & M(id) & \longrightarrow & X[1] & \longrightarrow & 0 \\ & & \downarrow f & \swarrow g & & & & & \\ & & Y & & & & & & \end{array}$$

where $gi = f$. Let $i_1^n : X[1]^n \longrightarrow M(id)^n$ be a canonical injection and $s^n : X[1]^{n-1} \longrightarrow Y^{n-1}$ such that $s^n = g^{n-1}i_1^{n-1}$ for all $n \in \mathbb{Z}$. Let u be the differential of the complex $M(id)$. Then we have the following diagram

$$\begin{array}{ccccc} X^{n-1} \oplus X^{n-2} & \xrightarrow{u^{n-2}} & X^n \oplus X^{n-1} & \xrightarrow{u^{n-1}} & X^{n+1} \oplus X^n \\ \downarrow g^{n-2} & & \downarrow g^{n-1} & & \downarrow g^n \\ Y^{n-2} & \xrightarrow{\gamma^{n-2}} & Y^{n-1} & \xrightarrow{\gamma^{n-1}} & Y^n \end{array}$$

$s^{n+1}\lambda^n+\gamma^{n-1}s^n = g^n i_1^n\lambda^n+\gamma^{n-1}g^{n-1}i_1^{n-1} = g^n i_1^n\lambda^n+g^n u^{n-1}i_1^{n-1} = g^n(i_1^n\lambda^n+u^{n-1}i_1^{n-1}) = g^n i^n = f^n$.
ii) Consider $id : Y \longrightarrow Y$ and the exact sequence $0 \longrightarrow Y[-1] \longrightarrow M(id)[-1] \longrightarrow$

$Y \longrightarrow 0$. Since X is an $\mathcal{X} - projective$ complex, we have the following commutative diagram:



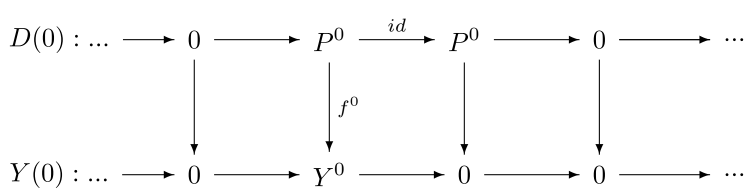
where $\pi g = f$. Let $\pi_1^n : M(id)[-1]^n \longrightarrow Y[-1]^n$ be a projection for all $n \in \mathbb{Z}$. Then if we take as $s^n = \pi_1^n g^n$, then for all $n \in \mathbb{Z}$, $s^{n+1}\lambda^n + \gamma^{n-1}s^n = f^n$ where λ and γ are boundary maps of the complexes of X and Y, respectively. So f is homotopic to zero. □

Proposition 3.2. *Let $({}^{\perp}\mathcal{X}, \mathcal{X})$ $((\mathcal{X}, \mathcal{X}^{\perp}))$ be a cotorsion pair. Then every $\mathcal{X}$-projective ($\mathcal{X}$-injective) complex is exact.*

Proof. By [2] we see that every $\mathcal{X}$-projective (injective) complex has an exact precover (preenvelope) with kernel (cokernel) in DG-injective (projective). The result follows. □

Lemma 3.3. *Let $\mathcal{X}$ be extension closed (and $({}^{\perp}\mathcal{X}, \mathcal{X})$ be a cotorsion pair). Let every R-module have an epic $\mathcal{X}$-projective-precover with kernel in $\mathcal{X}$. Then every bounded complex in $C(\mathcal{X}^*)$ has an epic exact $C(\mathcal{X}$-projective)-precover (which is also in $\varepsilon_{\mathcal{X}-projective}$) with kernel in $C(\mathcal{X}^*)$ (which is also in DG-$\mathcal{X}$-projective-injective=$(\varepsilon_{\mathcal{X}-projective})^{\perp}$). Thus every bounded $\mathcal{X}$-projective complex in $C(\mathcal{X}^*)$ is exact (which is also in $\varepsilon_{\mathcal{X}-projective}$ and every bounded complex in $C(\mathcal{X}^*)$ has an $\varepsilon_{\mathcal{X}-projective}$-precover).*

Proof. Let $Y(n) : ... \to 0 \to Y^0 \to Y^1 \to ... \to Y^n \to 0 \to ... \in C(\mathcal{X}^*)$. We use induction on n. Let $n = 0$, then we have the following commutative diagram:



where $P^0 \to Y^0 \to 0$ is an $\mathcal{X}$-projective-precover in $\mathcal{X}$ with kernel in $\mathcal{X}$ since $\mathcal{X}$ is extension closed, $D(0)$ is exact and $Ker(D(0) \to Y(0)) \in \mathcal{C}(\mathcal{X}^*)$. We consider the following diagram which is commutative:

$$\begin{array}{ccccccccc}
D(n): ...0 & \longrightarrow & P^0 & \xrightarrow{\lambda^0} & P^0 \oplus P^1 & \xrightarrow{\lambda^1} & ...P^{n-1} \oplus P^n & \xrightarrow{\lambda_1^n} & P^n ... \\
 & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle (0,f^1)} & & \downarrow{\scriptstyle (0,f^n)} & & \downarrow \\
Y(n): ...0 & \longrightarrow & Y^0 & \xrightarrow{a^0} & Y^1 & \xrightarrow{a^1} & ...Y^n & \longrightarrow & 0...
\end{array}$$

where λ_1^n is onto, $D(n)$ is an exact C($\mathcal{X}$-projective)-precover of $Y(n)$ such that $Ker(D(n) \to Y(n)) \in \mathcal{C}(\mathcal{X}^*)$ and the $P^i \to Y^i \to 0$ are $\mathcal{X}$-projective-precovers in $\mathcal{X}$ with kernels in $\mathcal{X}$ for $1 \leq i \leq n$. Since $D(n) \to Y(n) \to 0$ and $\overline{P^{n+1}} \to \underline{Y^{n+1}} \to 0$ are C($\mathcal{X}$-projective)-precovers, we have the following commutative diagram:

$$\begin{array}{ccc}
D(n) & \xrightarrow{s} & \overline{P^{n+1}} \\
\downarrow & & \downarrow \\
Y(n) & \longrightarrow & \underline{Y^{n+1}}
\end{array}$$

Thus we have the diagram:

$$\begin{array}{ccccccccc}
D(n): ...0 & \longrightarrow & P^0 & \xrightarrow{\lambda^0} & P^0 \oplus P^1 ... & \xrightarrow{\lambda^{n-1}} & P^{n-1} \oplus P^n & \xrightarrow{\lambda_1^n} & P^n ... \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle s^1} & & \downarrow{\scriptstyle s^2} \\
\overline{P^{n+1}}: ...0 & \longrightarrow & 0 & \longrightarrow & 0... & \longrightarrow & P^{n+1} & \xrightarrow{1} & P^{n+1} ...
\end{array}$$

where $s^2\lambda_1^n = s^1$ and $s^1\lambda^{n-1} = 0$. Moreover we see that $f^{n+1}s^1 = a^n(0, f^n)$ and $f^{n+1}s^2 = 0$ by the following diagrams:

$$\begin{array}{ccc}
P^{n-1} \oplus P^n & \xrightarrow{s^1} & P^{n+1} \\
\downarrow{\scriptstyle (0,f^n)} & & \downarrow{\scriptstyle f^{n+1}} \\
Y^n & \xrightarrow{a^n} & Y^{n+1}
\end{array}$$

$$\begin{array}{ccc}
P^n & \xrightarrow{s^2} & P^{n+1} \\
\downarrow & & \downarrow{\scriptstyle f^{n+1}} \\
0 & \longrightarrow & Y^{n+1}
\end{array}$$

Let $\lambda^n(x,y) = (\lambda_1^n(x,y), s^1(x,y))$, $\lambda_1^{n+1}(x,y) = s^2(x) - y$. Then we have the commutative diagram:

$$\begin{array}{ccccccccc}
D(n+1):\ldots & \longrightarrow & P^0\ldots & \longrightarrow & P^{n-1}\oplus P^n & \xrightarrow{\lambda^n} & P^n \oplus P^{n+1} & \xrightarrow{\lambda_1^{n+1}} & P^{n+1}\ldots \\
 & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle (0,f^n)} & & \downarrow{\scriptstyle (0,f^{n+1})} & & \downarrow \\
Y(n+1):\ldots & \longrightarrow & Y^0\ldots & \longrightarrow & Y^n & \xrightarrow{a^n} & Y^{n+1} & \longrightarrow & 0\ldots
\end{array}$$

where $Ker(D(n+1) \to Y(n+1)) \in C(\mathcal{X}^*)$ and since λ_1^{n+1} is onto, $Im(\lambda^n) = Ker(\lambda_1^{n+1})$, $Im(\lambda^{n-1}) = Ker(\lambda^n)$ and $D(n)$ is exact, $D(n+1)$ is exact. Therefore, $Y(n)$ has a C($\mathcal{X}$-projective)-precover. □

The following corollary is a direct consequence of Lemma 3.3.

Corollary 3.4. *i) Let $\mathcal{X}$ be extension closed. If $\mathcal{X}$-projective $\subseteq \mathcal{X}$ and every R-module has an epic $\mathcal{X}$-projective-precover with kernel in $\mathcal{X}$ (and $({}^{\perp}\mathcal{X}, \mathcal{X})$ is a cotorsion pair), then every bounded complex has an an epic exact C($\mathcal{X}$-projective)-precover (which is also in $\varepsilon_{\mathcal{X}-projective}$) with kernel in C($\mathcal{X}^*$). Thus if $({}^{\perp}\mathcal{X}, \mathcal{X})$ is a complete cotorsion pair, then $\varepsilon_{\mathcal{X}-projective}$ bounded complexes and $C(\mathcal{X}$-projective) bounded complexes are identical.*
ii)If $({}^{\perp}\mathcal{X}, \mathcal{X})$ is a complete cotorsion pair, then every bounded complex in C($\mathcal{X}^*$) has an $\varepsilon_{\mathcal{X}-projective}$-precover.

Lemma 3.5. *If $\mathcal{X}$ is extension closed and every R-module has a monic $\mathcal{X}$-injective-preenvelope with cokernel in $\mathcal{X}$ (and $(\mathcal{X}, \mathcal{X}^{\perp})$ is a cotorsion pair), then every bounded complex in C($\mathcal{X}^*$) has a monic exact C($\mathcal{X}$-injective)-preenvelope (which is also in $\varepsilon_{\mathcal{X}-injective}$) with cokernel in C($\mathcal{X}^*$) (which is also in DG-$\mathcal{X}$-injective-projective$={}^{\perp}(\varepsilon_{\mathcal{X}-injective})$).*
Thus every bounded $\mathcal{X}$-injective complex in C($\mathcal{X}^*$) is exact (which is also in $\varepsilon_{\mathcal{X}-injective}$ and hence every bounded complex in C($\mathcal{X}^*$) has an $\varepsilon_{\mathcal{X}-injective}$-preenvelope).

Proof. Let $Y(n) : \ldots \to 0 \to Y_n \to Y_{n-1} \to \ldots \to Y_0 \to 0 \to \ldots$. We use induction on n. Let $n = 0$, then we have the following commutative diagram:

$$\begin{array}{ccccccc}
Y(0):\ldots 0 & \longrightarrow & 0 & \longrightarrow & Y_0 & \longrightarrow & 0\ldots \\
\downarrow & & \downarrow & & \downarrow & & \\
E(0):\ldots 0 & \longrightarrow & E_0 & \xrightarrow{id} & E_0 & \longrightarrow & 0\ldots
\end{array}$$

where $0 \to Y_0 \to E_0$ is a monic preenvelope in $\mathcal{X}$ with cokernel in $\mathcal{X}$ and thus $E(0)$ is an exact preenvelope of $Y(0)$ with cokernel in $C(\mathcal{X}^*)$. We consider the following diagram which is commutative:

$$\begin{array}{ccccccccccc}
Y(n): \dots 0 & \longrightarrow & 0 & \longrightarrow & Y_n & \xrightarrow{a_n} & \cdots & \longrightarrow & Y_0 & \longrightarrow & 0\\
 & & \downarrow & & \downarrow{\scriptstyle (f_n,0)} & & & & \downarrow{\scriptstyle f_0} & & \\
E(n): \dots 0 & \longrightarrow & E_n & \xrightarrow{\lambda_n^1} & E_n \oplus E_{n-1} & \xrightarrow{\lambda_{n-1}} & \cdots & \longrightarrow & E_0 & \longrightarrow & 0
\end{array}$$

where the $0 \to Y_i \to E_i$ are $\mathcal{X}$-injective-preenvelopes in $\mathcal{X}$ with cokernel in $\mathcal{X}$ for $1 \leq i \leq n$, $E(n)$ is exact with cokernel $(Y(n) \to E(n)) \in C(\mathcal{X}^*)$. Since $0 \to Y_n \to E_n$ and $0 \to \underline{Y_{n+1}} \to \overline{E_{n+1}}$ are C($\mathcal{X}$-injective)-preenvelopes, we have the following commutative diagram:

$$\begin{array}{ccc}
\underline{Y_{n+1}} & \longrightarrow & Y(n)\\
\downarrow & & \downarrow\\
\overline{E_{n+1}} & \longrightarrow & E(n)
\end{array}$$

Then we have the diagram:

$$\begin{array}{ccccccccc}
\overline{E_{n+1}}: \dots 0 & \longrightarrow & E_{n+1} & \xrightarrow{1} & E_{n+1} & \longrightarrow & 0 & \longrightarrow & \cdots\\
 & & \downarrow{\scriptstyle s_{n+1}} & & \downarrow{\scriptstyle s_n} & & \downarrow & & \\
E(n): \dots 0 & \longrightarrow & E_n & \xrightarrow{\lambda_n^1} & E_n \oplus E_{n-1} & \xrightarrow{\lambda_{n-1}} & \cdots & \longrightarrow & \cdots
\end{array}$$

where $s_n = \lambda_n^1 s_{n+1}$ and $\lambda_{n-1} s_n = 0$. Moreover we see that $(f_n, 0)a_{n+1} = s_n f_{n+1}$ and $\lambda_n^1 s_{n+1} = s_n$ by the following diagrams:

$$\begin{array}{ccc}
Y_{n+1} & \xrightarrow{a_{n+1}} & Y_n\\
\downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle (f_n,0)}\\
E_{n+1} & \xrightarrow{s_n} & E_n \oplus E_{n-1}
\end{array}$$

$$\begin{array}{ccc}
E_{n+1} & \xrightarrow{s_{n+1}} & E_n\\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \lambda_n^1}\\
E_{n+1} & \xrightarrow{s_n} & E_n \oplus E_{n-1}
\end{array}$$

Let $\lambda^1_{n+1}(x) = (x, -s_{n+1}(x))$, $\lambda_n(x, y) = s_n(x) + \lambda^1_n(y)$. Then we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
Y(n+1): ...0 & \longrightarrow & 0 & \longrightarrow & Y_{n+1} & \xrightarrow{a_{n+1}} & Y_n... & \longrightarrow & Y_0... \\
 & & \downarrow & & \downarrow{\scriptstyle (f_{n+1},0)} & & \downarrow{\scriptstyle (f_n,0)} & & \downarrow{\scriptstyle f^0} \\
E(n+1): ...0 & \longrightarrow & E_{n+1} & \xrightarrow{\lambda^1_{n+1}} & E_{n+1} \oplus E_n & \xrightarrow{\lambda_n} & E_n \oplus E_{n-1}... & \longrightarrow & E_0...
\end{array}
$$

where $E(n+1)$ is exact with cokernel $(Y(n+1) \to E(n+1))$ in $C(\mathcal{X}^*)$. Therefore, $Y(n)$ has a C($\mathcal{X}$-injective)-preenvelope. □

Corollary 3.6. *i) Let $\mathcal{X}$ be extension closed. If $\mathcal{X}$-injective $\subseteq \mathcal{X}$ and every R-module has a monic $\mathcal{X}$-injective-preenvelope with kernel in $\mathcal{X}$ (and $(\mathcal{X}, \mathcal{X}^\perp)$ is a cotorsion pair), then every bounded complex has an a monic exact C($\mathcal{X}$-injective)-preenvelope (which is also in $\varepsilon_{\mathcal{X}-injective} \subseteq C(\mathcal{X}$-injective)) with kernel in C($\mathcal{X}^*$). Thus $\varepsilon_{\mathcal{X}-injective}$ and $C(\mathcal{X} - injective)$ bounded complexes are identical if $(\mathcal{X}, \mathcal{X}^\perp)$ is a cotorsion pair .*
ii) If $(\mathcal{X}, \mathcal{X}^\perp)$ is a complete cotorsion pair, then every bounded complex in C($\mathcal{X}^*$) has an $\varepsilon_{\mathcal{X}-injective}$-preenvelope.

We know that the direct (inverse) limit of exact complexes is also exact. Then we can give the following theorem.

Theorem 3.7. *Let $\mathcal{X}$ be closed under extensions. The following are satisfied:*
i) If every R-module has a monic (epic) $\mathcal{X}$-injective (projective)-preenvelope (precover) with cokernel (kernel) in $\mathcal{X}$, then every left (right) bounded complex in C($\mathcal{X}^*$) has a monic (epic) exact C($\mathcal{X}$-injective (projective))-preenvelope (precover) (which is also in $\varepsilon_{\mathcal{X}-injective(projective)}$ if $(\mathcal{X}, \mathcal{X}^\perp)$ $((^\perp\mathcal{X}, \mathcal{X}))$ is a cotorsion pair). Moreover if $\mathcal{X}$-injective (projective) $\subseteq \mathcal{X}$, then every left (right) bounded complex has a monic (epic) exact C($\mathcal{X}$-injective (projective))-preenvelope (precover).
ii) If every R-module has a monic (epic) $\mathcal{X}$-injective (projective)-preenvelope (precover) with cokernel (kernel) in $\mathcal{X}$, then every right (left) bounded complex in C($\mathcal{X}^*$) has a monic (epic) exact C($\mathcal{X}$-injective(projective))-preenvelope (precover).
Therefore every right (left) bounded $\mathcal{X}$-injective (projective) complex in C($\mathcal{X}^*$) is exact (which is also in $\varepsilon_{\mathcal{X}-injective(projective)}$ and every right (left) bounded complexes in C($\mathcal{X}^*$) has an $\varepsilon_{\mathcal{X}-injective(projective)}$-preenvelope (precover) if $(\mathcal{X}, \mathcal{X}^\perp)$ $((^\perp\mathcal{X}, \mathcal{X}))$ is a cotorsion pair). Moreover if $\mathcal{X}$-injective (projective) $\subseteq \mathcal{X}$, then every right (left) bounded complex has a monic (epic) exact C($\mathcal{X}$-injective (projective))-preenvelope (precover).

Proof. i) Let $Y : ... \to 0 \to Y^0 \to Y^1 \to ...$ and $E(n)$ be a C($\mathcal{X}$-injective)-preenvelope of $Y(n) : ... \to 0 \to Y^0 \to ... \to Y^n \to 0 \to ....$ Then $\varprojlim Y(n) = Y$. By Lemma 3.5, $Y(n)$ has a C($\mathcal{X}$-injective)-preenvelope $E(n)$ such that $0 \to$

$Y(n) \to E(n)$ is exact. Then by Theorem 1.5.13 in [3] and the proof of Lemma 3.5 $0 \to \varprojlim Y(n) \to \varprojlim E(n)$ is exact with cokernel $\varprojlim \frac{E(n)}{Y(n)} \in C(\mathcal{X}^*)$ which is also a direct transfinite limit of $DG(\mathcal{X})$-injective-projective complexes. Since $Ext^1(\frac{A}{B}, \varprojlim E(n)) = 0$ where $\frac{A}{B} \in C(\mathcal{X}^*)$ by Lemma 2.3 in [9], $\varprojlim E(n)$ is an exact C($\mathcal{X}$-injective)-preenvelope of Y. The other part is also proved similarly using C($\mathcal{X}$-projective) is closed under direct transfinite limits by Theorem 1.2 in [4].
ii) Let $Y : ... \to Y_2 \to Y_1 \to Y_0 \to 0 \to ...$ and $E(n)$ be a C($\mathcal{X}$-injective)-preenvelope of $Y(n) : ... \to 0 \to Y_n \to ... \to Y_1 \to Y_0 \to 0 \to ....$ Then $\varinjlim Y(n) = Y$. By Lemma 3.5, $Y(n)$ has a C($\mathcal{X}$-injective)-preenvelope $E(n)$ such that $0 \to Y(n) \to E(n)$ is exact. Then by Theorem 1.5.6 in [3] $0 \to \varinjlim Y(n) \to \varinjlim E(n)$ is exact with cokernel $\varinjlim \frac{E(n)}{Y(n)} \in C(\mathcal{X}^*)$ (which is also in $DG(\mathcal{X})$-injective-projective if $(\mathcal{X}, \mathcal{X}^\perp)$ is a cotorsion pair). Since $\varinjlim E(n)$ is also an inverse transfinite limit of some bounded $\mathcal{X}$-injective complexes, $Ext^1(\frac{A}{B}, \varinjlim E(n)) = 0$ where $\frac{A}{B} \in C(\mathcal{X}^*)$. So $\varinjlim E(n)$ is an exact C($\mathcal{X}$-injective)-preenvelope of Y. □

Corollary 3.8. *i) Let $\mathcal{X}$ be closed under extensions. If every R-module has a monic (epic) $\mathcal{X}$-injective (projective)-preenvelope (precover) with cokernel (kernel) in $\mathcal{X}$, then every complex in $C(\mathcal{X}^*)$ has a monic (epic) exact C($\mathcal{X}$-injective (projective))-preenvelope (precover) (which is also in $\varepsilon_{\mathcal{X}-injective(projective)}$ if $(\mathcal{X}, \mathcal{X}^\perp)$ $(({}^\perp\mathcal{X}, \mathcal{X}))$ is a cotorsion pair). Moreover if $\mathcal{X}$-injective (projective) $\subseteq \mathcal{X}$, then every complex has a monic (epic) exact C($\mathcal{X}$-injective (projective))-preenvelope (precover) (which is in $\varepsilon_{\mathcal{X}-injective}$ $(\varepsilon_{\mathcal{X}-projective})$ if $(\mathcal{X}, \mathcal{X}^\perp)$ $(({}^\perp\mathcal{X}, \mathcal{X}))$ is a cotorsion pair, thus $\varepsilon_{\mathcal{X}-injective(projective)}$ and C($\mathcal{X}$-injective (projective)) complexes are identical).*
ii) If $(\mathcal{X}, \mathcal{X}^\perp)$ is a complete cotorsion pair, then every complex in $C(\mathcal{X}^*)$ has a monic $\varepsilon_{\mathcal{X}-injective}$-preenvelope.

Example 3.9. Let $\mathcal{X}$ be a class of R-modules closed under quotients, extensions and direct sums (for the existence of such classes, if $\mathcal{X}$ is a class of injective modules on a hereditary noetherian ring which is constructed in [8], then $\mathcal{X}$ is closed under quotients, extensions and direct limits and moreover if $\mathcal{X}$ is the class of min-injective modules and simple ideals of ring R are projective, then it is closed under quotients, extensions and direct sums). If A and B are in $\mathcal{X}$ such that $\phi : A \to B$ is a homomorphism, then by Theorem 2.10 in [6], we have monic $\mathcal{X}$-injective-preenvelopes such that $f : A \to E_A$ and $g : B \to E_B$ with cokernels in $\mathcal{X}$. Then there exists a homomorphism $s : E_A \to E_B$ such that $g\phi = sf$. Using Lemma 3.5 we can determine an exact C($\mathcal{X}$-injective)-preenvelope $E(1)$ of complex $Y(1)$ as follows:

$$\begin{array}{ccccccccc} Y(1): ...0 & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow{\phi} & B & \longrightarrow & 0... \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle (f,0)} & & \downarrow{\scriptstyle g} & & \\ E(1): ...0 & \longrightarrow & E_A & \xrightarrow{\alpha} & E_A \oplus E_B & \xrightarrow{\beta} & E_B & \longrightarrow & 0... \end{array}$$

where $\alpha(x) = (x, -s(x))$ and $\beta(x, y) = s(x)+y$. Then every complex in $C(\mathcal{X}^*)$ has a monic exact C(X-injective)-preenvelope by Corollary 3.8.

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