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A NOTE ON FOUQUET-VANHERPE'S QUESTION AND FULKERSON CONJECTURE

F. CHEN

(Communicated by Amir Daneshgar)

Abstract. The excessive index of a bridgeless cubic graph G is the least integer k, such that G can be covered by k perfect matchings. An equivalent form of Fulkerson conjecture (due to Berge) is that every bridgeless cubic graph has excessive index at most five. Clearly, Petersen graph is a cyclically 4-edge-connected snark with excessive index at least 5, so Fouquet and Vanherpe asked whether Petersen graph is the only one with that property. Hägglund gave a negative answer to their question by constructing two graphs Blowup(K_4, C) and Blowup$(Prism, C_4)$. Based on the first graph, Esperet et al. constructed infinite families of cyclically 4-edge-connected snarks with excessive index at least five. Based on these two graphs, we construct infinite families of cyclically 4-edge-connected snarks $E_{0,1,2,\ldots,(k-1)}$ in which $E_{0,1,2}$ is Esperet et al.'s construction. In this note, we prove that $E_{0,1,2,3}$ has excessive index at least five, which gives a strongly negative answer to Fouquet and Vanherpe's question.

As a subcase of Fulkerson conjecture, Häggkvist conjectured that every cubic hypohamiltonian graph has a Fulkerson-cover. Motivated by a related result due to Hou et al.'s, in this note we prove that Fulkerson conjecture holds on some families of bridgeless cubic graphs.
Keywords: Fulkerson-cover, excessive index, snark, hypohamiltonian graph.
MSC(2010): Primary: 05C70; Secondary: 05C75, 05C40, 05C15.

1. Introduction

Let G be a simple graph (without loops or parallel edges) with vertex set $V(G)$ and edge set $E(G)$. A *perfect matching* of G is a 1-regular spanning subgraph of G. The *excessive index* of G (first introduced by Bonisoli and Cariolaro [3]), denoted by $\chi'_e(G)$, is the least integer k, such that G can be covered by k perfect matchings. We call these k perfect matchings as the *minimum perfect matching cover* of G.

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The following conjecture is due to Berge and Fulkerson, and first appeared in [6].

Conjecture 1.1 (Fulkerson conjecture, Fulkerson [6])**.** *If G is a bridgeless cubic graph, then G can be covered by six perfect matchings such that each edge is in exactly two of them.*

We call such 6 perfect matchings as the *Fulkerson-cover*. If Fulkerson conjecture is true, then deleting one perfect matching from the Fulkerson-cover would result in a covering of the graph by 5 perfect matchings. Thus, Berge conjectured that (unpublished and first appeared in [13])

Conjecture 1.2 (Berge, unpublished and first appeared in [13])**.** *If G is a bridgeless cubic graph, then $\chi'_e(G) \leq 5$.*

Mazzuoccolo [10] proved that Conjectures 1.1 and 1.2 are equivalent. But on a given graph, the equivalence of these two conjectures has not been proved.

A graph G is called *cyclically k-edge-connected* if at least k edges must be removed to disconnect it into two components, each of which contains a circuit.

Obviously, Conjectures 1.1 and 1.2 hold on 3-edge-colorable cubic graphs. So in this note, we only consider bridgeless non 3-edge-colorable cubic graphs, which are called *snarks*. For more details, see the book written by Zhang [14]. Fouquet and Vanherpe [5] proved that there are several infinite families of cyclically 3-edge-connected snarks with excessive index at least five. But for cyclically 4-edge-connected snarks, they only know Petersen graph. They proposed the following question.

Question 1.1 (Fouquet and Vanherpe [5])**.** *If G is a cyclically 4-edge-connected snark, then either G is Petersen graph or $\chi'_e(G) < 5$.*

Häggluund [7] gave a negative answer to Question 1.1 by constructing two graphs Blowup(K_4, C) and Blowup$(Prism, C_4)$. Based on Blowup(K_4, C), Esperet et al. [4] constructed infinite families of cyclically 4-edge-connected snarks with excessive index at least five. Based on these two graphs, in Section 2, we construct infinite families of bridgeless cubic graphs $M_{0,1,2,\ldots,(k-1)}$ and infinite families of cyclically 4-edge-connected snarks $E_{0,1,2,\ldots,(k-1)}$ ($k \geq 2$) where $E_{0,1,2}$ is Esperet et al.'s [4] construction.

In Section 3, we prove that each graph in $E_{0,1,2,3}$ (see Fig. 1) has excessive index at least five. This gives a strongly negative answer to Question 1.1. In Section 4, we prove that each graph in $M_{0,1,2,3}$ has a Fulkerson-cover.

Let $X \subseteq V(G)$ and $e = uv \in E(G)$. We use $G \setminus X$ to denote the subgraph of G obtained from G by deleting all the vertices of X and all the edges incident with X. Moreover if $X = \{x\}$, we simply write $G \setminus x$. Similarly, we use $G \setminus e$ to denote the subgraph of G obtained from G by deleting e. A *minor* of G is any graph obtained from G by means of a sequence of vertex and edge deletions and edge contractions. According to Hao et al. [8] and Hou et al. [9], we use $\overline{G}$

to denote the graph obtained from G by contracting all the vertices of degree 2.

A graph G is called *hypohamiltonian* if G itself doesn't have Hamilton circuits but $G \setminus v$ does for each vertex $v \in V(G)$. A graph G is called *Kotzig* if G has a 3-edge-coloring, each pair of which form a Hamilton circuit (the definition is defined by Häggkvist and Markström).

The research on Fulkerson conjecture has attracted more and more graph theorists, and in particular, Häggkvist [11] proposed the following conjecture in 2007.

Conjecture 1.3 (Häggkvist [11]). *If G is a cubic hypohamiltonian graph, then G has a Fulkerson-cover.*

There is little progress on Conjecture 1.3. Recently, Hou et al. [9] partially solved Conjecture 1.3 in the following theorem.

Theorem 1.1 (Hou, Lai and Zhang [9]). *Let G be a bridgeless cubic graph. If there exists a vertex $v \in V(G)$ such that $\overline{G \setminus v}$ is a Kotzig graph, then $\chi'_e(G) \leq 5$.*

Motivated by their results, in Section 5, we prove that

Theorem 1.2. *Let G be a bridgeless cubic graph. Then G has a Fulkerson-cover if one of the followings holds:*

(1) there exists a vertex $v \in V(G)$ such that $\overline{G \setminus v}$ is a Kotzig graph and $G \setminus e$ doesn't have Petersen graph as a minor for each edge e incident with v.

(2) there exists an edge $e \in E(G)$ such that $\overline{G \setminus e}$ is a Kotzig graph.

(3) for each $e \in E(G)$, $G \setminus e$ doesn't have Petersen graph as a minor.

Note that our proof is independent of Hou et al.'s [9].

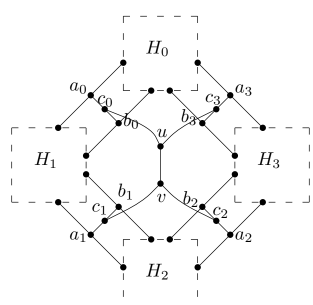

Fig. 1

2. **Preliminaries**

In this section, we will give some necessary definitions, constructions, lemmas and propositions.

Lemma 2.1 (Parity lemma, Blanuša [1])**.** *Let G be a cubic graph. If M is a perfect matching of G and T an edge-cut of G, then* $|M \cap T| \equiv |T| \pmod 2$.

Let X be a subset of $V(G)$. The *edge-cut* of G associated with X, denoted by $\partial_G(X)$, is the set of edges of G with exactly one end in X. The edge set $C = \partial_G(X)$ is called a *k-edge-cut* if $|\partial_G(X)| = k$.

Let G_i be a cyclically 4-edge-connected snark with excessive index at least 5, for $i = 0, 1$. Let $x_i y_i$ be an edge of G_i and x_i^0, x_i^1 (y_i^0, y_i^1) the neighbours of x_i (y_i). Let H_i be the graph obtained from G_i by deleting the vertices x_i and y_i. Let $\{G; G_0, G_1\}$ be the graph obtained from the disjoint union of H_0, H_1 by adding six vertices a_0, b_0, c_0, a_1, b_1, c_1 and 13 edges $a_0y_0^0$, $a_0x_1^0$, a_0c_0, c_0b_0, $b_0y_0^1$, $b_0x_1^1$, $b_1x_0^1$, $b_1y_1^1$, b_1c_1, c_1a_1, $a_1x_0^0$, $a_1y_1^0$, c_0c_1. We call graphs of this type as $E_{0,1}$ (see Fig. 2).

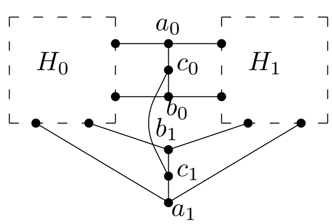

Fig. 2

Now we construct $E_{0,1,\ldots,(k-1)}$ ($k \geq 2$) as follows:

(1) $\{G; G_0, G_1\} \in E_{0,1}$ with $A_j = \{a_j, b_j, c_j\}$ for $j = 0, 1$.

(2) For $3 \leq i \leq k$, $\{G; G_0, G_1, \ldots, G_{i-1}\}$ is obtained from $\{G; G_0, G_1, \ldots, G_{i-2}\} \in E_{0,1,\ldots,(i-2)}$ by adding H_{i-1} and $A_{i-1} = \{a_{i-1}, b_{i-1}, c_{i-1}\}$ and by inserting a vertex v_{i-3} into e_0, such that

(i) G_{i-1} is a cyclically 4-edge-connected snark with excessive index at least 5 ($x_{i-1}y_{i-1}$ is an edge of G_{i-1} and x_{i-1}^0, x_{i-1}^1 (y_{i-1}^0, y_{i-1}^1) are the neighbours of x_{i-1} (y_{i-1}));

(ii) $H_{i-1} = G_{i-1} \setminus \{x_{i-1}, y_{i-1}\}$;

(iii) $e_0 \in E(\{G; G_0, G_1, \ldots, G_{i-2}\}) - \cup_{j=0}^{i-2} E(H_j) - \cup_{j=0}^{i-2}\{a_jc_j, c_jb_j\}$ and e_0 is incident with c_0;

(iv) a_{i-1} is adjacent to x_0^0 and y_{i-1}^0, b_{i-1} is adjacent to x_0^1 and y_{i-1}^1, a_{i-2} is adjacent to x_{i-1}^0 and y_{i-2}^0, b_{i-2} is adjacent to x_{i-1}^1 and y_{i-2}^1, c_{i-1} is adjacent to a_{i-1}, b_{i-1} and v_{i-3}, the other edges of $\{G; G_0, G_1, \ldots, G_{i-2}\}$ remain the same.

(v) $\{G; G_0, G_1, \ldots, G_{i-1}\} \in E_{0,1,\ldots,(i-1)}$.

If $k = 3$, then we obtain the class of graphs constructed by Esperet et al. [4]. If we ignore the excessive index and non 3-edge-colorability of G_i ($i = 0, 1, 2, \ldots, (k-1)$) and only assume that G_i has a Fulkerson-cover, then we obtain infinite families of bridgeless cubic graphs. We denote graphs of this type as $M_{0,1,2,\ldots,(k-1)}$ ($k \geq 2$).

Let $\{G; G_0, G_1, G_2, G_3\}$ be a graph in $E_{0,1,2,3}$. We consider how each perfect matching M of $\{G; G_0, G_1, G_2, G_3\}$ intersects $\partial_G(H_i)$ (see Fig. 1). Since $|\partial_G(H_i)| = 4$, by Lemma 2.1, we have that $|M \cap \partial_G(H_i)|$ is even. If $|M \cap \partial_G(H_i)| = 0$, then we say that M is of type 0 on H_i. If $|M \cap \partial_G(H_i)| = 2$, then we consider two cases: we say that M is of type 1 on H_i if $|M \cap \partial_G(H_i, A_i)| = |M \cap \partial_G(H_i, A_{i-1})| = 1$, while M is of type 2 on H_i, otherwise. If $|M \cap \partial_G(H_i)| = 4$, then we say that M is of type 4 on H_i. By observation, it's easy to obtain the following propositions.

Proposition 2.2. *If a perfect matching M contains uc_0, vc_1 (uc_3, vc_2), then at least one of the following holds:*

(1). M is of type 4 on H_1 (H_3), type 0 on H_0, H_2, type 1 on H_3 (H_1).
(2). M is of type 2 on H_0, H_1 (H_3), type 0 on H_2, type 1 on H_3 (H_1).
(3). M is of type 2 on H_1 (H_3), H_2, type 0 on H_0, type 1 on H_3 (H_1).
(4). M is of type 2 on H_0, H_2, type 0 on H_1 (H_3), type 1 on H_3 (H_1).
(5). M is of type 1 on H_0, H_1 (H_3), H_2, type 0 on H_3 (H_1).

Proposition 2.3. *If a perfect matching M contains uc_0, vc_2 (uc_3, vc_1), then at least one of the following holds:*

(1). M is of type 2 on H_0, type 0 on H_1 (H_3), type 1 on H_2, H_3 (H_1).
(2). M is of type 2 on H_1 (H_3), type 0 on H_0, type 1 on H_2, H_3 (H_1).
(3). M is of type 2 on H_3 (H_1), type 0 on H_2, type 1 on H_0, H_1 (H_3).
(4). M is of type 2 on H_2, type 0 on H_3 (H_1), type 1 on H_0, H_1 (H_3).

Proposition 2.4. *If a perfect matching M contains uv, then at least one of the following holds:*

(1). *M is of type 1 on H_0, H_2, type 0 on H_1, H_3.*
(2). *M is of type 1 on H_1, H_3, type 0 on H_0, H_2.*

It's easy to see that each perfect matching of type 0 on H_i corresponds to a perfect matching of G_i containing x_iy_i, while each perfect matching of type 1 on H_i corresponds to a perfect matching of G_i avoiding x_iy_i. Thus, we obtain the following proposition.

Proposition 2.5 (Esperet and Mazzuoccolo [4]). *If $\{G; G_0, G_1, G_2, G_3\}$ can be covered by k perfect matchings, and each of type 0 or 1 (not all of type 1) on H_i, for some $i \in \{0, 1, 2, 3\}$, then G_i can be covered by k perfect matchings.*

3. Each graph in $E_{0,1,2,3}$ has excessive index at least 5

From the construction of $E_{0,1,\dots,(k-1)}$, we have the following theorem.

Theorem 3.1. *Each graph in $E_{0,1,\dots,(k-1)}$ is a snark.*

Proof. If not, suppose that $\{G; G_0, G_1, \dots, G_{k-2}, G_{k-1}\} \in E_{0,1,\dots,(k-1)}$ has a 3-edge-coloring $\{M_1, M_2, M_3\}$. If M_1 is of type 2 or 4 on H_i, for some $i \in \{0, 1, 2, \dots, (k-1)\}$, without loss of generality, suppose that $|M_1 \cap \partial_G(H_i, A_i)| = 2$, then by the construction, $|M_1 \cap \partial_G(H_{i+1}, A_i)| = 0$, $|M_2 \cap \partial_G(H_{i+1}, A_i)| = |M_3 \cap \partial_G(H_{i+1}, A_i)| = 1$. By Lemma 2.1, both M_2 and M_3 are of type 1 on H_{i+1}, M_1 is of type 0 on H_{i+1}. By Proposition 2.5, G_{i+1} is 3-edge-colorable, a contradiction. Thus, M_j is of type 1 or 0 on H_i $(j = 1, 2, 3)$. But now by Lemma 2.1, we have that there exists an M_l $(l \in \{1, 2, 3\})$, such that M_l is of type 0 on H_i and the other two perfect matchings are of type 1 on H_i. Now by Proposition 2.5, G_i is 3-edge-colorable, a contradiction. □

From Theorem 3.1, it's easy to obtain the following theorem.

Theorem 3.2. *If $\{G; G_0, G_1, \dots, G_{k-2}, G_{k-1}\} \in E_{0,1,\dots,(k-1)}$, then the graph $\{G; G_0, G_1, \dots, G_{k-2}, G_{k-1}\}$ is a cyclically 4-edge-connected snark.*

Now we analyze the excessive index of $E_{0,1,\dots,(k-1)}$. First we consider the case $k = 2$.

Question 3.1. *If $\{G; G_0, G_1\} \in E_{0,1}$, then $\chi'_e(\{G; G_0, G_1\}) \geq 5$?*

Answer. The answer is no. Since if both G_0 and G_1 are the copies of Petersen graph, then $\{G; G_0, G_1\}$ has a perfect matching M_1, such that $E(\{G; G_0, G_1\}) - M_1$ is a set of two disjoint circuits C_0 and C_1, each of which contains 11 vertices. Furthermore, C_i contains all the vertices of $H_i \cup \{a_i, b_i, c_i\}$ for $i = 0, 1$. Let M_2 be a perfect matching of $\{G; G_0, G_1\}$ satisfying $x_0^0 a_1 \in M_2$ and $M_2 \setminus x_0^0 a_1 \subseteq E(C_0 \cup C_1)$. Let M_3 be a perfect matching of $\{G; G_0, G_1\}$ satisfying $a_0 x_1^0 \in M_3$ and $M_3 \setminus a_0 x_1^0 \subseteq E(C_0 \cup C_1)$. Let M_4 be a perfect matching of $\{G; G_0, G_1\}$ satisfying $c_0 c_1 \in M_4$ and $M_4 \setminus c_0 c_1 \subseteq E(C_0 \cup C_1)$. It's easy to verify that $\{G; G_0, G_1\}$ can be covered by $\{M_1, M_2, M_3, M_4\}$. Thus $\chi'_e(\{G; G_0, G_1\}) = 4$.

Esperet et al. [4] proved that for every graph $G \in E_{0,1,2}$, $\chi'_e(G) \geq 5$.

For the case $k = 4$, we have the following theorem.

Theorem 3.3. *If $\{G; G_0, G_1, G_2, G_3\} \in E_{0,1,2,3}$, then $\chi'_e(\{G; G_0, G_1, G_2, G_3\}) \geq 5$.*

Proof. If not, suppose that $\{G; G_0, G_1, G_2, G_3\} \in E_{0,1,2,3}$ is a counterexample, then by Theorem 3.1, $\chi'_e(\{G; G_0, G_1, G_2, G_3\}) = 4$. Assume that $\mathcal{F} = \{M_1, M_2, M_3, M_4\}$ is the minimum perfect matching cover of the graph $\{G; G_0, G_1, G_2, G_3\}$.

Claim 3.1. $\mathcal{F}$ has at most one element of type 4.

Proof. If not, without loss of generality, suppose that M_1 and M_2 are of type 4, then by Proposition 2.2 (1), M_1, M_2 are of type 0 on H_0 and H_2. By Proposition 2.5, M_3 and M_4 must be of type 2 on H_0 and H_2. But now uv can't be covered by $\mathcal{F}$, a contradiction. □

Claim 3.2. $\mathcal{F}$ has no element of type 4.

Proof. If not, without loss of generality, suppose that M_1 is of type 4 on H_1, then by Proposition 2.2 (1), M_1 is of type 0 on H_0, H_2, type 1 on H_3. Since $\mathcal{F}$ is the minimum perfect matching cover of $\{G; G_0, G_1, G_2, G_3\}$, without loss of generality, suppose that $uv \in M_2$. By Proposition 2.4, either M_2 is of type 1 on H_0, H_2, type 0 on H_1, H_3 or M_2 is of type 1 on H_1, H_3, type 0 on H_0, H_2.

If M_2 is of type 1 on H_1, H_3, type 0 on H_0, H_2, then by Proposition 2.5, M_3 and M_4 must be of type 2 on H_0, H_2. Now in this situation $\chi'_e(G_3) \leq 4$, a contradiction. Thus M_2 is of type 1 on H_0, H_2, type 0 on H_1, H_3. But now M_3 and M_4 are of type 0 on H_1. Otherwise either $\partial(H_i)$ can't be covered by $\mathcal{F}$ or $\chi'_e(G_i) \leq 4$, for some $i \in \{0, 2, 3\}$, a contradiction. Now by Propositions 2.2 (4)(5), 2.3 (1)(4) and 2.4 (1), each of M_3 and M_4 is of type 1 or 0 on H_3. Thus $\chi'_e(G_3) \leq 4$, a contradiction. □

Claim 3.3. Every element of $\mathcal{F}$ containing uv can't be of type 1 on H_1, H_3, type 0 on H_0, H_2.

Proof. If not, then assume that $uv \in M_1$ and M_1 is of type 1 on H_1, H_3, type 0 on H_0, H_2. Now there is at most one perfect matching of type 0 on H_1 or H_3. Since otherwise either $\partial_G(H_i)$ can't be covered by $\mathcal{F}$ or $\chi'_e(G_i) \leq 4$, for some $i \in \{1, 3\}$, a contradiction. By Propositions 2.2-2.4, there are at least two perfect matchings of type 0 on H_0 or H_2. But if there are 3 perfect matchings of type 0 on H_0 or H_2, then $\partial_G(H_0)$ or $\partial_G(H_2)$ can't be covered by $\mathcal{F}$, a contradiction. Thus there are exactly 2 perfect matchings of type 0 on H_0 or H_2. Without loss of generality, suppose that M_1 and M_2 are of type 0 on H_0. By Proposition 2.5, M_3 and M_4 are of type 2 on H_0.

If M_3 or M_4 is of type 2 on H_1 or H_3, then it's of type 0 on H_2. By Proposition 2.5, M_2 and M_4 or M_2 and M_3 are of type 2 on H_2. By relabelling, we may assume that M_2 and M_3 are of type 2 on H_2. Now M_2 is of type 2 on H_2, type 0 on H_0, M_3 is of type 2 on H_0, H_2, M_4 is of type 2 on H_0, H_1 or H_0, H_3. But now either $\partial_G(H_2)$ can't be covered by $\mathcal{F}$ or $\chi'_e(G_i) \leq 4$, for some $i \in \{1, 3\}$. Thus M_3 and M_4 can't be of type 2 on H_1 or H_3. But now, by Propositions 2.2-2.4, we have that each of M_3 and M_4 is either of type 1 on H_1, type 0 on H_3 or of type 0 on H_1, type 1 on H_3. By Proposition 2.5, we have that either M_2 is of type 2 on H_1 and H_3 or $\chi'_e(G_i) \leq 4$, for some $i \in \{1, 3\}$, a contradiction.

□

By Claim 3.2, $\mathcal{F}$ has no perfect matching of type 4. Since $\mathcal{F}$ is the minimum perfect matching cover of $\{G;\, G_0,\, G_1,\, G_2,\, G_3\}$, without loss of generality, suppose that $uv \in M_1$. By Proposition 2.4, either M_1 is of type 1 on H_1, H_3, type 0 on H_0, H_2 or M_1 is of type 1 on H_0, H_2, type 0 on H_1, H_3. By Claim 3.3, M_1 is of type 1 on H_0, H_2, type 0 on H_1, H_3. Similar to the proof of Claim 3.3, there are two perfect matchings of type 0 on H_1 or H_3. Suppose that M_1 and M_2 are of type 0 on H_1. By Proposition 2.5, M_3 and M_4 are of type 2 on H_1. Now by Propositions 2.2 (2)(3), 2.3 (2)(3), M_3 and M_4 are of type 1 on H_3. But now by Proposition 2.5, we have that M_2 is of type 2 on H_3, type 0 on H_1, a contradiction. Since this type of perfect matchings don't exist.

Therefore M_1 can't be of type 1 on H_0, H_2, type 0 on H_1, H_3, a contradiction to Proposition 2.4.

□

Theorem 3.3 gives a strongly negative answer to Question 1.1. It's natural to propose the following question.

Question 3.2. *If* $\{G;\, G_0, \ldots, G_{k-2},\, G_{k-1}\} \in E_{0,1,\ldots,(k-1)}$ $(k \geq 3)$*, then* $\chi'_e(\{G;\, G_0, \ldots, G_{k-2},\, G_{k-1}\}) \geq 5$*?*

4. Each graph in $M_{0,1,2,3}$ has a Fulkerson-cover

A *cycle* of G is a subgraph of G with each vertex of even degree. A *circuit* of G is a minimal 2-regular cycle of G.

The following theorem, due to Hao et al. [8], is very important in our main proof.

Theorem 4.1 (Hao, Niu, Wang, Zhang and Zhang [8])**.** *A bridgeless cubic graph G has a Fulkerson-cover if and only if there are two disjoint matchings E_1 and E_2, such that $E_1 \cup E_2$ is a cycle and $\overline{G \setminus E_i}$ is 3-edge colorable, for each $i = 1, 2$.*

Theorem 4.2. *If $\{G;\, G_0,\, G_1,\, G_2,\, G_3\} \in M_{0,1,2,3}$, then $\{G;\, G_0,\, G_1,\, G_2,\, G_3\}$ has a Fulkerson-cover.*

Proof. Since G_i has a Fulkerson-cover, for each $i = 0, 1, 2, 3$, suppose that $M_i^1, M_i^2, \ldots, M_i^6$ is the Fulkerson-cover of G_i. Let E_2^i be the set of edges covered twice by M_i^1, M_i^2, M_i^3, E_0^i be the set of edges not covered by M_i^1, M_i^2, M_i^3. Now $E_2^i \cup E_0^i$ is an even cycle, and $\overline{\{G;\, G_0,\, G_1,\, G_2,\, G_3\} \setminus E_2^i}$ can be colored by three colors 4, 5, 6, $\overline{\{G;\, G_0,\, G_1,\, G_2,\, G_3\} \setminus E_0^i}$ can be colored by three colors 1, 2, 3. Then E_2^i, E_0^i are the desired disjoint matchings as in Theorem 4.1. By choosing three perfect matchings of G_i, we could obtain two desired disjoint matchings E_2^i, E_0^i, such that either $x_i,\, y_i \in E_2^i \cup E_0^i$ or $x_i, y_i \notin E_2^i \cup E_0^i$. Now for each $i = 0$, 2 we choose three perfect matchings of G_i, such that $x_i,\, y_i \notin E_2^i \cup E_0^i$. For each $i = 1$, 3, we choose three perfect matchings of G_i, such that $x_i,\, y_i \in E_2^i \cup E_0^i$. Suppose that $x_1^0 x_1,\ y_1^0 y_1,\ x_3^0 x_3,\ y_3^1 y_3 \in E_2^i \cup E_0^i$. Replace $x_1^0 x_1$

and $y_3^1y_3$ by $x_1^0a_0c_0uc_3b_3y_3^1$, and replace $y_1^0y_1$ and $x_3^0x_3$ by $y_1^0a_1c_1vc_2a_2x_3^0$. Let C be the resulting cycle of $\{G; G_0, G_1, G_2, G_3\}$ through the above operation. Let E_1 and E_2 be two disjoint perfect matchings of C. It's easy to verify that $\overline{\{G; G_0, G_1, G_2, G_3\} \setminus E_i}$ is 3-edge colorable, for each $i = 1, 2$. Therefore by Theorem 4.1, $\{G; G_0, G_1, G_2, G_3\}$ has a Fulkerson-cover.

□

Similar to the proof of Theorem 4.2, we have the following theorem.

Theorem 4.3. *If $\{G; G_0, G_1\} \in M_{0,1}$, then $\{G; G_0, G_1\}$ has a Fulkerson-cover.*

Proof. Since G_i has a Fulkerson-cover, for each $i = 0, 1$, suppose that $M_i^1, M_i^2, \ldots, M_i^6$ is the Fulkerson-cover of G_i. Let E_2^i be the set of edges covered twice by $M_i^1,\ M_i^2,\ M_i^3$, E_0^i be the set of edges not covered by $M_i^1,\ M_i^2,\ M_i^3$, now $E_2^i \cup E_0^i$ is an even cycle, and $\overline{\{G; G_0, G_1\} \setminus E_2^i}$ can be colored by three colors 4, 5, 6, $\overline{\{G; G_0, G_1\} \setminus E_0^i}$ can be colored by three colors 1, 2, 3. Then E_2^i, E_0^i are the desired disjoint matchings as in Theorem 4.1. By choosing three perfect matchings of G_i, we could obtain two desired disjoint matchings E_2^i, E_0^i, such that $x_i,\ y_i \in E_2^i \cup E_0^i$. Now for each $i = 0,\ 1$, we choose three perfect matchings of G_i, such that $x_i,\ y_i \in E_2^i \cup E_0^i$. Suppose that $y_0^0y_0,\ x_1^0x_1,\ y_1^1y_1,\ x_0^1x_0 \in E_2^i \cup E_0^i$. Replace $y_0^0y_0$ and $x_1^0x_1$ by $x_1^0a_0y_0^0$ and replace $y_1^1y_1$ and $x_0^1x_0$ by $y_1^1b_1x_0^1$. Let C be the resulting cycle of $\{G; G_0, G_1\}$ through the above operation. Let E_1 and E_2 be two disjoint perfect matchings of C. It's easy to verify that $\overline{\{G; G_0, G_1\} \setminus E_i}$ is 3-edge colorable, for each $i = 1, 2$. Therefore by Theorem 4.1, $\{G; G_0, G_1\}$ has a Fulkerson-cover.

□

Since for $k = 2$ (by Theorem 4.3), $k = 3$ (Esperet et al. [4]) and $k = 4$ (by Theorem 4.2), $M_{0,1,2,\ldots,(k-1)}$ has a Fulkerson-cover. Thus it's natural to consider the following question.

Question 4.1. *If $\{G; G_0, G_1, \ldots, G_{k-1}\} \in M_{0,1,2,\ldots,(k-1)}$, then the graph $\{G; G_0, G_1, \ldots, G_{k-1}\}$ has a Fulkerson-cover?*

5. Proof of Theorem 1.2

In order to prove the main result, we first recall the following theorem that is important in our proof.

Theorem 5.1 (Robertson, Sanders, Seymour and Thomas [12])**.** *Let G be a bridgeless cubic graph. If G doesn't have Petersen graph as a minor, then G is 3-edge-colorable.*

1.2 (1). Suppose that $N(v) = \{v_1, v_2, v_3\}$ and $\{M_1, M_2, M_3\}$ is the 3-edge-coloring of $\overline{G \setminus v}$, such that $M_1 \cup M_2$, $M_1 \cup M_3$ and $M_2 \cup M_3$ are all Hamilton circuits.

If vv_1v_2v is a triangle of G, then since $\{M_1, M_2, M_3\}$ is the 3-edge-coloring of $\overline{G \setminus v}$, and $M_1 \cup M_2$, $M_1 \cup M_3$, $M_2 \cup M_3$ are all Hamilton circuits, we have that G has a Hamilton circuit. Thus G is 3-edge-colorable and therefore admits a Fulkerson-cover. So suppose that v is in no triangle of G.

Let a, b, c be the edges obtained from $G \setminus v$ by contracting v_1, v_2, v_3, respectively.

If $a \in M_1$, $b \in M_2$, $c \in M_3$, then let $C_1 = M_1 \cup M_2$, $C_2 = M_1 \cup M_3$, $C_3 = M_2 \cup M_3$. Let C_1' be the graph obtained from C_1 by inserting v_1 into a and v_2 into b. Let C_2' be the graph obtained from C_2 by inserting v_1 into a and v_3 into c. Let C_3' be the graph obtained from C_3 by inserting v_2 into b and v_3 into c. Now C_1', C_2' and C_3' are all circuits of length $|V(G)| - 2$ in G. Let M_1' and M_2' be two disjoint perfect matchings of C_1', M_3' and M_4' be two disjoint perfect matchings of C_2', M_5' and M_6' be two disjoint perfect matchings of C_3'. Now $\{M_1' \cup \{vv_3\}, M_2' \cup \{vv_3\}, M_3' \cup \{vv_2\}, M_4' \cup \{vv_2\}, M_5' \cup \{vv_1\}, M_6' \cup \{vv_1\}\}$ is a Fulkerson-cover of G.

If $a \in M_1$, $b \in M_2$, $c \in M_2$, then let $C = M_1 \cup M_2$ and C_1 be the graph obtained from C by inserting v_1 into a, v_2 into b and v_3 into c. Let $P(v_1, v_2)$ be a segment between v_1 and v_2 in C_1, such that $v_3 \notin P(v_1, v_2)$. Let $C_2 = vv_1P(v_1, v_2)v_2v$. Now the length of C_2 is even. Let E_1 and E_2 be two disjoint perfect matchings of C_2. Suppose that $E_1 \cap M_1 \neq \emptyset$, then $E_1 \cap M_2 = \emptyset$, $E_2 \cap M_2 \neq \emptyset$, and $E_2 \cap M_1 = \emptyset$. Now both $\overline{G \setminus E_1}$ and $\overline{G \setminus E_2}$ are bridgeless, since $M_2 \cup M_3$ and $M_1 \cup M_3$ are Hamilton circuits. Since $G \setminus vv_i$ $(i = 1, 2, 3)$ doesn't have Petersen graph as a minor, both $\overline{G \setminus E_1}$ and $\overline{G \setminus E_2}$ don't have Petersen graph as a minor. By Theorem 5.1, both $\overline{G \setminus E_1}$ and $\overline{G \setminus E_2}$ are 3-edge-colorable. Therefore, by Theorem 4.1, G has a Fulkerson-cover.

If a, b, $c \in M_1$, then $M_2 \cup M_3$ is an even circuit of G. Let E_1 be the graph obtained from M_1 by inserting v_1 into a, v_2 into b and v_3 into c. Since $E_1 \cup M_{5-i}$ is in $G \setminus M_i$ $(i = 2, 3)$, we have that $\overline{G \setminus M_i}$ is bridgeless and has at most 4 vertices of degree 3. By Theorem 5.1, $\overline{G \setminus M_i}$ is 3-edge-colorable. Therefore, by Theorem 4.1, G has a Fulkerson-cover. □

By Theorem 1.2 (1), we obtain the following corollary.

Corollary 5.2. *Let G be a bridgeless cubic graph. If there exists a vertex $v \in V(G)$ such that $G \setminus e$ doesn't have Petersen graph as a minor for each edge e incident with v and $\overline{G \setminus v}$ is uniquely 3-edge-colorable, then G has a Fulkerson-cover.*

Proof. Suppose that $\{M_1, M_2, M_3\}$ is the uniquely 3-edge-coloring of $\overline{G \setminus v}$. We claim that $M_1 \cup M_2$, $M_1 \cup M_3$ and $M_2 \cup M_3$ are all Hamilton circuits. Since if $M_1 \cup M_2$ isn't a Hamilton circuit, then $M_1 \cup M_2$ has another 2-edge-coloring M_1' and M_2'. Now $\{M_1', M_2', M_3\}$ is a 3-edge-coloring of $\overline{G \setminus v}$, which is different from $\{M_1, M_2, M_3\}$, a contradiction. Therefore, by Theorem 1.2 (1), G has a Fulkerson-cover. □

Proof of Theorem 1.2 (2). Suppose that $e = v_1v_2 \in E(G)$ and $\{M_1, M_2, M_3\}$ is the 3-edge-coloring of $\overline{G \setminus e}$, such that $M_1 \cup M_2$, $M_1 \cup M_3$ and $M_2 \cup M_3$ are all Hamilton circuits. Let a and b be the edges of $\overline{G \setminus e}$ obtained from $G - e$ by contracting v_1 and v_2, respectively.

If a, b are in the same matching M_i ($i \in \{1, 2, 3\}$), then without loss of generality, suppose that $a, b \in M_1$. Let C be the graph obtained from $M_1 \cup M_2$ by inserting v_1 into a and v_2 into b. Now C is a Hamilton circuit of G. Thus G is 3-edge-colorable and therefore G has a Fulkerson-cover.

If a, b aren't in the same matching M_i ($i \in \{1, 2, 3\}$), then without loss of generality, suppose that $a \in M_1$ and $b \in M_2$. Let C be the graph obtained from $M_1 \cup M_2$ by inserting v_1 into a and v_2 into b. Now C is a Hamilton circuit of G. Thus G is 3-edge-colorable and therefore G has a Fulkerson-cover. $\square$

Proof of Theorem 1.2 (3). If G itself doesn't have Petersen graph as a minor, then by Theorem 5.1, G is 3-edge-colorable. Therefore G has a Fulkerson-cover. So suppose that G has Petersen graph as a minor. But now, by assumption, G is Petersen graph. It's easy to check that Petersen graph satisfies the first condition of Theorem 1.2. Therefore G has a Fulkerson-cover. $\square$

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