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FULKERSON CONJECTURE

F. CHEN

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Abstract. The excessive index of a bridgeless cubic graph $G$ is the least integer $k$, such that $G$ can be covered by $k$ perfect matchings. An equivalent form of Fulkerson conjecture (due to Berge) is that every bridgeless cubic graph has excessive index at most five. Clearly, Petersen graph is a cyclically 4-edge-connected snark with excessive index at least five, so Fouquet and Vanherpe asked whether Petersen graph is the only one with that property. Håggkvist gave a negative answer to their question by constructing two graphs Blowup($K_4$, $C$) and Blowup($Prism$, $C_4$). Based on the first graph, Esperet et al. constructed infinite families of cyclically 4-edge-connected snarks with excessive index at least five. Based on these two graphs, we construct infinite families of cyclically 4-edge-connected snarks $E_{0,1,2,...,(k-1)}$ in which $E_{0,1,2}$ is Esperet et al.’s construction. In this note, we prove that $E_{0,1,2,3}$ has excessive index at least five, which gives a strongly negative answer to Fouquet and Vanherpe’s question.

As a subcase of Fulkerson conjecture, Håggkvist conjectured that every cubic hypohamiltonian graph has a Fulkerson-cover. Motivated by a related result due to Hou et al.’s, in this note we prove that Fulkerson conjecture holds on some families of bridgeless cubic graphs.

Keywords: Fulkerson-cover, excessive index, snark, hypohamiltonian graph.

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1. Introduction

Let $G$ be a simple graph (without loops or parallel edges) with vertex set $V(G)$ and edge set $E(G)$. A perfect matching of $G$ is a 1-regular spanning subgraph of $G$. The excessive index of $G$ (first introduced by Bonisoli and Cariolaro [3]), denoted by $\chi'_e(G)$, is the least integer $k$, such that $G$ can be covered by $k$ perfect matchings. We call these $k$ perfect matchings as the minimum perfect matching cover of $G$.

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The following conjecture is due to Berge and Fulkerson, and first appeared in [6].

**Conjecture 1.1** (Fulkerson conjecture, Fulkerson [6]). If $G$ is a bridgeless cubic graph, then $G$ can be covered by six perfect matchings such that each edge is in exactly two of them.

We call such 6 perfect matchings as the *Fulkerson-cover*. If Fulkerson conjecture is true, then deleting one perfect matching from the Fulkerson-cover would result in a covering of the graph by 5 perfect matchings. Thus, Berge conjectured that (unpublished and first appeared in [13])

**Conjecture 1.2** (Berge, unpublished and first appeared in [13]). If $G$ is a bridgeless cubic graph, then $\chi'_e(G) \leq 5$.

Mazzuoccolo [10] proved that Conjectures 1.1 and 1.2 are equivalent. But on a given graph, the equivalence of these two conjectures has not been proved.

A graph $G$ is called *cyclically k-edge-connected* if at least $k$ edges must be removed to disconnect it into two components, each of which contains a circuit.

Obviously, Conjectures 1.1 and 1.2 hold on 3-edge-colorable cubic graphs. So in this note, we only consider bridgeless non 3-edge-colorable cubic graphs, which are called *snarks*. For more details, see the book written by Zhang [14]. Fouquet and Vanherpe [5] proved that there are several infinite families of cyclically 3-edge-connected snarks with excessive index at least five. But for cyclically 4-edge-connected snarks, they only know Petersen graph. They proposed the following question.

**Question 1.1** (Fouquet and Vanherpe [5]). If $G$ is a cyclically 4-edge-connected snark, then either $G$ is Petersen graph or $\chi'_e(G) < 5$.

Häggblund [7] gave a negative answer to Question 1.1 by constructing two graphs Blowup($K_4, C$) and Blowup($Prism, C_4$). Based on Blowup($K_4, C$), Esperet et al. [4] constructed infinite families of cyclically 4-edge-connected snarks with excessive index at least five. Based on these two graphs, in Section 2, we construct infinite families of bridgeless cubic graphs $M_{0,1,2,\ldots,(k-1)}$ and infinite families of cyclically 4-edge-connected snarks $E_{0,1,2,\ldots,(k-1)}$ ($k \geq 2$) where $E_{0,1,2}$ is Esperet et al.’s [4] construction.

In Section 3, we prove that each graph in $E_{0,1,2,3}$ (see Fig. 1) has excessive index at least five. This gives a strongly negative answer to Question 1.1. In Section 4, we prove that each graph in $M_{0,1,2,3}$ has a Fulkerson-cover.

Let $X \subseteq V(G)$ and $e = uv \in E(G)$. We use $G \setminus X$ to denote the subgraph of $G$ obtained from $G$ by deleting all the vertices of $X$ and all the edges incident with $X$. Moreover if $X = \{x\}$, we simply write $G \setminus x$. Similarly, we use $G \setminus e$ to denote the subgraph of $G$ obtained from $G$ by deleting $e$. A *minor* of $G$ is any graph obtained from $G$ by means of a sequence of vertex and edge deletions and edge contractions. According to Hao et al. [8] and Hou et al. [9], we use $\overline{G}$
to denote the graph obtained from \( G \) by contracting all the vertices of degree 2.

A graph \( G \) is called hypohamiltonian if \( G \) itself doesn’t have Hamilton circuits but \( G \setminus v \) does for each vertex \( v \in V(G) \). A graph \( G \) is called Kotzig if \( G \) has a 3-edge-coloring, each pair of which form a Hamilton circuit (the definition is defined by Häggkvist and Markström).

The research on Fulkerson conjecture has attracted more and more graph theorists, and in particular, Häggkvist [11] proposed the following conjecture in 2007.

**Conjecture 1.3** (Häggkvist [11]). *If \( G \) is a cubic hypohamiltonian graph, then \( G \) has a Fulkerson-cover.*

There is little progress on Conjecture 1.3. Recently, Hou et al. [9] partially solved Conjecture 1.3 in the following theorem.

**Theorem 1.1** (Hou, Lai and Zhang [9]). *Let \( G \) be a bridgeless cubic graph. If there exists a vertex \( v \in V(G) \) such that \( G \setminus v \) is a Kotzig graph, then \( \chi'(G) \leq 5 \).

Motivated by their results, in Section 5, we prove that

**Theorem 1.2.** *Let \( G \) be a bridgeless cubic graph. Then \( G \) has a Fulkerson-cover if one of the followings holds:

1. there exists a vertex \( v \in V(G) \) such that \( G \setminus v \) is a Kotzig graph and \( G \setminus e \) doesn’t have Petersen graph as a minor for each edge \( e \) incident with \( v \).
2. there exists an edge \( e \in E(G) \) such that \( G \setminus e \) is a Kotzig graph.
3. for each \( e \in E(G) \), \( G \setminus e \) doesn’t have Petersen graph as a minor.*

Note that our proof is independent of Hou et al.’s [9].

![Fig. 1](image-url)
2. Preliminaries

In this section, we will give some necessary definitions, constructions, lemmas and propositions.

Lemma 2.1 (Parity lemma, Blanuša [1]). Let $G$ be a cubic graph. If $M$ is a perfect matching of $G$ and $T$ an edge-cut of $G$, then $|M \cap T| \equiv |T| \pmod{2}$.

Let $X$ be a subset of $V(G)$. The edge-cut of $G$ associated with $X$, denoted by $\partial_X(G)$, is the set of edges of $G$ with exactly one end in $X$. The edge set $C = \partial_X(G)$ is called a $k$-edge-cut if $|\partial_X(G)| = k$.

Let $G_i$ be a cyclically 4-edge-connected snark with excessive index at least 5, for $i = 0, 1$. Let $x_i y_i$ be an edge of $G_i$ and $x_i^0, x_i^1, y_i^0, y_i^1$ the neighbours of $x_i$ ($y_i$). Let $H_i$ be the graph obtained from $G_i$ by deleting the vertices $x_i$ and $y_i$. Let $\{G; G_0, G_1\}$ be the graph obtained from the disjoint union of $G_0, H_1$ by adding six vertices $a_0, b_0, c_0, a_1, b_1, c_1$ and 13 edges $a_0 y_0^0, a_0 x_0^1, a_0 c_0, c_0 b_0, b_0 y_0^1, b_0 x_1^1, b_1 x_1^1, b_1 y_1^1, b_1 c_1, c_1 a_1, a_1 x_0^0, a_1 y_0^1, c_0 c_1$. We call graphs of this type as $E_{0,1}$ (see Fig. 2).

[Diagram of $E_{0,1}$]

Fig. 2

Now we construct $E_{0,1},...,E_{k-1}$ ($k \geq 2$) as follows:

1. $\{G; G_0, G_1\} \in E_{0,1}$ with $A_j = \{a_j, b_j, c_j\}$ for $j = 0, 1$.
2. For $3 \leq i \leq k$, $\{G; G_0, G_1, ..., G_{i-2}\}$ is obtained from $\{G; G_0, G_1, ..., G_{i-2}\}$ by adding $H_{i-1}$ and $A_{i-1} = \{a_{i-1}, b_{i-1}, c_{i-1}\}$ and by inserting a vertex $v_{i-3}$ into $e_0$, such that
   (i) $G_{i-1}$ is a cyclically 4-edge-connected snark with excessive index at least 5 ($x_{i-1} y_{i-1}$ is an edge of $G_{i-1}$ and $x_{i-1}^0, x_{i-1}^1, y_{i-1}^0, y_{i-1}^1$ are the neighbours of $x_{i-1}, y_{i-1}$);
   (ii) $H_{i-1} \subseteq G_{i-1} \setminus \{x_{i-1}, y_{i-1}\}$;
   (iii) $e_0 \in E((G; G_0, G_1, ..., G_{i-2})) - \cup_{j=0}^{i-2} E(H_j) - \cup_{j=0}^{i-2} \{a_j c_j, c_j b_j\}$ and $e_0$ is incident with $c_0$;
   (iv) $a_{i-1}$ is adjacent to $x_{i-1}^0$ and $y_{i-1}^0$, $b_{i-1}$ is adjacent to $x_{i-1}^0$ and $y_{i-1}^1$, $c_{i-1}$ is adjacent to $x_{i-1}^1$ and $y_{i-2}^1$, $b_{i-2}$ is adjacent to $x_{i-1}^1$ and $y_{i-2}^1$, $c_{i-1}$ is adjacent...
to $a_{i-1}, b_{i-1}$ and $v_{i-3}$, the other edges of \{G; G_0, G_1, \ldots, G_{i-2}\} remain the same.

(v) $\{G; G_0, G_1, \ldots, G_{i-1}\} \in E_{0,1,\ldots,(i-1)}$.

If $k = 3$, then we obtain the class of graphs constructed by Esperet et al. [4]. If we ignore the excessive index and non 3-edge-colorability of $G_i$ ($i = 0, 1, 2, \ldots, (k - 1)$) and only assume that $G_i$ has a Fulkerson-cover, then we obtain infinite families of bridgeless cubic graphs. We denote graphs of this type as $M_{0,1,2,\ldots,(k-1)}$ ($k \geq 2$).

Let $\{G; G_0, G_1, G_2, G_3\}$ be a graph in $E_{0,1,2,3}$. We consider how each perfect matching $M$ of $\{G; G_0, G_1, G_2, G_3\}$ intersects $\partial_G(H_i)$ (see Fig. 1). Since $|\partial_G(H_i)| = 4$, by Lemma 2.1, we have that $|M \cap \partial_G(H_i)|$ is even. If $|M \cap \partial_G(H_i)| = 0$, then we say that $M$ is of type 0 on $H_i$. If $|M \cap \partial_G(H_i)| = 2$, then we consider two cases: we say that $M$ is of type 1 on $H_i$ if $|M \cap \partial_G(H_i, A_i)| = |M \cap \partial_G(H_i, A_{i-1})| = 1$, while $M$ is of type 2 on $H_i$, otherwise. If $|M \cap \partial_G(H_i)| = 4$, then we say that $M$ is of type 4 on $H_i$. By observation, it’s easy to obtain the following propositions.

**Proposition 2.2.** If a perfect matching $M$ contains $uv_0$, $vc_1$ ($uc_3$, $vc_2$), then at least one of the following holds:

1. $M$ is of type 4 on $H_1$ ($H_3$), type 0 on $H_0$, $H_2$, type 1 on $H_3$ ($H_1$).
2. $M$ is of type 2 on $H_0$, $H_1$ ($H_3$), type 0 on $H_2$, type 1 on $H_3$ ($H_1$).
3. $M$ is of type 2 on $H_1$ ($H_3$), $H_2$, type 0 on $H_0$, type 1 on $H_3$ ($H_1$).
4. $M$ is of type 2 on $H_0$, $H_2$, type 0 on $H_1$ ($H_3$), type 1 on $H_3$ ($H_1$).
5. $M$ is of type 1 on $H_0$, $H_1$ ($H_3$), $H_2$, type 0 on $H_3$ ($H_1$).

**Proposition 2.3.** If a perfect matching $M$ contains $uv_0$, $vc_2$ ($uc_3$, $vc_1$), then at least one of the following holds:

1. $M$ is of type 2 on $H_0$, type 0 on $H_1$ ($H_3$), type 1 on $H_2$, $H_3$ ($H_1$).
2. $M$ is of type 2 on $H_1$ ($H_3$), type 0 on $H_0$, type 1 on $H_2$, $H_3$ ($H_1$).
3. $M$ is of type 2 on $H_3$ ($H_1$), type 0 on $H_2$, type 1 on $H_0$, $H_1$ ($H_3$).
4. $M$ is of type 2 on $H_2$, type 0 on $H_3$ ($H_1$), type 1 on $H_0$, $H_1$ ($H_3$).

**Proposition 2.4.** If a perfect matching $M$ contains $uv$, then at least one of the following holds:

1. $M$ is of type 1 on $H_0$, $H_2$, type 0 on $H_1$, $H_3$.
2. $M$ is of type 1 on $H_1$, $H_3$, type 0 on $H_0$, $H_2$.

It’s easy to see that each perfect matching of type 0 on $H_i$ corresponds to a perfect matching of $G_i$ containing $x_iy_i$, while each perfect matching of type 1 on $H_i$ corresponds to a perfect matching of $G_i$ avoiding $x_iy_i$. Thus, we obtain the following proposition.

**Proposition 2.5** (Esperet and Mazzuoccolo [4]). If $\{G; G_0, G_1, G_2, G_3\}$ can be covered by $k$ perfect matchings, and each of type 0 or 1 (not all of type 1) on $H_i$, for some $i \in \{0, 1, 2, 3\}$, then $G_i$ can be covered by $k$ perfect matchings.
3. Each graph in $E_{0,1,2,3}$ has excessive index at least $5$

From the construction of $E_{0,1,2,3}$, we have the following theorem.

**Theorem 3.1.** Each graph in $E_{0,1,2,3}$ is a snark.

**Proof.** If not, suppose that $\{G; G_0, G_1, \ldots, G_{k-2}, G_{k-1}\} \in E_{0,1,2,3}$ has a 3-edge-coloring $\{M_1, M_2, M_3\}$. If $M_1$ is of type 2 or 4 on $H_i$, for some $i \in \{0, 1, 2, \ldots, (k-1)\}$, without loss of generality, suppose that $|M_1 \cap \partial G(H_i, A_i)| = 2$, then by the construction, $|M_1 \cap \partial G(H_{i+1}, A_i)| = 0$, $|M_2 \cap \partial G(H_{i+1}, A_i)| = |M_3 \cap \partial G(H_{i+1}, A_i)| = 1$. By Lemma 2.1, both $M_2$ and $M_3$ are of type 1 on $H_{i+1}$, $M_1$ is of type 0 on $H_{i+1}$. By Proposition 2.5, $G_{i+1}$ is 3-edge-colorable, a contradiction. Thus, $M_1$ is of type 1 or 0 on $H_i$ ($j = 1, 2, 3$). But now by Lemma 2.1, we have that there exists an $M_l$ ($l \in \{1, 2, 3\}$), such that $M_l$ is of type 0 on $H_i$ and the other two perfect matchings are of type 1 on $H_i$. Now by Proposition 2.5, $G_l$ is 3-edge-colorable, a contradiction. \(\Box\)

From Theorem 3.1, it’s easy to obtain the following theorem.

**Theorem 3.2.** If $\{G; G_0, G_1, \ldots, G_{k-2}, G_{k-1}\} \in E_{0,1,2,3}$, then the graph $\{G; G_0, G_1, \ldots, G_{k-2}, G_{k-1}\}$ is a cyclically 4-edge-connected snark.

Now we analyze the excessive index of $E_{0,1,2,3}$. First we consider the case $k = 2$.

**Question 3.1.** If $\{G; G_0, G_1\} \in E_{0,1}$, then $\chi'_e(\{G; G_0, G_1\}) \geq 5$?

**Answer.** The answer is no. Since both $G_0$ and $G_1$ are the copies of Petersen graph, then $\{G; G_0, G_1\}$ has a perfect matching $M_1$, such that $E(\{G; G_0, G_1\}) = M_1$ is a set of two disjoint circuits $C_0$ and $C_1$, each of which contains 11 vertices. Furthermore, $C_i$ contains all the vertices of $H_i \cup \{a_i, b_i, c_i\}$ for $i = 0, 1$. Let $M_2$ be a perfect matching of $\{G; G_0, G_1\}$ satisfying $x_0^0x_1^0 \subseteq E(C_0 \cup C_1)$. Let $M_3$ be a perfect matching of $\{G; G_0, G_1\}$ satisfying $a_0x_1^0 \subseteq M_3$ and $M_3 \setminus a_0x_1^0 \subseteq E(C_0 \cup C_1)$. Let $M_4$ be a perfect matching of $\{G; G_0, G_1\}$ satisfying $c_0c_1 \subseteq M_4$ and $M_4 \setminus c_0c_1 \subseteq E(C_0 \cup C_1)$. It’s easy to verify that $\{G; G_0, G_1\}$ can be covered by $\{M_1, M_2, M_3, M_4\}$. Thus $\chi'_e(\{G; G_0, G_1\}) = 4$.

Esperet et al. [4] proved that for every graph $G \in E_{0,1,2,3}$, $\chi'_e(G) \geq 5$.

For the case $k = 4$, we have the following theorem.

**Theorem 3.3.** If $\{G; G_0, G_1, G_2, G_3\} \in E_{0,1,2,3}$, then $\chi'_e(\{G; G_0, G_1, G_2, G_3\}) \geq 5$.

**Proof.** If not, suppose that $\{G; G_0, G_1, G_2, G_3\} \in E_{0,1,2,3}$ is a counterexample, then by Theorem 3.1, $\chi'_e(\{G; G_0, G_1, G_2, G_3\}) = 4$. Assume that $F = \{M_1, M_2, M_3, M_4\}$ is the minimum perfect matching cover of the graph $\{G; G_0, G_1, G_2, G_3\}$.

**Claim 3.1.** $F$ has at most one element of type 4.
Proof. If not, without loss of generality, suppose that $M_1$ and $M_2$ are of type 4, then by Proposition 2.2 (1), $M_1$, $M_2$ are of type 0 on $H_0$ and $H_2$. By Proposition 2.5, $M_3$ and $M_4$ must be of type 2 on $H_0$ and $H_2$. But now $uv$ can’t be covered by $\mathcal{F}$, a contradiction. 

Claim 3.2. $\mathcal{F}$ has no element of type 4.

Proof. If not, without loss of generality, suppose that $M_1$ is of type 4 on $H_1$, then by Proposition 2.2 (1), $M_1$ is of type 0 on $H_0$, $H_2$, type 1 on $H_3$. Since $\mathcal{F}$ is the minimum perfect matching cover of $\{G; G_0, G_1, G_2, G_3\}$, without loss of generality, suppose that $uv \in M_2$. By Proposition 2.4, either $M_2$ is of type 1 on $H_0$, $H_2$, type 0 on $H_1$, $H_3$ or $M_2$ is of type 1 on $H_1$, $H_3$, type 0 on $H_0$, $H_2$.

If $M_2$ is of type 1 on $H_1$, $H_3$, type 0 on $H_0$, $H_2$, then by Proposition 2.5, $M_3$ and $M_4$ must be of type 2 on $H_0$, $H_2$. Now in this situation $\chi'_v(G_3) \leq 4$, a contradiction. Thus $M_2$ is of type 1 on $H_0$, $H_2$, type 0 on $H_1$, $H_3$. But now $M_3$ and $M_4$ are of type 0 on $H_1$. Otherwise either $\partial(H_2)$ can’t be covered by $\mathcal{F}$ or $\chi'_v(G_1) \leq 4$, for some $i \in \{0, 2, 3\}$, a contradiction. Now by Propositions 2.2 (4)(5), 2.3 (1)(4) and 2.4 (1), each of $M_3$ and $M_4$ is of type 1 or 0 on $H_2$. Thus $\chi'_v(G_3) \leq 4$, a contradiction.

Claim 3.3. Every element of $\mathcal{F}$ containing $uv$ can’t be of type 1 on $H_1$, $H_3$, type 0 on $H_0$, $H_2$.

Proof. If not, then assume that $uv \in M_1$ and $M_1$ is of type 1 on $H_1$, $H_3$, type 0 on $H_0$, $H_2$. Now there is at most one perfect matching of type 0 on $H_1$ or $H_3$. Since otherwise either $\partial_C(H_1)$ can’t be covered by $\mathcal{F}$ or $\chi'_v(G_i) \leq 4$, for some $i \in \{1, 3\}$, a contradiction. By Propositions 2.2-2.4, there are at least two perfect matchings of type 0 on $H_0$ or $H_2$. But if there are 3 perfect matchings of type 0 on $H_0$ or $H_2$, then $\partial_C(H_0)$ or $\partial_C(H_2)$ can’t be covered by $\mathcal{F}$, a contradiction. Thus there are exactly 2 perfect matchings of type 0 on $H_0$ or $H_2$. Without loss of generality, suppose that $M_1$ and $M_2$ are of type 0 on $H_0$. By Proposition 2.5, $M_3$ and $M_4$ are of type 2 on $H_0$.

If $M_3$ or $M_4$ is of type 2 on $H_1$ or $H_3$, then it’s of type 0 on $H_2$. By Proposition 2.5, $M_2$ and $M_4$ or $M_2$ and $M_3$ are of type 2 on $H_2$. By relabelling, we may assume that $M_2$ and $M_3$ are of type 2 on $H_2$. Now $M_2$ is of type 2 on $H_2$, type 0 on $H_0$, $M_3$ is of type 2 on $H_0$, $H_2$, $M_4$ is of type 2 on $H_0$, $H_1$ or $H_0$, $H_3$. But now either $\partial_C(H_2)$ can’t be covered by $\mathcal{F}$ or $\chi'_v(G_i) \leq 4$, for some $i \in \{1, 3\}$. Thus $M_3$ and $M_4$ can’t be of type 2 on $H_1$ or $H_3$. But now, by Propositions 2.2-2.4, we have that each of $M_3$ and $M_4$ is either of type 1 on $H_1$, type 0 on $H_3$ or of type 0 on $H_1$, type 1 on $H_3$. By Proposition 2.5, we have that either $M_2$ is of type 2 on $H_1$ and $H_3$ or $\chi'_v(G_i) \leq 4$, for some $i \in \{1, 3\}$, a contradiction.
By Claim 3.2, \( \mathcal{F} \) has no perfect matching of type 4. Since \( \mathcal{F} \) is the minimum perfect matching cover of \( \{G; G_0, G_1, G_2, G_3\} \), without loss of generality, suppose that \( uv \in M_1 \). By Proposition 2.4, either \( M_1 \) is of type 1 on \( H_1, H_3 \), type 0 on \( H_0, H_2 \) or \( M_1 \) is of type 1 on \( H_0, H_2 \), type 0 on \( H_1, H_3 \). By Claim 3.3, \( M_1 \) is of type 1 on \( H_0, H_2 \), type 0 on \( H_1, H_3 \). Similar to the proof of Claim 3.3, there are two perfect matchings of type 0 on \( H_1 \) or \( H_3 \). Suppose that \( M_1 \) and \( M_2 \) are of type 0 on \( H_1 \). By Proposition 2.5, \( M_3 \) and \( M_4 \) are of type 2 on \( H_1 \). Now by Propositions 2.2 (2)(3), 2.3 (2)(3), \( M_3 \) and \( M_4 \) are of type 1 on \( H_3 \). But now by Proposition 2.5, we have that \( M_2 \) is of type 2 on \( H_3 \), type 0 on \( H_1 \), a contradiction. Since this type of perfect matchings don’t exist.

Therefore \( M_1 \) can’t be of type 1 on \( H_0, H_2 \), type 0 on \( H_1, H_3 \), a contradiction to Proposition 2.4.

\[ \Box \]

Theorem 3.3 gives a strongly negative answer to Question 1.1. It’s natural to propose the following question.

**Question 3.2.** If \( \{G; G_0, \ldots, G_{k-2}, G_{k-1}\} \in E_{0,1,\ldots,(k-1)} \) \( (k \geq 3) \), then

\[ \chi_e(\{G; G_0, \ldots, G_{k-2}, G_{k-1}\}) \geq 5 \]

4. Each graph in \( M_{0,1,2,3} \) has a Fulkerson-cover

A cycle of \( G \) is a subgraph of \( G \) with each vertex of even degree. A circuit of \( G \) is a minimal 2-regular cycle of \( G \).

The following theorem, due to Hao et al. [8], is very important in our main proof.

**Theorem 4.1** (Hao, Niu, Wang, Zhang and Zhang [8]). A bridgeless cubic graph \( G \) has a Fulkerson-cover if and only if there are two disjoint matchings \( E_1 \) and \( E_2 \), such that \( E_1 \cup E_2 \) is a cycle and \( \overline{G \setminus E_i} \) is 3-edge colorable, for each \( i = 1, 2 \).

**Theorem 4.2.** If \( \{G; G_0, G_1, G_2, G_3\} \in M_{0,1,2,3} \), then \( \{G; G_0, G_1, G_2, G_3\} \) has a Fulkerson-cover.

**Proof.** Since \( G_i \) has a Fulkerson-cover, for each \( i = 0, 1, 2, 3 \), suppose that \( M_i^1, M_i^2, \ldots, M_i^6 \) is the Fulkerson-cover of \( G_i \). Let \( E_i^2 \) be the set of edges covered twice by \( M_i^1, M_i^2, M_i^3 \), \( E_i^0 \) be the set of edges not covered by \( M_i^1, M_i^2, M_i^3 \). Now \( E_i^2 \cup E_i^0 \) is an even cycle, and \( \{G; G_0, G_1, G_2, G_3\} \setminus E_i^2 \) can be colored by three colors 4, 5, 6, \( \{G; G_0, G_1, G_2, G_3\} \setminus E_i^0 \) can be colored by three colors 1, 2, 3. Then \( E_i^2, E_i^0 \) are the desired disjoint matchings as in Theorem 4.1. By choosing three perfect matchings of \( G_i \), we could obtain two desired disjoint matchings \( E_i^2, E_i^0 \), such that either \( x_i, y_i \in E_i^2 \cup E_i^0 \) or \( x_i, y_i \notin E_i^2 \cup E_i^0 \). Now for each \( i = 0, 2 \) we choose three perfect matchings of \( G_i \), such that \( x_i, y_i \notin E_i^2 \cup E_i^0 \). For each \( i = 1, 3 \), we choose three perfect matchings of \( G_i \), such that \( x_i, y_i \in E_i^2 \cup E_i^0 \). Suppose that \( x_i^1y_1, x_i^3y_3, x_3^2x_3, y_3^1y_3 \in E_i^2 \cup E_i^0 \). Replace \( x_i^1y_1 \)
and $y_3^4y_3$ by $x_2^3a_0c_0uc_3b_3y_3^4$, and replace $y_1^0y_1$ and $x_2^3x_3$ by $y_1^0a_1c_1ve_2a_2x_3^0$. Let $C$ be the resulting cycle of \{G; G_0, G_1, G_2, G_3\} through the above operation. Let $E_1$ and $E_2$ be two disjoint perfect matchings of $C$. It’s easy to verify that \{G; G_0, G_1, G_2, G_3\} \setminus E_i is 3-edge colorable, for each $i = 1, 2$. Therefore by Theorem 4.1, \{G; G_0, G_1, G_2, G_3\} has a Fulkerson-cover.

\[\square\]

Similar to the proof of Theorem 4.2, we have the following theorem.

**Theorem 4.3.** If \{G; G_0, G_1\} $\in M_{0,1}$, then \{G; G_0, G_1\} has a Fulkerson-cover.

**Proof.** Since $G_i$ has a Fulkerson-cover, for each $i = 0, 1$, suppose that $M_1^i, M_2^i, \ldots, M_6^i$ is the Fulkerson-cover of $G_i$. Let $E_2^i$ be the set of edges covered twice by $M_1^i, M_2^i, M_3^i$. $E_0^i$ be the set of edges not covered by $M_1^i, M_2^i, M_3^i$, now $E_2^i \cup E_0^i$ is an even cycle, and \{G; G_0, G_1\} \setminus E_2^i can be colored by three colors 4, 5, 6, \{G; G_0, G_1\} \setminus E_0^i can be colored by three colors 1, 2, 3. Then $E_2^i, E_0^i$ are the desired disjoint matchings as in Theorem 4.1. By choosing three perfect matchings of $G_i$, we could obtain two desired disjoint matchings $E_2^i, E_0^i$, such that $x_i, y_i \in E_2^i \cup E_0^i$. Now for each $i = 0, 1$, we choose three perfect matchings of $G_i$, such that $x_i, y_i \in E_2^i \cup E_0^i$. Suppose that $y_0^iy_0, x_0^iy_0, x_0^iy_0, x_0^iy_0 \in E_2^i \cup E_0^i$. Replace $y_0^iy_0$ and $x_0^iy_0$ by $y_0^ia_0y_0^i$ and replace $y_0^iy_0$ and $x_0^iy_0$ by $y_0^ib_1x_0^i$. Let $C$ be the resulting cycle of \{G; G_0, G_1\} through the above operation. Let $E_1$ and $E_2$ be two disjoint perfect matchings of $C$. It’s easy to verify that \{G; G_0, G_1\} \setminus E_i is 3-edge colorable, for each $i = 1, 2$. Therefore by Theorem 4.1, \{G; G_0, G_1\} has a Fulkerson-cover.

\[\square\]

Since for $k = 2$ (by Theorem 4.3), $k = 3$ (Esperet et al. [4]) and $k = 4$ (by Theorem 4.2), $M_{0,1,2,\ldots, (k-1)}$ has a Fulkerson-cover. Thus it’s natural to consider the following question.

**Question 4.1.** If \{G; G_0, G_1, \ldots, G_k-1\} $\in M_{0,1,2,\ldots, (k-1)}$, then the graph \{G; G_0, G_1, \ldots, G_k-1\} has a Fulkerson-cover?

5. **Proof of Theorem 1.2**

In order to prove the main result, we first recall the following theorem that is important in our proof.

**Theorem 5.1** (Robertson, Sanders, Seymour and Thomas [12]). Let $G$ be a bridgeless cubic graph. If $G$ doesn’t have Petersen graph as a minor, then $G$ is 3-edge-colorable.

**1.2 (1).** Suppose that $N(v) = \{v_1, v_2, v_3\}$ and $\{M_1, M_2, M_3\}$ is the 3-edge-coloring of $G \setminus v$, such that $M_1 \cup M_2, M_1 \cup M_3$ and $M_2 \cup M_3$ are all Hamilton circuits.
If \(vv_1v_2v\) is a triangle of \(G\), then since \(\{M_1, M_2, M_3\}\) is the 3-edge-coloring of \(G\setminus v\), and \(M_1 \cup M_2, M_1 \cup M_3, M_2 \cup M_3\) are all Hamilton circuits, we have that \(G\) has a Hamilton circuit. Thus \(G\) is 3-edge-colorable and therefore admits a Fulkerson-cover. So suppose that \(v\) is in no triangle of \(G\).

Let \(a, b, c\) be the edges obtained from \(G\setminus v\) by contracting \(v_1, v_2, v_3\), respectively.

If \(a \in M_1, b \in M_2, c \in M_3\), then let \(C_1 = M_1 \cup M_2, C_2 = M_1 \cup M_3, C_3 = M_2 \cup M_3\). Let \(C'_1\) be the graph obtained from \(C_1\) by inserting \(v_1\) into \(a\) and \(v_2\) into \(b\). Let \(C'_2\) be the graph obtained from \(C_2\) by inserting \(v_1\) into \(a\) and \(v_3\) into \(c\). Let \(C'_3\) be the graph obtained from \(C_3\) by inserting \(v_2\) into \(b\) and \(v_3\) into \(c\). Now \(C'_1, C'_2\) and \(C'_3\) are all circuits of length \(|V(G)| - 2\) in \(G\). Let \(M'_1\) and \(M'_2\) be two disjoint perfect matchings of \(C'_1, M'_3\) and \(M'_4\) to be two disjoint perfect matchings of \(C'_2, M'_5\) and \(M'_6\) be two disjoint perfect matchings of \(C'_3\). Now \(\{M'_1 \cup \{vv_3\}, M'_2 \cup \{vv_3\}, M'_3 \cup \{vv_2\}, M'_4 \cup \{vv_2\}, M'_5 \cup \{vv_1\}, M'_6 \cup \{vv_1\}\}\) is a Fulkerson-cover of \(G\).

If \(a \in M_1, b \in M_2, c \in M_3\), then let \(C = M_1 \cup M_2\) and \(C_1\) be the graph obtained from \(C\) by inserting \(v_1\) into \(a\), \(v_2\) into \(b\) and \(v_3\) into \(c\). Let \(P(v_1, v_2)\) be a segment between \(v_1\) and \(v_2\) in \(C_1\), such that \(v_3 \notin P(v_1, v_2)\). Let \(C_2 = vv_1P(v_1, v_2)v_2v\). Now the length of \(C_2\) is even. Let \(E_1\) and \(E_2\) be two disjoint perfect matchings of \(C_2\). Suppose that \(E_1 \cap M_1 \neq \emptyset\), then \(E_1 \cap M_2 = \emptyset, E_2 \cap M_2 \neq \emptyset, \) and \(E_2 \cap M_1 = \emptyset\). Now both \(G\setminus E_1\) and \(G\setminus E_2\) are bridgeless, since \(M_2 \cup M_3\) and \(M_1 \cup M_3\) are Hamilton circuits. Since \(G\setminus vv_1 (i = 1, 2, 3)\) doesn’t have Petersen graph as a minor, both \(G\setminus E_1\) and \(G\setminus E_2\) don’t have Petersen graph as a minor. By Theorem 5.1, both \(G\setminus E_1\) and \(G\setminus E_2\) are 3-edge-colorable. Therefore, by Theorem 4.1, \(G\) has a Fulkerson-cover.

If \(a, b, c \in M_1\), then \(M_2 \cup M_3\) is an even circuit of \(G\). Let \(E_1\) be the graph obtained from \(M_1\) by inserting \(v_1\) into \(a\), \(v_2\) into \(b\) and \(v_3\) into \(c\). Since \(E_1 \cup M_{5-i}\) is in \(G\setminus M_i (i = 2, 3)\), we have that \(G\setminus M_i\) is bridgeless and has at most 4 vertices of degree 3. By Theorem 5.1, \(G\setminus M_i\) is 3-edge-colorable. Therefore, by Theorem 4.1, \(G\) has a Fulkerson-cover. □

By Theorem 1.2 (1), we obtain the following corollary.

**Corollary 5.2.** Let \(G\) be a bridgeless cubic graph. If there exists a vertex \(v \in V(G)\) such that \(G\setminus v\) doesn’t have Petersen graph as a minor for each edge \(e\) incident with \(v\) and \(G\setminus v\) is uniquely 3-edge-colorable, then \(G\) has a Fulkerson-cover.

**Proof.** Suppose that \(\{M_1, M_2, M_3\}\) is the uniquely 3-edge-coloring of \(G\setminus v\). We claim that \(M_1 \cup M_2, M_1 \cup M_3, M_2 \cup M_3\) are all Hamilton circuits. Since if \(M_1 \cup M_2\) isn’t a Hamilton circuit, then \(M_1 \cup M_2\) has another 2-edge-coloring \(M'_1\) and \(M'_2\). Now \(\{M'_1, M'_2, M_3\}\) is a 3-edge-coloring of \(G\setminus v\), which is different from \(\{M_1, M_2, M_3\}\), and a contradiction. Therefore, by Theorem 1.2 (1), \(G\) has a Fulkerson-cover. □
Proof of Theorem 1.2 (2). Suppose that $e = v_1v_2 \in E(G)$ and $\{M_1, M_2, M_3\}$ is the 3-edge-coloring of $G \setminus e$, such that $M_1 \cup M_2, M_1 \cup M_3$ and $M_2 \cup M_3$ are all Hamilton circuits. Let $a$ and $b$ be the edges of $G - e$ obtained from $G - e$ by contracting $v_1$ and $v_2$, respectively.

If $a, b$ are in the same matching $M_i (i \in \{1, 2, 3\})$, then without loss of generality, suppose that $a, b \in M_1$. Let $C$ be the graph obtained from $M_1 \cup M_2$ by inserting $v_1$ into $a$ and $v_2$ into $b$. Now $C$ is a Hamilton circuit of $G$. Thus $G$ is 3-edge-colorable and therefore $G$ has a Fulkerson-cover.

If $a, b$ aren’t in the same matching $M_i (i \in \{1, 2, 3\})$, then without loss of generality, suppose that $a \in M_1$ and $b \in M_2$. Let $C$ be the graph obtained from $M_1 \cup M_2$ by inserting $v_1$ into $a$ and $v_2$ into $b$. Now $C$ is a Hamilton circuit of $G$. Thus $G$ is 3-edge-colorable and therefore $G$ has a Fulkerson-cover. □

Proof of Theorem 1.2 (3). If $G$ itself doesn’t have Petersen graph as a minor, then by Theorem 5.1, $G$ is 3-edge-colorable. Therefore $G$ has a Fulkerson-cover. So suppose that $G$ has Petersen graph as a minor. But now, by assumption, $G$ is Petersen graph. It’s easy to check that Petersen graph satisfies the first condition of Theorem 1.2. Therefore $G$ has a Fulkerson-cover. □

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References


(Fuyuan Chen) **INSTITUTE OF STATISTICS AND APPLIED MATHEMATICS, ANHUI UNIVERSITY OF FINANCE AND ECONOMICS, BENGBU, ANHUI, 233030, P. R. CHINA.**

*E-mail address*: chenfuyuan19871018@163.com