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OPERATOR-VALUED TENSORS ON MANIFOLDS

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Abstract. In this paper we try to extend geometric concepts in the context of operator valued tensors. To this end, we aim to replace the field of scalars $\mathbb{R}$ by self-adjoint elements of a commutative $C^\star$-algebra, and reach an appropriate generalization of geometrical concepts on manifolds. First, we put forward the concept of operator-valued tensors and extend semi-Riemannian metrics to operator valued metrics. Then, in this new geometry, some essential concepts of Riemannian geometry such as curvature tensor, Levi-Civita connection, Hodge star operator, exterior derivative, divergence,... will be considered.
Keywords: Operator-valued tensors, operator-valued semi-Riemannian metrics, Levi-Civita connection, curvature, Hodge star operator.
MSC(2010): Primary: 65F05; Secondary: 46L05, 11Y50.

1. **Introduction**

Tensors have a vast application in Mathematics and Physics. Tensors as operators are scalar valued, and extension of tensors to vector valued tensors arised naturally in Mathematics and Physics. Frölicher and Nijenhuis, defined vector valued differential forms in [7]. This concept was very powerful and they could characterize degreed derivations on the exterior algebra of differential forms.

Since, scalars constitute an algebra, a natural extension of scalar valued tensors is occurred when we replace the field of scalars by an algebra. For example, Lie algebra 1-forms on a principle bundle are related to principal connections. If we consider local functions on a manifold with value in the exterior algebra of a vector space, we find the notion of super-manifolds. Non-commutative Geometry arises when we use non-commutative algebras [3].

Since, C*-algebras are very similar to the complex numbers, C*-algebra valued tensors are very similar to ordinary tensors and can play a significant role

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in Physics and Mathematics. In fact, elements of a C*-algebra can be viewed as linear operators on a Hilbert space and observables in a quantum system corresponds to self-adjoint operators, so to describe a quantum system we should deal with operators as scalars. Therefore, operator valued tensors can be used for describing quantum systems.

The first aim of the present paper is to extend the theory of semi-Riemannian metrics to operator-valued semi-Riemannian metrics. Spaces of linear operators are directly related to C*-algebras, so C*-algebras are considered as a basic concept in this article. The first definition of C*-algebra valued positive definite inner products can be found in [14]. These spaces are called pre Hilbert modules and are frequently used in the theory of operator algebras. Hilbert C*-modules provide a natural generalization of Hilbert spaces arising when the field of scalars $\mathbb{C}$ is replaced by an arbitrary C*-algebra. This generalization, in the case of commutative C*-algebras appeared in the paper of Kaplansky [9], however the non-commutative case seemed too complicated at that time. The general theory of Hilbert C*-modules appeared in the pioneering papers of W. Paschke [14] and M. Rieffel [16]. The theory of Hilbert C*-modules may also be considered as a non-commutative generalization of the theory of vector bundles and non-commutative geometry [3].

A number of results on geometrical structures of Hilbert C*-modules and their operators have been obtained [6]. Henceforth, Hilbert C*-modules are generalizations of inner product spaces that on the level of manifolds, provide a generalization of Riemannian manifolds. Due to the physical applications, we extend the definition to non-positive case and replace the positive definiteness of the inner product by non-degeneracy. In this case, we should restrict ourselves to finitely generated modules, because our application is intended for free finite dimensional modules and the general case is more complicated. The content of this paper might also provide a framework for field quantization which is not elaborated in this paper. The main idea for quantization is to replace scalars by operators on a Hilbert space. In the field of quantum mechanics, spectrum of operators plays the role of values of the measurements. So replacements of scalars by operators is the first step for quantization. In this direction, some works have been done [1,2], but it seems C*-algebras are the best candidate to play the role of scalars and we must deal with Modules over C*-algebras.

In this article, we only consider commutative C*-algebras for many reasons. Non-commutative algebras can be used in the realm of non-commutative Geometry, and it is not our aim to enter in this realm. Many basic definition such as vector field can not be extended properly for non-commutative C*-algebras, because the set of derivations of an algebra is a module on the center of that algebra. So, extended vector fields are modules on the center of that C*-algebra and we need the center of that algebra be equal to itself, so the algebra must be commutative. Definition of the inner product encounters the same problem.

Bilinear maps over a module whose scalars are non-commutative are very restrictive. From physical point of view, commutativity means operators in the C*-algebra represent quantities that are simultaneously measurable and this assumption is not very restrictive.

The content of the present paper is structured as follows: Section 1 contains the preliminary facts about the C*-algebras needed to explain our concepts, Section 2 covers the definition of extended tangent bundle, operator-valued vector fields, and operator-valued tensors and explains some of their basic properties. The Pettis-integral of operator-valued volume forms are defined and Stokes' theorem is proved in Section 3. Section 4 is devoted to operator-valued connections and curvature, and the definition of the covariant derivative of operator-valued vector fields. In section 5, operator-valued inner product and some of its basic properties will be illustrated. The existence, uniqueness and the properties of the Hodge star operator for operator-valued inner product spaces are the goal of Section 6. In Section 7 the concepts of Section 5 extend to manifolds. The existence and uniqueness of Levi-Civita connection of operator-valued semi-Riemannian metrics is proved in Section 8. In the last section Ricci tensor, scalar curvature, and sectional curvature are discussed.

2. **Review of C*-algebras**

In this section we review some definitions and results from C*-algebras that we need in the sequel.

Definition 2.1. A Banach $*$-algebra is a complex Banach algebra $\mathfrak{A}$ with a conjugate linear involution $*$ which is an anti-isomorphism. That is, for all $a, b \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$,

$$(a+b)^* = a^* + b^*, \quad (\lambda a)^* = \overline{\lambda} a^*, \quad a^{**} = a, \quad (ab)^* = b^* a^*.$$

Definition 2.2. A C*-algebra $\mathfrak{A}$, is a Banach $*$-algebra with the additional norm condition, for all $a \in \mathfrak{A}$

$$\|a^* a\| = \|a\|^2 .$$

For example the space of all bounded linear operators on a Hilbert space of $\mathcal{H}$ is a C*-algebra. This C*-algebra is denoted by $\mathfrak{B}(\mathcal{H})$.

Remark 2.3. We only consider unital C*-algebras, and denote the unit element by 1.

By Gelfand-Naimark theorem, all unital commutative C*-algebras have the form $C(X)$, in which X is a compact Hausdorff space.

Definition 2.4. An element a of a C*-algebra is said to be **self-adjoint** if $a^* = a$, **normal** if $a^* a = aa^*$, and **unitary** if $a^* a = aa^* = 1$.

For now on, $\mathfrak{A}$ is a C*-algebra. The set of all self-adjoint elements of $\mathfrak{A}$ is denoted by $\mathfrak{A}_{\mathbb{R}}$.

Definition 2.5. The spectrum of an element a in a C*-algebra is the set

$$\sigma(a) = \{z \in \mathbb{C} : z1 - a \text{ is not invertible}\}.$$

Theorem 2.6. *The spectrum $\sigma(a)$ of any element a of a* C**-algebra is a non empty compact set and contained in the $\{z \in \mathbb{C} : |z| \leqslant \|A\|\}$, if $a \in \mathfrak{A}_{\mathbb{R}}$, then $\sigma(a) \subseteq \mathbb{R}$ [8, 17].*

Remark 2.7. For any $a \in \mathfrak{A}$, if $\lambda \in \sigma(a)$, then $|\lambda| \leq \|a\|$.

For each normal element $a \in \mathfrak{A}$ there is a smallest C*-subalgebra C*(a, 1) of $\mathfrak{A}$ which contains a, 1, and is isomorphic to $C(\sigma(a))$.

An element $a \in \mathfrak{A}_{\mathbb{R}}$ is positive if $\sigma(a) \subseteq \mathbb{R}_+$. The set of positive elements of $\mathfrak{A}_{\mathbb{R}}$ is denoted by $\mathfrak{A}_{\mathbb{R}_+}$. For any $a \in \mathfrak{A}$, a^*a is positive.

Theorem 2.8. *If $a \in \mathfrak{A}_{\mathbb{R}_+}$, then there exists a unique element $b \in \mathfrak{A}_{\mathbb{R}_+}$ such that $b^2 = a$ [12].*

We denote by $\sqrt{a}$ the unique positive element b such that $b^2 = a$. If a is a self-adjoint element, then a^2 is positive, and we set $|a| = \sqrt{a^2}$, $a^+ = \frac{1}{2}(|a| + a)$, $a^- = \frac{1}{2}(|a| - a)$. The elements $|a|$, a^+, and a^- are positive and $a = a^+ - a^-$, $a^+a^- = 0$. If $a, b \in \mathfrak{A}_{\mathbb{R}}$, then $|ab| = |a||b|$ (cf. [12]).

3. **Extending tangent bundle**

Throughout this paper, $\mathfrak{A}$ is a commutative unital C*-algebra which according to the **Gelfand-Naimark** second theorem can be thought as a C*-subalgebra of some $\mathfrak{B}(\mathcal{H})$. Let M be a smooth manifold, we set $TM^{\mathfrak{A}} = \bigcup\limits_{p \in M}(T_pM \otimes_{\mathbb{R}} \mathfrak{A})$, so $TM^{\mathfrak{A}}$ is a bundle of free $\mathfrak{A}$-modules over M. Smooth functions from M to $\mathfrak{A}$ can be defined and the set of them is denoted by $C^\infty(M, \mathfrak{A})$. Addition, scalar multiplication, and multiplication of functions in $C^\infty(M, \mathfrak{A})$ are defined pointwise. The involution of $\mathfrak{A}$ can be extended to $C^\infty(M, \mathfrak{A})$ as follows:

$$\begin{aligned} * : C^\infty(M, \mathfrak{A}) &\longrightarrow C^\infty(M, \mathfrak{A}) \\ f \ &\longmapsto f^* \end{aligned}$$

where $f^*(x) = f(x)^*$.

Definition 3.1. A $\mathfrak{A}$-vector field $\tilde{X}$ over M is a section of the bundle $TM^{\mathfrak{A}}$.

The set of all smooth $\mathfrak{A}$-vector fields on M is denoted by $\mathfrak{X}(M)^{\mathfrak{A}}$, in fact

$$\mathfrak{X}(M)^{\mathfrak{A}} = \mathfrak{X}(M) \otimes_{C^\infty(M)} C^\infty(M, \mathfrak{A}).$$

Smooth $\mathfrak{A}$-vector fields can be multiplied by smooth $\mathfrak{A}$-valued functions and $\mathfrak{X}(M)^{\mathfrak{A}}$ is a module over the $\star$-algebra $C^\infty(M, \mathfrak{A})$. For a vector field $X \in \mathfrak{X}(M)$ and a function $f \in C^\infty(M, \mathfrak{A})$, define $X \otimes f$ as a $\mathfrak{A}$-vector field by

$$(X \otimes f)_p = X_p \otimes f(p).$$

These fields are called simple, and any $\mathfrak{A}$-vector field can be written locally as a finite sum of simple $\mathfrak{A}$-vector fields. We can identify X and $X \otimes 1$, so $\mathfrak{X}(M)$ is a $C^\infty(M)$-subspace of $\mathfrak{X}(M)^{\mathfrak{A}}$.

A smooth vector field $X \in \mathfrak{X}(M)$ defines a derivation on $C^\infty(M, \mathfrak{A})$ by $f \longmapsto Xf$. In fact, for any integral curve $\alpha : I \longrightarrow M$ of X, we have, $(Xf)(\alpha(t)) = \frac{\mathrm{d}}{\mathrm{d}t} f(\alpha(t))$.

This operation can be extended to $\mathfrak{A}$-vector fields, such that for simple elements of $\mathfrak{X}(M)^{\mathfrak{A}}$ such as $X \otimes h$ we have

$$(X \otimes h)(f) = h\,(Xf) \qquad f \in C^\infty(M, \mathfrak{A}).$$

If $\mathfrak{A}$ is non-commutative, this definition is not well-defined. This definition implies that for $f, h \in C^\infty(M, \mathfrak{A})$ and $\widetilde{X} \in \mathfrak{X}(M)^{\mathfrak{A}}$ we have

$$(h\widetilde{X})(f) = h.(\widetilde{X}f).$$

The Lie bracket of ordinary vector fields can be extended to $\mathfrak{A}$-vector fields as the following, if $X, Y \in \mathfrak{X}(M)$ and $h, k \in C^\infty(M, \mathfrak{A})$ then,

$$[X \otimes h\ , Y \otimes k] = [X, Y] \otimes (hk) + Y \otimes (hX(k)) - X \otimes (kY(h)).$$

The verification of main properties of the Lie bracket are routine.
An involution on $\mathfrak{X}(M)^{\mathfrak{A}}$ for simple $\mathfrak{A}$-vector field, such as $X \otimes f$, is defined by

$$(X \otimes f)^* = X \otimes f^*.$$

For $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(M)^{\mathfrak{A}}, f \in C^\infty(M, \mathfrak{A})$, we have

$$\widetilde{X}(f)^* = \widetilde{X}^*(f^*), \left[\widetilde{X}, \widetilde{Y}\right]^* = \left[\widetilde{X}^*, \widetilde{Y}^*\right]. \tag{3.1}$$

Definition 3.2. An $\mathfrak{A}$-valued covariant tensor field of order k on M is an operator $T : \mathfrak{X}(M)^{\mathfrak{A}} \times \cdots \times \mathfrak{X}(M)^{\mathfrak{A}} \longrightarrow C^\infty(M, \mathfrak{A})$ which is $k - C^\infty(M, \mathfrak{A})$-linear.

Contravariant and mixed $\mathfrak{A}$-valued tensors can be defined in a similar way. Alternating covariant $\mathfrak{A}$-tensor fields are called $\mathfrak{A}$-differential forms, and the set of all $\mathfrak{A}$-differential forms of order k is denoted by $A^k(M, \mathfrak{A})$. $\mathfrak{A}$-differential forms and exterior product and exterior derivation of these forms are special case of vector valued differential forms. The only difference is that $\mathfrak{A}$ maybe infinite dimensional.

Lie derivation of $\mathfrak{A}$-tensor fields along $\mathfrak{A}$-vector fields is defined naturally. For $\tilde{X} \in \mathfrak{X}(M)^{\mathfrak{A}}$ and a covariant $\mathfrak{A}$-tensor field of order k, such as $\tilde{\mathrm{T}}$, Lie derivation of $\tilde{\mathrm{T}}$ along $\tilde{X}$ is also a covariant $\mathfrak{A}$-tensor field of order k and defined as follows. For $\tilde{Y_1}, \cdots, \tilde{Y_k} \in \mathfrak{X}(M)^{\mathfrak{A}}$ we have

$$(\mathcal{L}_{\tilde{X}}\tilde{\mathrm{T}})(\tilde{Y_1}, \cdots, \tilde{Y_k}) = \tilde{X}(\tilde{\mathrm{T}}(\tilde{Y_1}, \cdots, \tilde{Y_k})) - \sum_{i=1}^{k} \tilde{\mathrm{T}}(\tilde{Y_1}, \cdots, [\tilde{X}, \tilde{Y_i}], \cdots, \tilde{Y_k})$$

If $\tilde{\mathrm{T}}$ is a differential form, then its Lie derivation along any $\mathfrak{A}$-vector field is also a differential form and Cartan formula for its derivation holds .i.e.,

$$\mathcal{L}_{\tilde{X}}\tilde{\mathrm{T}} = \mathrm{d}(\mathfrak{i}_{\tilde{X}}\tilde{\mathrm{T}}) + \mathfrak{i}_{\tilde{X}}(\mathrm{d}\tilde{\mathrm{T}})$$

De Rham cohomology can be defined in the context of $\mathfrak{A}$-valued differential forms similarly. Let $B^k(M,\mathfrak{A})$ be the space of closed forms, and $Z^k(M,\mathfrak{A})$ be the space of exact forms in $A^k(M,\mathfrak{A})$, then these spaces are $\mathfrak{A}$-modules and $Z^k(M,\mathfrak{A}) \subseteq B^k(M,\mathfrak{A})$ and set $H^k_{\mathfrak{A}}(M) = \frac{B^k(M,\mathfrak{A})}{Z^k(M,\mathfrak{A})}$. The space $H^k_{\mathfrak{A}}(M)$ is an $\mathfrak{A}$-module and its relation to ordinary cohomology is as follows.

As a computational tool, we can always assume $\mathfrak{A} = C(X)$ for some compact Hausdorff space X. In this case, any $\mathfrak{A}$-valued k-differential form $\tilde{\omega}$, in a coordinate system, can be written as $\tilde{\omega} = \tilde{\omega}_{i_1\cdots i_k}\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$ in which $\tilde{\omega}_{i_1\cdots i_k} \in C^\infty(M, C(X))$. For any $x \in X$, set $\tilde{\omega}_x = \tilde{\omega}_{i_1\cdots i_k}(x)\,\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$ that is an ordinary k-differential form. We can show that $(\mathrm{d}\tilde{\omega})_x = \mathrm{d}(\tilde{\omega}_x)$, so if $\tilde{\omega}$ is closed (exact) then every $\tilde{\omega}_x$ is closed (exact) too. conversely, If all $\tilde{\omega}_x$ is closed then $\tilde{\omega}$ is closed, but for exactness this property is not straightforward.

Problem 3.3. *For a $C(X)$-valued differential form $\tilde{\omega}$ on a compact manifold, if all $\tilde{\omega}_x$ are exact, is it true that $\tilde{\omega}$ is exact?*

Theorem 3.4. *For a compact manifold M, $H^k_{\mathfrak{A}}(M)$ naturally contains $H^k(M) \otimes_{\mathbb{R}} \mathfrak{A}$.*

Proof. For $\omega \in B^k(M)$, let $[\omega]$ be its equivalent class in $B^k(M)$, and for $\tilde{\omega} \in B^k(M,\mathfrak{A})$, let $[\tilde{\omega}]_{\mathfrak{A}}$ be its equivalent class in $B^k(M,\mathfrak{A})$. Define:

$$\begin{array}{rccc} \Phi: & H^k(M) \otimes_{\mathbb{R}} \mathfrak{A} & \longrightarrow & H^k_{\mathfrak{A}}(M) \\ & [\omega] \otimes a & \longmapsto & [a\omega]_{\mathfrak{A}} \end{array}$$

Clearly, Φ is well defined and is $\mathfrak{A}$-linear. To complete the proof, suppose $\mathfrak{A} = C(X)$. Choose $\omega_1, \cdots, \omega_m \in B^k(M)$ such that $\{[\omega_1], \cdots, [\omega_m]\}$ is a basis for $H^k(M)$. To prove Φ is one to one, suppose $\Phi([\omega_1]\otimes a_1 + \cdots + [\omega_m]\otimes a_m) = 0$. So $[a_1\omega_1 + \cdots + a_k\omega_m]_{\mathfrak{A}} = 0$ and $\tilde{\omega} = a_1\omega_1 + \cdots + a_k\omega_m$ is exact. Therefore, for each $x \in X$, $\tilde{\omega}_x$ is exact and $[a_1(x)\omega_1 + \cdots + a_k(x)\omega_m] = 0$. Since $\{[\omega_1], \cdots, [\omega_m]\}$ is a basis, for any $x \in X$ we have $a_1(x) = \cdots = a_m(x) = 0$, so $a_1 = \cdots = a_m = 0$ and $[\omega_1] \otimes a_1 + \cdots + [\omega_m] \otimes a_m = 0$. ☐

4. Integration of $\mathfrak{A}$-valued volume forms and Stokes' theorem

We remind the notion of integral of vector valued functions, called **Pettis**-integral [5]. The Borel σ-algebra over $\mathbb{R}^n$ is denoted by $\mathcal{B}_n = \mathcal{B}(\mathbb{R}^n)$, and suppose that μ is the Lebesgue measure.

Definition 4.1. Suppose $V \in \mathcal{B}_n$. A measurable function $f : V \longrightarrow \mathfrak{A}$ is called

(i) weakly integrable if $\Lambda(f)$ is Lebesgue integrable for every $\Lambda \in \mathfrak{A}^*$
(ii) Pettis integrable if there exists $x \in \mathfrak{A}$ such that $\Lambda(x) = \int_V \Lambda(f)\,\mathrm{d}\mu$, for every $\Lambda \in \mathfrak{A}^*$.

If f is Pettis-integrable over $V \in \mathcal{B}_n$ then x is unique and is called Pettis-integral of f over V. We use the notations $\int_V f \,\mathrm{d}\mu$ or $(P)\int_V f \,\mathrm{d}\mu$ to show the Pettis-integral of f over V. It is proved that each function $f \in C_c(\mathbb{R}^n, \mathfrak{A})$ is Pettis integrable over any $V \in \mathcal{B}_n$ (cf. [17]).

Theorem 4.2 (Change of Variables)**.** *Suppose D and D' are open domains of integration in $\mathbb{R}^n$, and $G : D \longrightarrow D'$ is a diffeomorphism . For every Pettis-integrable function $f : D' \longrightarrow \mathfrak{A}$,*

$$\int_{D'} f \,\mathrm{d}\mu = \int_D (f \circ G)\, |\det(DG)| \,\mathrm{d}\mu.$$

Proof. Applying the Pettis-integral's definition and using classical change of variables theorem, one can conclude the desired result. ☐

Definition 4.3. An $\mathfrak{A}$-valued n-form on M ($n = \dim M$) is called an $\mathfrak{A}$-valued volume form on M.

In the canonical coordinate system, an $\mathfrak{A}$-valued volume form $\tilde{\omega}$ on $\mathbb{R}^n$ is written as follows:

$$\widetilde{\omega} = f \,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n, \tag{4.1}$$

where $f \in C^\infty(\mathbb{R}^n, \mathfrak{A})$.

Definition 4.4. Let $\tilde{\omega}$ be a compactly supported $\mathfrak{A}$-valued n-form on $\mathbb{R}^n$. Define the integral of $\tilde{\omega}$ over $\mathbb{R}^n$ by,

$$\int_{\mathbb{R}^n} \widetilde{\omega} = (P)\int_{\mathbb{R}^n} f \;\mathrm{d}x_1 \cdots \mathrm{d}x_n$$

where $\tilde{\omega}$ is defined as in (4.1).

Definition 4.5. For $\Lambda \in \mathfrak{A}^*$, define the operator $\Lambda : \mathcal{A}^k(M, \mathfrak{A}) \longrightarrow \mathcal{A}^k(M)$ by

$$(\Lambda\tilde{\omega})(X_1, ..., X_k) = \Lambda(\widetilde{\omega}(X_1, ..., X_k)),$$

where $\widetilde{\omega} \in \mathcal{A}^k(M, \mathfrak{A})$, $X_i \in \mathfrak{X}(M)$.

Definition 4.6. Let M be an oriented smooth n-manifold, and let $\tilde{\omega}$ be an $\mathfrak{A}$-valued n-form on M. First suppose that the support of $\tilde{\omega}$ is compact and is included in the domain of a single chart (U, φ) which is positively oriented. We define the integral of $\tilde{\omega}$ over M as

$$\int_M \widetilde{\omega} = (P)\int_{\varphi(U)} (\varphi^{-1})^*(\widetilde{\omega}). \tag{4.2}$$

By using the change of variable's theorem one can prove that $\int_M \widetilde{\omega}$ does not depend on the choice of chart whose domain contains $\mathrm{supp}(\widetilde{\omega})$. To integrate an arbitrary compact support $\mathfrak{A}$-valued n-form, we can use partition of unity as the same as ordinary n-forms.

Lemma 4.7. *For any $\Lambda \in \mathfrak{A}^*$ we have,*

(i) *if* $f : M \longrightarrow N$ *is a smooth map, then for each* $\widetilde{\omega} \in \mathcal{A}^n(N, \mathfrak{A})$, $\Lambda f^*(\widetilde{\omega}) = f^*(\Lambda\widetilde{\omega})$,
(ii) *for a compactly supported* $\mathfrak{A}$*-valued volume* n*-form* $\widetilde{\omega}$ *on* M, $\Lambda(\int_M \widetilde{\omega}) = \int_M \Lambda(\widetilde{\omega})$,
(iii) *for* $\widetilde{\omega} \in \mathcal{A}^k(M, \mathfrak{A})$, $\Lambda(\mathrm{d}\widetilde{\omega}) = \mathrm{d}(\Lambda\widetilde{\omega})$.

Proof. To prove (i) suppose that $X_1, \cdots, X_k \in \mathfrak{X}M$, then

$$\begin{aligned} f^*(\Lambda\tilde{\omega})(X_1, \cdots, X_k) =& (\Lambda\tilde{\omega})(f_*(X_1), \cdots, f_*(X_k)) = \Lambda(\widetilde{\omega}(f_*(X_1), \cdots, f_*(X_k)) \\ =& \Lambda((f^*\widetilde{\omega})(X_1, \cdots, X_k)) = (\Lambda(f^*\widetilde{\omega}))(X_1, \cdots, X_k). \end{aligned}$$

To prove (ii) suppose that $\tilde{\omega}$ is compactly supported in the domain of a single chart (U, φ) that is positively oriented, thus

$$\Lambda(\int_M \widetilde{\omega}) = \Lambda(\int_{\mathbb{R}^n} (\varphi^{-1})^*\widetilde{\omega}) = \int_{\mathbb{R}^n} \Lambda((\varphi^{-1})^*\widetilde{\omega}) = \int_{\mathbb{R}^n} (\varphi^{-1})^*(\Lambda\widetilde{\omega}) = \int_M \Lambda\tilde{\omega}.$$

The general case follows from the above result and using partition of unity.

(iii) Note that in the special case $k = 0$, $\mathcal{A}^0(M, \mathfrak{A}) = C^\infty(M, \mathfrak{A})$ and the continuity of Λ implies (iii) for the elements of $C^\infty(M, \mathfrak{A})$. The general case follows from this one. □

Theorem 4.8 (Stokes' Theorem)**.** *Let* M *be an oriented smooth* n*-manifold with boundary and orientations of* M *and* ∂M *are compatible, and let* $\widetilde{\omega}$ *be a compactly supported smooth valued* $(n-1)$*-form on* M*.Then*

$$\int_M \mathrm{d}\widetilde{\omega} = \int_{\partial M} \widetilde{\omega}.$$

Proof. By Lemma 4.7 for each $\Lambda \in \mathfrak{A}^*$ we have

$$\Lambda(\int_M \mathrm{d}\widetilde{\omega}) = \int_M \Lambda(\mathrm{d}\widetilde{\omega}) = \int_M \mathrm{d}(\Lambda\tilde{\omega}) = \int_{\partial M} (\Lambda\tilde{\omega}) = \Lambda(\int_{\partial M} \widetilde{\omega})$$

By use of the Han-Banach theorem, it follows that $\int_M \mathrm{d}\widetilde{\omega} = \int_{\partial M} \widetilde{\omega}$. □

5. Connection and curvature

The notion of covariant derivation of $\mathfrak{A}$-vector fields can be defined as follows.

Definition 5.1. An $\mathfrak{A}$-connection ∇ on M is a bilinear map

$$\nabla : \mathfrak{X}(M)^{\mathfrak{A}} \times \mathfrak{X}(M)^{\mathfrak{A}} \longrightarrow \mathfrak{X}(M)^{\mathfrak{A}}$$

such that for all $\tilde{X}, \tilde{Y} \in \mathfrak{X}(M)^{\mathfrak{A}}$ and any $f \in C^\infty(M, \mathfrak{A})$,

(i) $\nabla_{f\tilde{X}}\tilde{Y} = f\nabla_{\tilde{X}}\tilde{Y}$
(ii) $\nabla_{\tilde{X}}f\tilde{Y} = \tilde{X}(f)\tilde{Y} + f\nabla_{\tilde{X}}\tilde{Y}$.

Every ordinary connection on M can be extended uniquely to an $\mathfrak{A}$-connection on M. For an ordinary connection on M such as ∇, its extension as an $\mathfrak{A}$-connection is defined as follows. If $X, Y \in \mathfrak{X}M$ and $h, k \in C^\infty(M, \mathfrak{A})$, then define

$$\nabla_{X\otimes h}(Y \otimes k) = h((\nabla_X Y) \otimes k + Y \otimes X(k))$$

The torsion tensor of a $\mathfrak{A}$-connection ∇ is defined by

$$T(\tilde{X}, \tilde{Y}) = \nabla_{\tilde{X}}\tilde{Y} - \nabla_{\tilde{Y}}\tilde{X} - [\tilde{X}, \tilde{Y}] \qquad \tilde{X}, \tilde{Y} \in \mathfrak{X}(M)^{\mathfrak{A}}$$

If $T = 0$, then ∇ is called torsion-free.

Definition 5.2. Let ∇ be an $\mathfrak{A}$-connection on M. The function $\mathcal{R} : \mathfrak{X}(M)^{\mathfrak{A}} \times \mathfrak{X}(M)^{\mathfrak{A}} \times \mathfrak{X}(M)^{\mathfrak{A}} \longrightarrow \mathfrak{X}(M)^{\mathfrak{A}}$ given by

$$\mathcal{R}(\tilde{X}, \tilde{Y})(\tilde{Z}) = \nabla_{\tilde{X}}\nabla_{\tilde{Y}}\tilde{Z} - \nabla_{\tilde{Y}}\nabla_{\tilde{X}}\tilde{Z} - \nabla_{[\tilde{X},\tilde{Y}]}\tilde{Z}$$

is a $(1, 3)$ $\mathfrak{A}$-tensor on M and is called the curvature tensor of ∇.

Proposition 5.3 (The Bianchi identities)**.** *If $\mathcal{R}$ is the curvature of a torsion free $\mathfrak{A}$-connection ∇ on M, then for all $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(M)^{\mathfrak{A}}$,*

(i) $\mathcal{R}(\tilde{X}, \tilde{Y})\tilde{Z} + \mathcal{R}(\tilde{Y}, \tilde{Z})\tilde{X} + \mathcal{R}(\tilde{Z}, \tilde{X})\tilde{Y} = 0$;
(ii) $(\nabla_{\tilde{X}}\mathcal{R})(\tilde{Y}, \tilde{Z}) + (\nabla_{\tilde{Y}}\mathcal{R})(\tilde{Z}, \tilde{X}) + (\nabla_{\tilde{Z}}\mathcal{R})(\tilde{X}, \tilde{Y}) = 0$.

Proof. The proof is similar to the one for ordinary connections (cf. [15]). □

If ∇^1, ∇^2 are $\mathfrak{A}$-connections on M, then, the operator $T = \nabla^1 - \nabla^2$ is a $(1, 2)$ $\mathfrak{A}$-tensor and the space of $\mathfrak{A}$-connections on M is an affine space which is modeled on the space of $(1, 2)$ $\mathfrak{A}$-tensors.

6. Operator-valued inner product

In this section, we introduce the notion $\mathfrak{A}$-valued inner products on $\mathfrak{A}$-modules, and investigate some of their basic properties . For the case of Hilbert C*-modules refer to [10, 11, 14]. For an $\mathfrak{A}$-module V, denote the collection of all $\mathfrak{A}$-linear mappings from V into $\mathfrak{A}$ by $V^\sharp$. This space is also an $\mathfrak{A}$-module. If V is a free finite dimensional module, then $V^\sharp$ is also a free finite dimensional module of the same dimension.

In the following, we define the notion of $\mathfrak{A}$-valued inner products. In this definition, inner products are not necessarily positive definite, so we must restrict ourselves to finite dimensional or finitely generated modules. This topic, for arbitrary modules, is more complicated and is far from the scope of this article.

Definition 6.1. Let V be a finitely generated $\mathfrak{A}$-module. A mapping $\langle\cdot,\cdot\rangle : V \times V \longrightarrow \mathfrak{A}$ is called an $\mathfrak{A}$-valued inner product on V if for all $a \in \mathfrak{A}$, and $x, y, z \in V$ the following conditions hold:

(i) $\langle x, y\rangle^* = \langle y, x\rangle$

(ii) $\langle ax+y,z\rangle = a\langle x,z\rangle + \langle y,z\rangle$
(iii) $\forall w \in V \langle w,x\rangle = 0 \Rightarrow x = 0$
(iv) $\forall T \in V^\sharp\ \exists x \in V\ \forall y \in V\ (T(y) = \langle y,x\rangle)$

Conditions (iii) and (iv) are called nondegeneracy of the inner product. Note that for each $x \in V$ the mapping $\hat{x} : V \longrightarrow \mathfrak{A}$, defined by $\hat{x}(y) = \langle y,x\rangle$ belongs to $V^\sharp$. One can see that the mapping $x \longmapsto \hat{x}$ from V into $V^\sharp$ is conjugate $\mathfrak{A}$-linear and nondegenracy of the inner product is equivalent to the bijectivity of this map. If for all $x \in V$, $\langle x,x\rangle$ is a positive selfadjoint element of $\mathfrak{A}$, this inner product is called positive.

In the context of Hilbert modules, condition (iv) is not part of the definition and this condition is called self-duality of Hilbert modules [9]. This property, automatically, holds for free finite dimensional Hilbert modules. Here, because of non-positivity of inner product, this property is part of the definition.

Theorem 6.2. *Let V be a finite dimensional free $\mathfrak{A}$-module and $\langle\cdot,\cdot\rangle :$ $V \times V \longrightarrow \mathfrak{A}$ satisfy (i) and (ii) of the above-mentioned definition, and let $\{e_i\}$ be a basis for V. Then, $\langle\cdot,\cdot\rangle$ is nondegenerate if and only if, $\det(\langle e_i,e_j\rangle)$ is invertible in $\mathfrak{A}$.*

Proof. First, assume that $\det(\langle e_i,e_j\rangle)$ is invertible in $\mathfrak{A}$. Set $g_{ij} = \langle e_i,e_j\rangle$, so (g_{ij}) is an invertible $\mathfrak{A}$-valued matrix. Denote the inversion of this matrix by (g^{ij}). Set $e^i = g^{ij}e_j$; $\{e^i\}$ is also a basis and is called the reciprocal base of $\{e_i\}$. The characteristic property of the reciprocal base is that $\langle e_i,e^j\rangle = \delta_i^j$, so for every $x \in V$, if $x = \lambda^i e_i$, then $\lambda^i = \langle x,e^i\rangle$. To prove nondegenracy, suppose for all $w \in V$ we have $\langle w,x\rangle = 0$. So, for all index i we have $\langle x,e^i\rangle = 0$ that implies $\lambda^i = 0$, so $x = 0$. For any $T \in V^\sharp$, set $\lambda_i = T(e_i)$ so, for $x = \bar{\lambda}_i e^i$ we have $\hat{x} = T$.

Conversely, suppose that $\langle\cdot,\cdot\rangle$ is nondegenerate. For each i, consider the $\mathfrak{A}$-linear map $T^i : V \longrightarrow \mathfrak{A}$ defined by $T^i(e_j) = \delta_j^i$. By nondegenarcy, the map $x \longmapsto \hat{x}$ is bijective and there exists $u^i \in V$ such that $\widehat{u^i} = T^i$, so $\langle e_j,u^i\rangle = \delta_j^i$. Any u^i can be written as $u^i = a^{ij}e_j$ for some $a_{ij} \in \mathfrak{A}$. An easy computation shows that the matrix $A = (a^{ij})$ is the inverse of the matrix $D = (g_{ij})$, In fact,

$$\delta_j^i = \langle e_j,u^i\rangle = \langle u^i,e_j\rangle = \langle a^{ik}e_k,e_j\rangle = a^{ik}\langle e_k,e_j\rangle = a^{ik}g_{kj}$$

Therefore $AD = I$, so $\det(A)\det(D) = 1$ and $\det(D)$ is invertible. □

Let V be an $\mathfrak{A}$ module. V, may have an involution that is a conjugate $\mathfrak{A}$-linear isomorphism $* : V \longrightarrow V$ such that $*^2 = 1_V$. An inner product $\langle\cdot,\cdot\rangle$ on V is called compatible with the involution if for all $x,y \in V$

$$\langle x^*,y^*\rangle = \langle x,y\rangle^*.$$

In these spaces, if x and y are self-conjugate elements of V, then $\langle x,y\rangle$ is self-adjoint.

Example 6.3. Let W be a finite dimensional real vector space and $\langle\cdot,\cdot\rangle : W \times W \longrightarrow \mathfrak{A}_{\mathbb{R}}$ be a symmetric bilinear map such that for some basis $\{e_i\}$ of W, $\det(\langle e_i, e_j\rangle)$ is invertible in $\mathfrak{A}$. The tensor product $W^{\mathfrak{A}} = W \otimes_{\mathbb{R}} \mathfrak{A}$ is a free $\mathfrak{A}$-module and $\langle\cdot,\cdot\rangle$ can be extended uniquely to a $\mathfrak{A}$-valued inner product on it as follows. For all $u, v \in W$ and $a, b \in \mathfrak{A}$ set

$$\langle u \otimes a \ , v \otimes b\rangle = a\, b^* \,\langle u \ , v\rangle.$$

The $\mathfrak{A}$-module $W^{\mathfrak{A}}$ has a natural involution that is $(u \otimes a)^* = u \otimes a^*$ and the extended inner product on $W^{\mathfrak{A}}$ is compatible with this involution. In fact, any inner product on $W^{\mathfrak{A}}$ which is compatible with this involution can be obtained by the above method.

For an inner product on a free finite dimensional $\mathfrak{A}$-module, we define an element of $\mathfrak{A}$ as its signature. In the ordinary cases, signature is a scalar that is ± 1 and is defined by orthonormal bases. Here, we must define signature by arbitrary bases.

Theorem 6.4. *Let V be a free finite dimensional $\mathfrak{A}$-module and $\langle\cdot,\cdot\rangle$ be an inner product on it. If $\{e_i\}$ is a basis on V, set $g_{ij} = \langle e_i, e_j\rangle$ and $g = \det(g_{ij})$. Then, g is selfadjoint and $\dfrac{|g|}{g}$ does not depend on the choice of the base.*

Proof. Put $G = (g_{ij})$, since $g_{ij}^* = g_{ji}$ we find ${}^tG = G^*$, therefore

$$g = \det(G) = \det({}^tG) = \det(G^*) = \det(G)^* = g^*.$$

Now, suppose that $\{e'_i\}$ is another basis. Set $g'_{ij} = \langle e'_i, e'_j\rangle$ and $G' = (g'_{ij})$ and $g' = \det(G')$. For some matrix $A = (a_i^j)$ we have $e'_i = a_i^j e_j$, so

$$g'_{ij} = \langle e'_i, e'_j\rangle = \langle a_i^k e_k, a_j^l e_l\rangle = a_i^k a_j^{l*}\langle e_k, e_l\rangle = a_i^k a_j^{l*} g_{kl}.$$

This equality implies that $G' = AG({}^tA^*)$, so $g' = g\det(A)\det(A)^*$. Since $\det(A)\det(A)^*$ is a positive selfadjoint element of $\mathfrak{A}$ we deduce that $|g'| = |g|\det(A)\det(A)^*$. Consequently,

$$\frac{|g'|}{g'} = \frac{|g|\det(A)\det(A)^*}{g\ \det(A)\det(A)^*} = \frac{|g|}{g}.$$

□

The value $\nu = \dfrac{|g|}{g}$ which does not depend on the choice of the base, is called the signature of the inner product. Note that $\sigma(\nu) \subseteq \{-1, 1\}$ and $\nu^{-1} = \nu$. In the ordinary metrics, ν is exactly $(-1)^q$ where q is the index of the inner product.

Note that in free finite dimensional $\mathfrak{A}$ modules such as V that has an $\mathfrak{A}$-inner product, V and $V^\sharp$ are naturally isomorphic and this isomorphism induces an inner product on $V^\sharp$. So, all results about V, can be stated for $V^\sharp$ too.

7. The hodge star operator

To define the Hodge star operator, first we need the notion of orientation of free modules. Let V be a free finite dimensional $\mathfrak{A}$-module. By definition two ordered bases of V have the same orientation if determinant of the transition matrix between them is a positive element of $\mathfrak{A}$. This is an equivalence relation between the bases of V, but there exist many equivalence classes. We choose one of these classes as an orientation for V and we call it an orientation for V. In this case, we call V an orientated space and every basis in the orientation, is called a proper base of V. Note that there are many orientations on V and it is not appropriate to call some of them positive.

Definition 7.1. Let V be an oriented free n-dimensional $\mathfrak{A}$-module that has an $\mathfrak{A}$-inner product. For each proper base $\{e_i\}$ with reciprocal base $\{e^i\}$, and $g = \det(g_{ij})$, set $\Omega = \sqrt{|g|}\, e^1 \wedge \cdots \wedge e^n$. This tensor is called the canonical volume form of the inner product.

Theorem 7.2. *The canonical volume form does not depend on the choice of the proper base.*

Proof. Assume that $\{u_i\}$ is another proper base with reciprocal base $\{u^j\}$. For some matrix $A = (a_i^j)$, we have $u_i = a_i^j\, e_j$. The same orientation of $\{e_i\}$ and $\{u_i\}$ implies that $\det(A)$ is positive. If $A^{-1} = (\beta_i^j)$, then $e_i = \beta_i^j\, u_j$. According to the proof of Theorem 6.4 if

$$g_{ij} = \langle e_i, e_j\rangle,\ u_{ij} = \langle u_i, u_j\rangle,\ g = \det(g_{ij}),\ u = \det(u_{ij}),$$

then, $u = \det(A)\det(A^*)g = (\det A)^2\, g$. Positivity of $\det(A)$ yields $\sqrt{|u|} = \det(A)\sqrt{|g|}$, henceforth

$$\begin{aligned}
\sqrt{|g|}\, e^1 \wedge \cdots \wedge e^n &= \frac{\sqrt{|g|}}{g}\, e_1 \wedge \cdots \wedge e_n = \frac{\sqrt{|g|}}{g}\, (\beta_1^{i_1}\ u_{i_1}) \wedge \cdots \wedge (\beta_n^{i_n}\ u_{i_n}) \\
&= \frac{\sqrt{|g|}}{g}\ \det(A^{-1})\, u_1 \wedge \cdots \wedge u_n \\
&= \frac{\sqrt{|g|}}{g\ \det(A)}\, (u_{1i_1}\ u^{i_1}) \wedge \cdots \wedge (u_{ni_n}\ u^{i_n}) \\
&= \frac{\sqrt{|g|}\ u}{g\ \det(A)}\, u^1 \wedge \cdots \wedge u^n \\
&= \frac{\sqrt{|g|}\ u\ \det(A)}{g\ (\det A)^2}\, u^1 \wedge \cdots \wedge u^n = \sqrt{|u|}\, u^1 \wedge \cdots \wedge u^n.
\end{aligned}$$

□

Any $\mathfrak{A}$-inner product on the free n-dimensional $\mathfrak{A}$-module V, in a natural way, can be extended to the space of each exterior powers $\Lambda^k V$. For $\alpha = u_1 \wedge \cdots \wedge u_k$ and $\beta = v_1 \wedge \cdots \wedge v_k$, set $\langle \alpha, \beta\rangle = \det(\langle u_i, v_j\rangle)$. If $\{e_i\}$ is a base of V with reciprocal base $\{e^j\}$, then it is straightforward to check that $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{1 \leq i_1 < \ldots < i_k \leq n}$ is a basis of $\Lambda^k V$ and its reciprocal base is $\{e^{i_1} \wedge \cdots \wedge e^{i_k}\}_{1 \leq i_1 < \ldots < i_k \leq n}$. If the inner product on V is positive, then the induced inner product on $\Lambda^k V$ is also positive.

Remark 7.3. Note that $\langle \Omega, \Omega \rangle = \nu$.

We now define the operation $\star$, called the Hodge star operator which is similar to the Hodge star operator for ordinary metrics. This is a conjugate linear isomorphism from $\Lambda^k V$ into $\Lambda^{n-k} V$. This operator depends on the inner product and also on the orientation of V. For each $\beta \in \Lambda^k V$, define $\mu_\beta \in (\Lambda^{n-k} V)^\sharp$ by

$$\mu_\beta : \Lambda^{n-k} V \longrightarrow \mathfrak{A}, \qquad \alpha \mapsto \nu \, \langle \beta \wedge \alpha, \Omega \rangle.$$

Because of the non-degeneracy of the inner product on $\Lambda^{n-k} V$, it follows that there is a unique $\star\beta \in \Lambda^{n-k} V$ such that $\mu_\beta(\alpha) = \langle \alpha, \star\beta \rangle$, that is; $\langle \alpha, \star\beta \rangle = \nu \, \langle \beta \wedge \alpha, \Omega \rangle$. So $\star : \Lambda^k V \longrightarrow \Lambda^{n-k} V$ is an operator that for all $\beta \in \Lambda^k V$ and $\alpha \in \Lambda^{n-k} V$ we have

$$\langle \alpha, \star\beta \rangle = \nu \, \langle \beta \wedge \alpha, \Omega \rangle.$$

The above equation shows that $\star$ is a conjugate linear map, that is

$$\star(\beta_1 + \beta_2) = \star(\beta_1) + \star(\beta_2), \qquad \star(a\beta) = a^* \star \beta.$$

Since $\beta \wedge \alpha$ is a multiple of Ω, and the coefficient is $\nu \, \langle \beta \wedge \alpha, \Omega \rangle$, we find that

$$\forall \, \beta \in \Lambda^k V, \; \forall \, \alpha \in \Lambda^{n-k} V, \qquad \beta \wedge \alpha = \langle \alpha, \star\beta \rangle \Omega.$$

We now summarize the properties of the operator $\star$ in the following theorem.

Theorem 7.4. *If V is an oriented free n-dimensional $\mathfrak{A}$-module that has an $\mathfrak{A}$-valued inner product with signature ν, and $\{e_i\}$ is a proper base of V, then the operator $\star$ satisfies*

(i) $\star(e^{\sigma(1)} \wedge \cdots \wedge e^{\sigma(k)}) = \mathrm{sgn}(\sigma) \, \frac{1}{\sqrt{|g|}} \, e_{\sigma(k+1)} \wedge \cdots \wedge e_{\sigma(n)}$; $(\sigma \in S_n)$

(ii) $\star(e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(k)}) = \mathrm{sgn}(\sigma) \, \frac{g}{\sqrt{|g|}} \, e^{\sigma(k+1)} \wedge \cdots \wedge e^{\sigma(n)}$; $(\sigma \in S_n)$

(iii) $\alpha \wedge \star\beta = \nu \, \langle \alpha, \beta \rangle \, \Omega \quad \alpha, \beta \in \Lambda^k V$;

(iv) $\star(1) = \nu \, \Omega$;

(v) $\star(\Omega) = 1$;

(vi) $\star \star (\alpha) = (-1)^{k(n-k)} \nu \, \alpha, \quad \alpha \in \Lambda^k V$;

(vii) $\langle \star\alpha, \star\beta \rangle = \nu \, \langle \alpha, \beta \rangle^*, \quad \alpha, \beta \in \Lambda^k V$.

Proof. The proof is a straightforward computation and is similar to the one in ordinary case. □

Note that, all these results hold for $V^\sharp$.

8. **Operator-valued metrics on manifolds**

We now consider operator valued metrics on manifolds.

Definition 8.1. An $\mathfrak{A}$-valued semi-Riemannian metric on a smooth manifold M is a smooth map $\langle .,. \rangle : TM^{\mathfrak{A}} \oplus TM^{\mathfrak{A}} \longrightarrow \mathfrak{A}$ such that for each $p \in M$, $\langle .,. \rangle$ restricts to an $\mathfrak{A}$-valued inner product on $T_pM^{\mathfrak{A}}$ that is compatible with its natural involution. If restrictions of $\langle .,. \rangle$ are positive inner products, we call it an $\mathfrak{A}$-valued Riemannian metric.

This definition implies that the inner product of any two ordinary vector fields on M is an $\mathfrak{A}_{\mathbb{R}}$-valued function on M.

For each $p \in M$, denote the signature of the inner product on T_pM by ν_p. The map $\nu : p \mapsto \nu_p$ is called the signature function of the metric. This function is continuous on M and we can prove that it is constant on the each connected components of M.

Theorem 8.2. *Let M be a connected manifold and $\langle .,. \rangle$ be a semi-Riemannian $\mathfrak{A}$-valued metric on it. If ν is the signature function of the metric, then ν is constant.*

Proof. First, consider the following subset of $\mathfrak{A}$:

$$N = \{a \in \mathfrak{A} \mid a^* = a \ , \ a^2 = 1\}.$$

Clearly, N is nonempty and the values of ν are in N. We can prove that the distance of any two distinct elements of N is greater than 2. Suppose $a, b \in N$ and $a \neq b$. So,

$$\|a - b\|^2 = \|(a - b)^2\| = \|a^2 + b^2 - 2ab\| = 2\|(1 - ab)\|.$$

Since $(ab)^2 = a^2b^2 = 1$, we have $\sigma(ab) \subset \{-1, 1\}$. -1 must be in $\sigma(ab)$, otherwise $\sigma(ab) = \{1\}$ that implies $ab = 1$, hence $a = b$ that is contrary to the assumption. So, $2 \in \sigma(1 - ab)$ that implies $\|(1 - ab)\| \geq 2$. Consequently,

$$\|a - b\|^2 = 2\|(1 - ab)\| \geq 4 \ \Rightarrow \ \|a - b\| \geq 2.$$

This property of N implies that the induced topology on N is discrete, on the other hand $\nu(M)$ must be connected, so $\nu(M)$ is a singleton and ν is constant. $\square$

If M is an oriented manifold, then for each $p \in M$ we can use any positive oriented basis in T_pM to define an orientation for $T_pM^{\mathfrak{A}}$. Let Ω_p be the canonical volume form on $T_pM^{\mathfrak{A}}$, this gives rise to a globally defined $\mathfrak{A}$-valued volume form $\widetilde{\Omega}$ over M. In a positively oriented coordinate system $(U, x^1, \cdots, x^n)$, if we put $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$ and $g = \det(g_{ij})$, then

$$\widetilde{\Omega} = \sqrt{|g|}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

We could also define the Hodge star operator on $\mathfrak{A}$-valued differential forms. Here, for each $p \in M$ we should consider the induced inner product on $(T_pM^{\mathfrak{A}})^\sharp$.

Definition 8.3. The Hodge star operator $\star : \mathcal{A}^k(M,\mathfrak{A}) \to \mathcal{A}^{n-k}(M,\mathfrak{A})$ maps any k-form $\alpha \in \mathcal{A}^k(M,\mathfrak{A})$ to the $(n-k)$-form $\star\alpha \in \mathcal{A}^{n-k}(M,\mathfrak{A})$ such that for any $\beta \in \mathcal{A}^{n-k}(M,\mathfrak{A})$,

$$\alpha \wedge \beta = \langle \beta, \star\alpha \rangle \, \widetilde{\Omega}.$$

Our previous results now imply that

$$\star(1) = \nu \, \widetilde{\Omega}, \quad \star(\widetilde{\Omega}) = 1, \quad \alpha \wedge \star\beta = (\beta \wedge \star\alpha)^*.$$

For each $\alpha \in \mathcal{A}^k(M,\mathfrak{A})$, Also we have the identity

$$\star\star(\alpha) = (-1)^{k(n-k)} \, \nu \, \alpha. \tag{8.1}$$

Using the Hodge star operator, one can define a new operator, called the coderivative, denoted by δ.

Definition 8.4. The co-differential of $\alpha \in \mathcal{A}^k(M,\mathfrak{A})$ is $\delta\alpha \in \mathcal{A}^{k-1}(M,\mathfrak{A})$ that is defined by

$$\delta\alpha = (-1)^{n(k+1)+1} \nu \, (\star \, \mathrm{d} \star \alpha).$$

Note that the pointwise inner product, induces an $\mathfrak{A}$-valued inner product on $\mathcal{A}_c^k(M,\mathfrak{A})$ by

$$(\alpha, \beta) = \int_M \alpha \wedge \star\beta = \int_M \nu \, \langle \alpha, \beta \rangle \, \widetilde{\Omega}.$$

If the $\mathfrak{A}$-valued metric on M is Riemannian, then this inner product is positive. The next theorem states that with respect to this inner product, the co-differential operator is adjoint to the differential operator.

Theorem 8.5. *For any $\alpha \in \mathcal{A}_c^k(M,\mathfrak{A})$ and $\beta \in \mathcal{A}_c^{k-1}(M,\mathfrak{A})$, $(\mathrm{d}\beta, \alpha) = (\beta, \delta\alpha)$*

Proof. We have

$$\begin{aligned}
(\mathrm{d}\beta, \alpha) &= \int_M \mathrm{d}\beta \wedge \star\alpha = \int_M \mathrm{d}(\beta \wedge \star\alpha) - (-1)^{k-1} \beta \wedge \mathrm{d} \star \alpha \\
&= (-1)^k \int_M \beta \wedge \mathrm{d} \star \alpha,
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\beta, \delta\alpha) &= (\beta, (-1)^{n(k+1)+1} \nu \, (\star \, \mathrm{d} \star \alpha)) = \int_M \beta \wedge (-1)^{n(k+1)+1} \nu \, (\star\star \, \mathrm{d} \star \alpha) \\
&= (-1)^{n(k+1)+1} \nu \, (-1)^{(n-k+1)(k-1)} \nu \int_M \beta \wedge \mathrm{d} \star \alpha \\
&= (-1)^k \int_M \beta \wedge \mathrm{d} \star \alpha,
\end{aligned}$$

therefore, $(\mathrm{d}\beta, \alpha) = (\beta, \delta\alpha)$. ∎

Using the operator δ, we can define Laplace operator Δ on $\mathfrak{A}$-valued differential forms, as follows

$$\Delta = (\mathrm{d}\delta + \delta\mathrm{d}) : \mathcal{A}^k(M,\mathfrak{A}) \longrightarrow \mathcal{A}^k(M,\mathfrak{A})$$

In particular, for a function f (0-form) we find

$$\Delta f = -\nu \star \mathrm{d} \star \mathrm{d}f = -\frac{1}{\sqrt{|g|}} \sum \partial_i (g^{ij} \sqrt{|g|}\, \partial_j f).$$

An $\mathfrak{A}$-valued differential form ω is called harmonic, if $\Delta(\omega) = 0$. The question of relation between De Rham cohomolgy class and $\mathfrak{A}$-valued harmonic forms are answered here partially.

Lemma 8.6. *If M is a compact $\mathfrak{A}$-valued Riemannian manifold and ω is a harmonic form, then $\mathrm{d}\omega = 0$ and $\delta\omega = 0$.*

Proof.

$$0 = (\Delta\omega, \omega) = ((\mathrm{d}\delta + \delta\mathrm{d})(\omega), \omega) = (\delta\omega, \delta\omega) + (\mathrm{d}\omega, \mathrm{d}\omega).$$

Since, $(\delta\omega, \delta\omega)$, and $(\mathrm{d}\omega, \mathrm{d}\omega)$ are positive elements of $\mathfrak{A}$, we deduce that $(\delta\omega, \delta\omega) = 0$ and $(\mathrm{d}\omega, \mathrm{d}\omega) = 0$, so $\mathrm{d}\omega = 0$ and $\delta\omega = 0$. □

Lemma 8.7. *If M is a compact $\mathfrak{A}$-valued Riemannian manifold, then the only exact harmonic form is zero.*

Proof. Suppose $\omega = \mathrm{d}\eta$ and ω is harmonic. We have:

$$(\omega, \omega) = (\omega, \mathrm{d}\eta) = (\delta\omega, \eta) = (0, \eta) = 0 \Longrightarrow \omega = 0.$$

□

Corollary 8.8. *If M is a compact $\mathfrak{A}$-valued Riemannian manifold and two $\mathfrak{A}$-valued harmonic forms ω_1 and ω_2 are in the same cohomology class, then $\omega_1 = \omega_2$.*

9. The Levi-Civita connection of operator-valued metrics

Suppose that M is an $\mathfrak{A}$-valued semi-Riemannian manifold. We say that an $\mathfrak{A}$-connection ∇ is compatible with $\langle .,. \rangle$ if for any three vector fields $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(M)^{\mathfrak{A}}$,

$$\tilde{X}\langle \tilde{Y}, \tilde{Z} \rangle = \langle \nabla_{\tilde{X}} \tilde{Y}, \tilde{Z} \rangle + \langle \tilde{Y}, \nabla_{\tilde{X}^*} \tilde{Z} \rangle. \tag{9.1}$$

This equality holds iff it holds for ordinary vector fields in $\mathfrak{X}M$.

Lemma 9.1. *Suppose that M is an $\mathfrak{A}$-valued semi-Riemannian manifold. For $\widetilde{V} \in \mathfrak{X}(M)^{\mathfrak{A}}$, let $\widetilde{V}^\flat$ be the $\mathfrak{A}$-valued one-form on M given by*

$$\widetilde{V}^\flat(\tilde{X}) = \langle \widetilde{V}, \tilde{X}^* \rangle, \ \textit{for all} \quad \tilde{X} \in \mathfrak{X}(M)^{\mathfrak{A}}.$$

Then the map $\widetilde{V} \longrightarrow \widetilde{V}^\flat$ is a $C^\infty(M,\mathfrak{A})$-module isomorphism from $\mathfrak{X}(M)^{\mathfrak{A}}$ onto $\mathcal{A}^1(M,\mathfrak{A})$.

Proof. Nondegeneracy of the metric gives the result. □

Theorem 9.2. *If M is an $\mathfrak{A}$-valued semi-Riemannian manifold, then there exists a unique torsion free $\mathfrak{A}$-connection ∇ that is compatible with the metric.*

Proof. Fix $\tilde{X}, \tilde{Y} \in \mathfrak{X}(M)^{\mathfrak{A}}$, and define $\mu_{\tilde{X},\tilde{Y}} : \mathfrak{X}(M)^{\mathfrak{A}} \longrightarrow C^{\infty}(M, \mathfrak{A})$ by

$$\begin{aligned} \mu_{\tilde{X},\tilde{Y}}(\tilde{Z}) = \quad & \tilde{X}\langle \tilde{Y}, \tilde{Z}^* \rangle + \tilde{Y}\langle \tilde{Z}, \tilde{X}^* \rangle - \tilde{Z}\langle \tilde{X}, \tilde{Y}^* \rangle \\ & + \langle [\tilde{X}, \tilde{Y}], \tilde{Z}^* \rangle - \langle [\tilde{Y}, \tilde{Z}], \tilde{X}^* \rangle + \langle [\tilde{Z}, \tilde{X}], \tilde{Y}^* \rangle. \end{aligned}$$

A straightforward computation shows that the map $\tilde{Z} \mapsto \mu_{X,Y}(\tilde{Z})$ is $C^{\infty}(M, \mathfrak{A})$-linear and is a $\mathfrak{A}$-valued one-form. By the lemma 9.1, there is a unique $\mathfrak{A}$-vector field, denoted by $2\,\nabla_{\tilde{X}}\tilde{Y}$, such that $\mu_{\tilde{X},\tilde{Y}}(\tilde{Z}) = 2\,\langle \nabla_{\tilde{X}}\tilde{Y}, \tilde{Z}^* \rangle$ for all $\tilde{Z} \in \mathfrak{X}(M)^{\mathfrak{A}}$. Now, standard argument shows that ∇ is the unique torsion free $\mathfrak{A}$-connection that is compatible with the metric. This connection is called the Levi-Civita connection of the metric. □

Proposition 9.3. *If $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{V} \in \mathfrak{X}(M)^{\mathfrak{A}}$, then for the $\mathfrak{A}$-curvature tensor of the Levi-Civita connection, we have*

(i) $\langle \mathcal{R}(\tilde{X}, \tilde{Y})(\tilde{Z}), \tilde{V} \rangle = -\langle \mathcal{R}(\tilde{X}, \tilde{Y})(\tilde{V}^*), \tilde{Z}^* \rangle$;

(ii) $\langle \mathcal{R}(\tilde{X}, \tilde{Y})(\tilde{Z}), \tilde{V} \rangle = \langle \mathcal{R}(\tilde{Z}, \tilde{V}^*)(\tilde{X}), \tilde{Y}^* \rangle$.

Proof. The method of proof is similar to the one for Semi-Riemannian manifolds (cf. [13]). □

On $\mathfrak{A}$-valued semi-Riemannian manifolds we can straightly generalize differential operators such as gradient, and divergence.

Definition 9.4. The gradient of a function $f \in C^{\infty}(M, \mathfrak{A})$ is the $\mathfrak{A}$-vector field that is equivalent to the 1-differential form $\mathrm{d}f \in \mathcal{A}^1(M, \mathfrak{A})$, Thus

$$\langle \nabla f, \tilde{X}^* \rangle = \tilde{X}(f) \qquad \forall\, \tilde{X} \in \mathfrak{X}(M)^{\mathfrak{A}}$$

in terms of a coordinate system, $\nabla f = \dfrac{\partial f}{\partial x^i}\, g^{ij} \dfrac{\partial}{\partial x^j}$.

Definition 9.5. The Hessian of a function $f \in C^{\infty}(M, \mathfrak{A})$ is its second covariant derivative $\mathrm{Hess}(f) = \nabla(\mathrm{d}f)$.

The Hessian of f is a symmetric $(0, 2)$ $\mathfrak{A}$-tensor field and its operation on vector fields $\tilde{X}, \tilde{Y} \in \mathfrak{X}(M)^{\mathfrak{A}}$ is defined as follows:

$$\mathrm{Hess}(f)(\tilde{X}, \tilde{Y}) = \tilde{X}(\tilde{Y}f) - \langle \nabla f, \nabla_{\tilde{X}^*}\tilde{Y}^* \rangle = \langle \nabla_{\tilde{X}}(\nabla f), \tilde{Y}^* \rangle.$$

Definition 9.6. If $\widetilde{X}$ is an $\mathfrak{A}$-vector field, the contraction of its covariant differential is called divergence of $\widetilde{X}$ and is denoted by $\mathrm{div}(\widetilde{X}) \in C^{\infty}(M, \mathfrak{A})$. In a coordinate system, $\mathrm{div}(\widetilde{X}) = g^{ij}\,\langle \nabla_{\partial_i}\widetilde{X}, \partial_j \rangle$.

Theorem 9.7. *Let M be a an oriented $\mathfrak{A}$-valued semi-Riemannian manifold, and $\tilde{\Omega}$ be its canonical volume form. Then, for any $\mathfrak{A}$-vector field $\tilde{X} \in \mathfrak{X}(M)^{\mathfrak{A}}$ we have*

$$\mathcal{L}_{\tilde{X}}\tilde{\Omega} = \operatorname{div}(\tilde{X})\tilde{\Omega}.$$

Proof. Computations, as in [18], show that the equality holds in the case of scalar metrics. But, all these computations, without any change, are also valid in the case of $\mathfrak{A}$-valued metrics, except that the functions we encounter are $\mathfrak{A}$-valued. □

10. **Ricci, scalar curvature, and sectional curvature**

In the past sections, we have presented the basic notions and facts about the curvature of the Levi-Civita connection of a given $\mathfrak{A}$-valued semi-Riemannian manifold. We begin to consider some invariants that truly characterize curvature. In this section, M is an $\mathfrak{A}$-valued semi-Riemannian manifold with the $\mathfrak{A}$-Levi-Civita connection ∇.

Definition 10.1. For each $p \in M$, the Ricci curvature tensor,
$\mathcal{R}\mathrm{ic}_p : T_pM^{\mathfrak{A}} \times T_pM^{\mathfrak{A}} \longrightarrow \mathfrak{A}$ is given by

$$\mathcal{R}\mathrm{ic}_p(\tilde{u}, \tilde{v}) = \operatorname{trace}\big(\ \tilde{w} \longrightarrow \mathcal{R}(\tilde{w}, \tilde{u})\tilde{v}\ \big),$$

and the scalar curvature $\mathcal{S}$ is the trace of $\mathcal{R}\mathrm{ic}$.

In coordinate systems,

$$\mathcal{R}\mathrm{ic}(\widetilde{X}, \widetilde{Y}) = g^{ij}\langle \mathcal{R}(\frac{\partial}{\partial x^i}, \widetilde{X})\widetilde{Y}, \frac{\partial}{\partial x^j}\rangle, \quad \mathcal{S} = g^{ij}\mathcal{R}\mathrm{ic}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$$

Thus, $\mathcal{R}ic$ is a symmetric $(0, 2)$ tensor on M.

A two-dimensional free $\mathfrak{A}$-submodule Π of $T_pM^{\mathfrak{A}}$ is called an $\mathfrak{A}$-tangent plane to M at p. For $p \in M$, $\tilde{u}, \tilde{v} \in T_pM^{\mathfrak{A}}$, define $\mathcal{Q}(\tilde{u}, \tilde{v}) = \langle \tilde{u}, \tilde{u}\rangle\langle \tilde{v}, \tilde{v}\rangle - \langle \tilde{u}, \tilde{v}\rangle \langle \tilde{u}, \tilde{v}\rangle^*$. In fact, $\mathcal{Q}(\tilde{u}, \tilde{v}) = \langle \tilde{u} \wedge \tilde{v}, \tilde{u} \wedge \tilde{v}\rangle$.

Definition 10.2. A $\mathfrak{A}$-tangent plane Π to M is called nondegenerate if for some base $\{\tilde{u}, \tilde{v}\}$ of Π, $\mathcal{Q}(\tilde{u}, \tilde{v})$ is invertible in $\mathfrak{A}$.

The invertiblility of $\mathcal{Q}(\tilde{u}, \tilde{v})$ does not depend on the choice of the base. If $\{\tilde{u}, \tilde{v}\}$ is a base $T_pM^{\mathfrak{A}}$, then

$$\mathcal{K}(\tilde{u}, \tilde{v}) := \frac{\langle \mathcal{R}(\tilde{u}, \tilde{v})\tilde{v}^*, \tilde{u}\rangle}{\mathcal{Q}(\tilde{u}, \tilde{v})}$$

is well-defined and only depends on the 2-dimensional submodule determined by $\tilde{u}$ and $\tilde{v}$.

Definition 10.3. We refer to $\mathcal{K}(\tilde{u}, \tilde{v})$ as the sectional curvature of the 2-plane determined by $\tilde{u}$ and $\tilde{v}$.

Corollary 10.4. *If M has constant curvature c, then*

$$\mathcal{R}(\tilde{u}, \tilde{v})\tilde{w} = c\{\langle \tilde{v}, \tilde{w}^*\rangle\tilde{u} - \langle \tilde{u}, \tilde{w}^*\rangle\tilde{v}\}.$$

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