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AUTOREGRESSIVE GAUSSIAN RANDOM VECTORS OF FIRST ORDER

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Abstract. Autoregressive Gaussian random vectors of order one are introduced, characterized and studied. The characterization involves the existence and structural identification of the covariance matrix. Prediction for future values together with necessary and sufficient conditions for the stationarity are established. Some basic statistical properties are also presented. This class of random vectors appears to be suitable for modeling samples of small size of values with short range dependency and autoregressive property where an observation has persistent effect on its subsequent observation.

1. Introduction

In this work we will introduce and study a class of Gaussian random vectors, that are called here "*autoregressive Gaussian random vectors of order 1*". Let us define such a random vector: A Gaussian random vector $\mathbf{X}_n = (X_1, \cdots, X_n)'$ is said to be autoregressive of order 1 if there is a real number α for which $X_2 - \alpha X_1, \cdots, X_n - \alpha X_{n-1}$ are independent and identically distributed (i.i.d in short). We refer to the parameter α as the coefficient. If we let $Z_i = X_i - \alpha X_{i-1}$, then $X_i = \alpha X_{i-1} + Z_i$,

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$i = 2, \cdots, n$; thereby justifying the term autoregressive. The term is familiar in time series. An autoregressive process of order one, AR(1), is a process with short range dependency where the immediate future value is a finite linear combination, with fixed coefficients, of the present value and an innovation. The process is also stationary if and only if $|\alpha| < 1$, see Brockwell and Davis (1991). The process has intensively been applied to model data in different applied fields, specially in economy and hydrology. To build an AR(1) model evidently, data should come from an ideal environment where stationarity is feasible for a long enough period of time and the dependency structure does not vary in time. In practice, assuming such an ideal state is not plausible. The global warming, the economy disaster of 1998 in Asia which caused a sharp decrease in the price of the crude oil, and affected its forthcoming economic figures, and also the Tsunami of 2004 in East Asia that has depressed the tourist industry indices in its subsequent periods are examples among many. Another issue in modeling of an AR(1) is the sample size. Such a models usually is not reliable for series of small sizes, also early observations are of less value in fitting an AR(1); the correlation between X_1, X_i $(\rho(X_1, X_i))$ decreases to zero exponentially, as i increases. In the cited situations, if components of $\mathbf{X}_n = (X_1, \cdots, X_n)$ explain n immediate economic data after the incident, then it will not be realistic to assume a sharp decrease in $\rho(X_1, X_i)$ as $i \longrightarrow n$. In order to overcome such discrepancies while still holding on to the autoregressiveness, as an alternative to the AR(1) model, we propose *autoregressive Gaussian random vectors of order 1* (ARGRV(1) in short). As we will exhibit in the proceeding sections in contrast to AR(1) models, the effect of X_1 on forthcoming values in an ARGRV(1) remains feasible and can be controlled to decrease slowly.

As we learned from a referee, modeling long lasting effects of certain events in time series has been discussed in the literature, which is referred to as "intervention analysis". The issue is discussed and modeled in Chapter 12 of Box, Jenkins and Reinsel (1994), based on the work of Box and Tiao (1975). In their model, the intervention event occurs at a known point in time T, causing a time series X_t to be depressed by an intervention component, say I_t, and become $Y_t = I_t + X_t,\ t \geq T$. The intervention component in the cited work is taken to be a certain analytic operator acting on the step or the pulse functions. More importantly, intervention component is taken to be deterministic (non-random). Thus,

in contrast to ARGRV, the underlying stochastic phenomena of the series are not affected by the intervention event in the Box-Tiao model. Indeed, in their model, the intervention factor only affects the mean, and consequently the correlation between the observation at the time of the intervention and any of the subsequent observations, in the series X_t is the same as that in the series Y_t. But as we discussed earlier, autocorrelations in an ARGRV are different from those of the corresponding AR, exhibiting stochastic and statistical effects of an intervention event on its successive values.

Another issue, as pointed out by a referee, is that similar to the second order AR processes, one can define autoregressive second order random vectors AR(SO)RV; assuming $\mathbf{X}_n = (X_1, \cdots, X_n)$ to be a second order random vector, i.e. every linear combination of its components possesses the second moment, that $X_2 - \alpha X_1, \cdots, X_n - \alpha X_{n-1}$ are uncorrelated. In Remark 2.3 and in Section 5 we will discuss this model. The Gaussian assumption will provide the joint distribution of the vector which is a powerful tool in any statistical analysis.

AR(1) models and some of its variations have intensively been studied by different authors such as Andel (1988), Andrews (1993), Abraham and Balakrishnan (1999) and Zielinski (1999) among others.

The paper is organized as follows. In Section 2 we will provide a necessary and sufficient condition for the existence of an ARGRV(1). Regression and prediction are established in Section 3. Section 4 is devoted to examples, where stationary ARGRV(1) is introduced. We conclude the article by Section 5 which is for a discussion and future works.

2. **Characterization of ARGRV(1)**

Let $\mathbf{Z}_n = (Z_1, Z_2, ..., Z_n)'$ be a multivariate Gaussian (normal) random vector with the zero mean vector and a variance-covariance matrix $\Sigma_{\mathbf{Z}_n}$, i.e. $\mathbf{Z}_n \sim \mathrm{N}_n(\mathbf{0}, \Sigma_{\mathbf{Z}_n})$. Suppose $\Sigma_{\mathbf{Z}_n}$ has the following structure

$$\Sigma_{\mathbf{Z}_n} = \begin{bmatrix} \tau^2 & \mathbf{L}_n{}' \\ \mathbf{L}_n & \sigma^2 \mathbf{I}_{n-1} \end{bmatrix}, \tag{2.1}$$

where $\tau^2 = \mathrm{Var}(Z_1)$, $\sigma^2 = \mathrm{Var}(Z_i)$, $\mathbf{L}_n{}' = (\zeta_2, ..., \zeta_n)$, $\zeta_i = \mathrm{Cov}(Z_1, Z_i)$, $i = 2, ..., n$. with "Var" and "Cov" standing for the variance and the covariance, respectively, and $\mathbf{I}_n = [\delta_{ij}]_{i,j=1,...,n}$, $\delta_{ij} = 1$; $i = j$, $\delta_{ij} = 0$;

$i \neq j$. In the following we present conditions under which $\Sigma_{\mathbf{Z_n}}$ is non-negative definite. Equivalently, the determinant of every principle minor of order k, of $\Sigma_{\mathbf{Z}_n}$, $\Sigma_{\mathbf{Z}_k}$, $k = 2, ..., n$ must be non-negative, i.e.,

$$\begin{aligned}
& \det \begin{bmatrix} \tau^2 & (\zeta_2, ..., \zeta_k) \\ (\zeta_2, ..., \zeta_k)' & \sigma^2 \mathbf{I}_{k-1} \end{bmatrix} \\
= \; & \tau^{2k} \det \begin{bmatrix} 1 & \frac{1}{\tau^2}(\zeta_2, ..., \zeta_k) \\ \frac{1}{\tau^2}(\zeta_2, ..., \zeta_k)' & \frac{\sigma^2}{\tau^2}\mathbf{I}_{k-1} \end{bmatrix} \\
= \; & \tau^{2k} \det[\frac{\sigma^2}{\tau^2}\mathbf{I}_{k-1} - \frac{1}{\tau^4}(\zeta_2, ..., \zeta_k)'(\zeta_2, ..., \zeta_k)] \\
= \; & \tau^2 \sigma^{2(k-1)} \det[\mathbf{I}_{k-1} - \frac{1}{\tau^2\sigma^2}(\zeta_2, ..., \zeta_k)'(\zeta_2, ..., \zeta_k)] \\
= \; & \tau^2 \sigma^{2(k-1)} (1 - \frac{1}{\tau^2\sigma^2} \sum_{i=2}^{k} \zeta_i^2) \geq 0,
\end{aligned}$$

see Goulb and Van Loan (1989). Therefore $\Sigma_{\mathbf{Z}_n}$is non-negative definite if and only if

$$\tau^2\sigma^2 \geq \sum_{i=2}^{k} \zeta_i^2, k = 2, 3, ..., n,$$

or equivalently

$$\tau^2\sigma^2 \geq \sum_{i=2}^{n} \zeta_i^2. \tag{2.2}$$

Thus we have proved the following lemma.

Lemma 2.1. *The matrix $\Sigma_{\mathbf{Z}_n}$ is non-negative definite if and only if (2.2) is fulfilled.*

Remark 2.1. Note that for every value of τ^2 and σ^2, we can find n, $\zeta_2, ..., \zeta_n$ such that (2.2) is satisfied. This indicates that the class of random vectors of type $\mathbf{Z}_n$ is rich enough.

In the following we present two examples where such vectors arrive in more natural ways.

Example 2.1. Let $Y_1, ..., Y_n$ be i.i.d $\mathrm{N}(0, \sigma^2)$. Define the random vector $(Z_1, ..., Z_n)'$ as follows:

$$\begin{aligned} Z_1 &= Y_1, \\ Z_i &= \left(Y_i - (1/(i-1)) \sum_{k=1}^{i-1} Y_k \right) \sqrt{\frac{i-1}{i}}, \quad i = 2, ..., n. \end{aligned}$$

We note that Z_i is the normalization of the deviation of the immediate future value Y_i from the current mean $\frac{1}{i-1} \sum_{k=1}^{i-1} Y_k$. Clearly $\mathrm{E}(Z_i) = 0$, where E stands for the expected value, and

$$\begin{aligned} \mathrm{Var}(Z_i) &= \left(\frac{i-1}{i} \right) \left[\sigma^2 + \frac{\sigma^2}{i-1} \right] \\ &= \sigma^2, \quad i = 1, ..., n. \end{aligned}$$

Also note that for $i > j > 1$,

$$\begin{aligned} \mathrm{Cov}(Z_i, Z_j) &= \mathrm{Cov} \left((Y_i - \frac{\sum_{k=1}^{i-1} Y_k}{i-1}) \sqrt{\frac{i-1}{i}}, (Y_j - \frac{\sum_{k=1}^{j-1} Y_k}{j-1}) \sqrt{\frac{j-1}{j}} \right) \\ &= \sqrt{\frac{(i-1)(j-1)}{ij}} Cov \left(Y_i - \frac{\sum_{k=1}^{i-1} Y_k}{i-1}, Y_j - \frac{\sum_{k=1}^{j-1} Y_k}{j-1} \right) \\ &= \sqrt{\frac{(i-1)(j-1)}{ij}} \left[-\frac{\sigma^2}{i-1} + \frac{\sigma^2 (j-1)}{(i-1)(j-1)} \right] \\ &= 0. \end{aligned}$$

Similarly

$$\zeta_i = \mathrm{Cov}(Z_i, Z_1) = -\sigma^2 \sqrt{\frac{1}{i\,(i-1)}}\,, \quad i = 2, ..., n.$$

Hence $\tau^2 = \sigma^2$ and $\sum_{i=2}^{n} \zeta_i^2 = \sigma^4 \left(1 - \frac{1}{n}\right)$, which imply (2.2), for every $n \geq 2$.

Example 2.2. Let $(Y_1, ..., Y_n)' \sim N_n(\mathbf{0}, \mathbf{B})$, in which the covariance matrix $\mathbf{B}$ assumes the following form:

$$\mathbf{B} = \begin{bmatrix} 1 & \theta & \cdots & \theta \\ \theta & 1 & \cdots & \theta \\ \vdots & \vdots & \cdots & \vdots \\ \theta & \theta & \cdots & 1 \end{bmatrix} = (1-\theta)\mathbf{I}_n + \theta \mathbf{1}\mathbf{1}' ,$$

where $\mathbf{1}' = (1, ..., 1)$ and θ is restricted to $\theta \geq -\frac{1}{n-1}$. This matrix often describes the correspondence among certain biological variables such as the sizes of living things, see Johnoson and Wichern (1988), page 349.

By using Johnoson and Wichern (1988), it readily follows that

$$\mathrm{E}\left(Y_i | Y_{i-1}, Y_{i-2}, ..., Y_2\right) = \frac{\theta}{1+(i-3)\,\theta} \sum_{k=2}^{i-1} Y_k, \ i \geq 3,$$

and

$$\mathrm{Var}\left(Y_i | Y_{i-1}, Y_{i-2}, ..., Y_2\right) = 1 - \frac{(i-2)\,\theta^2}{1+(i-3)\,\theta}, \ i \geq 3.$$

Define the random vector $(Z_1, Z_2, ..., Z_n)$ as $Z_1 = Y_1, \ Z_2 = Y_2$ and

$$Z_i = \left(1 - \frac{(i-2)\,\theta^2}{1+(i-3)\,\theta}\right)^{-\frac{1}{2}} \left(Y_i - \mathrm{E}\left(Y_i | Y_{i-1}, Y_{i-2}, ..., Y_2\right)\right), i = 3, ..., n.$$

Therefore $\mathrm{E}(Z_i) = 0$, and for $i = 3, ..., n$, we have

$$\begin{aligned} \mathrm{Var}(Z_i) &= \left(1 - \frac{(i-2)\,\theta^2}{1+(i-3)\,\theta}\right)^{-1} \mathrm{E}(Y_i - \mathrm{E}\left(Y_i | Y_{i-1}, Y_{i-2}, ..., Y_2\right))^2 \\ &= \left(1 - \frac{(i-2)\,\theta^2}{1+(i-3)\,\theta}\right)^{-1} \left[\mathrm{Var}\left(Y_i | Y_{i-1}, Y_{i-2}, ..., Y_2\right)\right] \\ &= \left(1 - \frac{(i-2)\,\theta^2}{1+(i-3)\,\theta}\right)^{-1} \left[1 - \frac{(i-2)\,\theta^2}{1+(i-3)\,\theta}\right] \\ &= 1. \end{aligned}$$

Also note that $Z_2, ..., Z_n$ are normally distributed successive innovations (co-projections) and therefore are independent. Furthermore

$$\zeta_2 = \mathrm{Cov}(Z_2, Z_1) = \mathrm{Cov}\left(Y_2, Y_1\right) = \theta,$$

and

$$\begin{aligned}
\zeta_i &= \mathrm{Cov}(Z_i, Z_1) \\
&= \mathrm{Cov}\left(Y_i - \frac{\theta}{1+(i-3)\,\theta}\sum_{k=2}^{i-1} Y_k, Y_1\right)\left(1 - \frac{(i-2)\,\theta^2}{1+(i-3)\,\theta}\right)^{-\frac{1}{2}} \\
&= \left(\theta - \frac{(i-2)\,\theta^2}{1+(i-3)\,\theta}\right)\left(1 - \frac{(i-2)\,\theta^2}{1+(i-3)\,\theta}\right)^{-\frac{1}{2}} \\
&= \left(\frac{(1-\theta)\,\theta}{1+(i-3)\,\theta}\right)\left(\frac{(1-\theta)\,(1+(i-2)\,\theta)}{1+(i-3)\,\theta}\right)^{-\frac{1}{2}} \\
&= \theta\sqrt{\frac{1-\theta}{(1+(i-3)\,\theta)\,(1+(i-2)\,\theta)}}, \quad i = 3, ..., n.
\end{aligned}$$

Therefore for every $i = 2, 3, ..., n$,

$$\zeta_i = \theta\sqrt{\frac{1-\theta}{(1+(i-3)\,\theta)\,(1+(i-2)\,\theta)}}.$$

On the other hand $\tau^2 = \sigma^2 = 1$, and

$$\begin{aligned}
\sum_{i=2}^{n} \zeta_i^2 &= \theta\,(1-\theta)\sum_{i=2}^{n}\left(\frac{1}{1+(i-3)\theta} - \frac{1}{1+(i-2)\theta}\right) \\
&= \theta\,(1-\theta)\left(\frac{1}{1-\theta} - \frac{1}{1+(n-2)\theta}\right) \\
&= \theta\,(1-\theta)\left(\frac{(n-1)\,\theta}{(1-\theta)\,(1+(n-2)\theta)}\right) \\
&= \frac{(n-1)\,\theta^2}{1+(n-2)\theta}.
\end{aligned}$$

Therefore (2.2) is equivalent to $\frac{(n-1)\theta^2}{1+(n-2)\theta} \leq 1$ which is satisfied if and only if $\theta \geq -\frac{1}{n-1}$.

Now we are in a position to characterize an ARGRV(1). The following theorem provides the details.

Theorem 2.1. *A random vector* $\mathbf{X}_n = (X_1, X_2, ..., X_n)'$ *is an ARGRV(1) if and only if*

$$\mathbf{X}_n = \mathbf{A}^{-1}\mathbf{Z}_n, \tag{2.3}$$

where $\mathbf{Z}_n \sim N_n(\mathbf{0}, \Sigma_{\mathbf{Z}_n}), \Sigma_{\mathbf{Z}_n}$ *is given by (2.1) and satisfies (2.2), and*

$$\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n} : a_{ii} = 1, a_{i(i-1)} = -\alpha, a_{ij} = 0, \text{ for } j \neq i, i-1. \quad (2.4)$$

Proof. If $\mathbf{X}_n = (X_1, \cdots, X_n)'$ is an ARGRV(1), then according to the definition presented in Section 1, $\mathbf{X}_n \sim \mathrm{N}_n(\mathbf{0}\ , \Sigma_{\mathbf{X}_n})$ and there is a real number α for which $X_2 - \alpha X_1, \cdots, X_n - \alpha X_{n-1}$ are i.i.d. If we let $Z_1 = X_1$ and $Z_i = X_i - \alpha X_{i-1}, \ \ i = 2, \cdots, n$, then $\mathbf{Z}_n = \mathbf{A}\mathbf{X}_n \sim \mathrm{N}_n(\mathbf{0}\ , \Sigma_{\mathbf{Z}_n})$, where $\Sigma_{\mathbf{Z}_n}$ has the form given by (2.1). But $\Sigma_{\mathbf{Z}_n} = A\Sigma_{\mathbf{X}_n}\mathbf{A}'$ is non-negative definite, thus according to Lemma 2.1 it satisfies (2.2). Also note that since $\mathbf{A}$ is nonsingular ($\det(\mathbf{A}) = 1$), $\mathbf{X}_n = \mathbf{A}^{-1}\mathbf{Z}_n$.

On the other hands, let $\mathbf{X}_n = \mathbf{A}^{-1}\mathbf{Z}_n$, where $\mathbf{A}$ is given by (2.4), $\mathbf{Z}_n \sim \mathrm{N}_n(\mathbf{0}, \Sigma_{\mathbf{Z}_n})$, $\Sigma_{\mathbf{Z}_n}$ is given by (2.1) and satisfies (2.2). It readily follows that $Z_1 = X_1$, $Z_i = X_i - \alpha X_{i-1}, \ \ i = 2, ..., n$. Thus $X_i - \alpha X_{i-1}, \ \ i = 2, ..., n$ are independent, and consequently will be $\mathbf{X}_n$ ARGRV(1). □

Notation. We write $\mathbf{X}_n \sim \mathrm{ARG}(\mathbf{0},\ \alpha,\ \Sigma\ )$, if $\mathbf{X}_n = \mathbf{A}^{-1}\mathbf{Z}_n$, $\mathbf{Z}_n \sim \mathrm{N}_n(\mathbf{0}, \Sigma)$, $\mathbf{A}$ is given by (2.4) and Σ is given by (2.1) satisfying (2.2).

Remark 2.2. It follows from (2.3) and (2.4) that

$$X_i = \sum_{j=1}^{i} \alpha^{i-j} Z_j, \quad i = 1, ..., n, \quad (2.5)$$

thus $\{X_i\}_{i=1,\ldots,n}$ is a non-anticipating moving average, as in AR(1), but with different dependency structure.

It follows from (2.5) that

$$\sigma_i^2 = \mathrm{Var}\,(X_i) = \alpha^{2(i-1)}\tau^2 + \left(\frac{\alpha^{2(i-1)} - 1}{\alpha^2 - 1}\right)\sigma^2 + 2\sum_{k=2}^{i} \alpha^{2i-k-1}\zeta_k, \quad (2.6)$$

and for $i > j \geq 2$,

$$\begin{aligned}
\sigma_{ij} &= \mathrm{Cov}(X_i, X_j) = \mathrm{Cov}\left(\sum_{k=1}^{i} \alpha^{i-k} Z_k, \sum_{k=1}^{j} \alpha^{j-k} Z_k\right) \\
&= \sum_{k=1}^{j} \alpha^{i-k}\alpha^{j-k}\mathrm{Var}\,(Z_k) + 2\sum_{k=2}^{j} \alpha^{i+j-k-1}\mathrm{Cov}\,(Z_1, Z_k) \\
&\quad + \sum_{k=j+1}^{i} \alpha^{i+j-k-1}\mathrm{Cov}\,(Z_1, Z_k) \\
&= \alpha^{i+j-2}\tau^2 + (\sum_{k=2}^{j} \alpha^{i+j-2k})\sigma^2 + 2\sum_{k=2}^{j} \alpha^{i+j-k-1}\zeta_k \\
&\quad + \sum_{k=j+1}^{i} \alpha^{i+j-k-1}\zeta_k, \\
&= \alpha^{i+j-2}\tau^2 + (\frac{\alpha^{i+j-2} - \alpha^{i-j}}{\alpha^2 - 1})\sigma^2 + 2\sum_{k=2}^{j} \alpha^{i+j-k-1}\zeta_k \\
&\quad + \sum_{k=j+1}^{i} \alpha^{i+j-k-1}\zeta_k. \qquad (2.7)
\end{aligned}$$

It follows from (2.7) that

$$\sigma_{ij} - \alpha\sigma_{ij-1} = \mathrm{Cov}(X_i, Z_j) = \alpha^{i-j}\sigma^2 + \alpha^{i-1}\zeta_j, \qquad (2.8)$$

and

$$\sigma_{1i} = \alpha^{i-1}\tau^2 + \sum_{k=2}^{i} \alpha^{i-k}\zeta_k. \qquad (2.9)$$

Therefore,

$$\rho\,(X_1, X_i) = \frac{\alpha^{i-1}\tau^2 + \sum\limits_{k=2}^{i} \alpha^{i-k}\zeta_k}{\tau\left(\alpha^{2(i-1)}\tau^2 + \left(\frac{\alpha^{2(i-1)}-1}{\alpha^2-1}\right)\sigma^2 + 2\sum\limits_{k=2}^{i} \alpha^{2i-k-1}\zeta_k\right)^{\frac{1}{2}}}. \qquad (2.10)$$

In Example 2.1, by (2.5) we obtain

$$\begin{aligned} X_1 &= Z_1 = Y_1, \\ X_i &= \sum_{j=1}^{i} \alpha^{i-j} \left(Y_j - (1/(j-1)) \sum_{k=1}^{j-1} Y_k \right) \sqrt{\frac{j-1}{j}}, \quad i = 2, ..., n. \end{aligned}$$

Hence it follows from (2.10) that

$$\rho(X_1, X_i) = \alpha^{i-1} \left(\frac{1 - \alpha \sum_{k=2}^{i} 1/\alpha^k \sqrt{k(k-1)}}{\sqrt{(\alpha^{2i}-1)/(\alpha^2-1) - 2\alpha^{2i-1} \sum_{k=2}^{i} 1/\alpha^k \sqrt{k(k-1)}}} \right).$$

Similarly in Example 2.2, we have

$$\rho(X_1, X_i) = \alpha^{i-1} \left(\frac{1+\alpha\theta\sqrt{1-\theta} \sum_{k=2}^{i} 1/\alpha^k \sqrt{(1+(k-3)\theta)(1+(k-2)\theta)}}{\sqrt{(\alpha^{2i}-1)/(\alpha^2-1) + 2\alpha^{2i-1}\theta\sqrt{1-\theta} \sum_{k=2}^{i} 1/\alpha^k \sqrt{(1+(k-3)\theta)(1+(k-2)\theta)}}} \right).$$

These correlations together with the corresponding correlations in an AR(1) are plotted in Figure 1. As it indicates $\rho(X_1, X_i)$ vanishes for $i \geq 8$ in AR(1), but in ARGRV(1), $\rho(X_1, X_i)$ is significantly different from zero, $i \leq 20$.

Remark 2.3. Similar to the second order processes, it is possible to define an autoregressive second order random vector as follows. Assume $\mathbf{X}_n = (X_1, \cdots, X_n)$ is a mean zero random vector whose components possess second moments. Furthermore assume $X_2 - \alpha X_1, \cdots, X_n - \alpha X_{n-1}$ are uncorrelated. Then the linear space $\mathcal{H}$ generated by $X_1, X_2 - \alpha X_1, \cdots, X_n - \alpha X_{n-1}$, equipped with the inner product $\mathrm{E}XY$, $X, Y \in \mathcal{H}$, is a Hilbert space of dimension n, assuming that X_1 is not a linear combination of $X_2 - \alpha X_1, \cdots, X_n - \alpha X_{n-1}$. Conversely, if $\mathcal{H}$ is an n-dimensional Hilbert space of mean zero random variables, where the inner product is given by the covariance, then an AR(SO)RV can easily be formed as follows. Let $Z_1, \cdots, Z_n$ be an orthonormal basis in $\mathcal{H}$, and X_1 an arbitrary element in $\mathcal{H}$ which is not a linear combination of $Z_2, \cdots, Z_n$, then the random vector $\mathbf{X}_n = (X_1, \cdots, X_n)$ in which

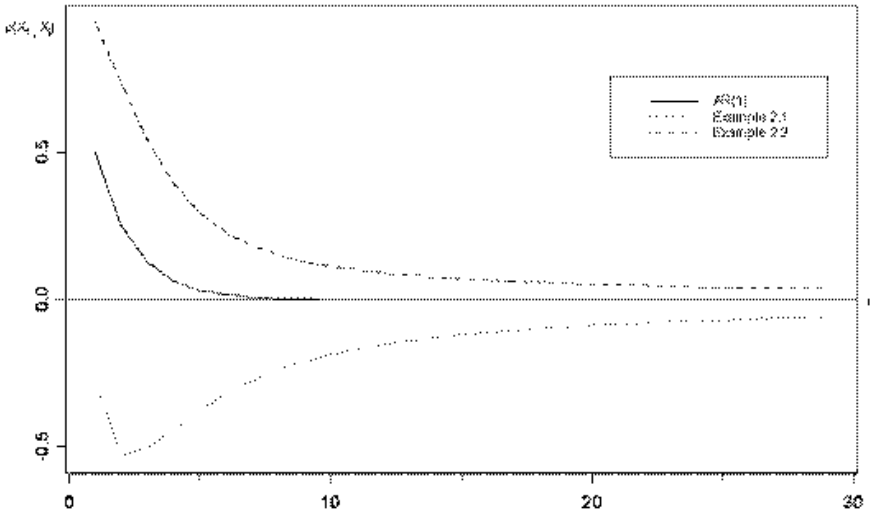

FIGURE 1. Plots of $\rho(X_1, X_i)$, $i = 1, ..., 30$; $\alpha = 0.5$ for ARGRV(1); Example 2.1 and Example 2.2 ($\theta = .7$); and in AR(1); the subjects in Remark 2.2.

$X_j = \alpha X_{j-1} + Z_j$, $j = 2, \cdots, n$, for any given nonzero real number α, is an AR(SO)RV in $\mathcal{H}$ that generates $\mathcal{H}$.

3. **Regression and prediction**

It is well-known that if $\mathbf{X}_n = (X_1, X_2, ..., X_n)' \sim \mathrm{N}_n(\mathbf{0}, \Sigma_{\mathbf{X}_n})$, then $\mathbf{X}_{i+1} = (X_1, X_2, ..., X_{i+1})' \sim \mathrm{N}_{i+1}(\mathbf{0}, \Sigma_{\mathbf{X}_{i+1}})$, $i = 0, ..., n-1$, where

$$\Sigma_{\mathbf{X}_{i+1}} = \begin{bmatrix} \Sigma_{\mathbf{X}_i} & \Sigma_{12} \\ \Sigma_{12}' & \sigma_{i+1}^2 \end{bmatrix},$$

and $\Sigma_{12}' = (\sigma_{1(i+1)}, \ ..., \ \sigma_{i(i+1)})$, $\sigma_{j(i+1)} = \mathrm{Cov}(X_j, \ X_{i+1})$, $\sigma_{i+1}^2 = \mathrm{Var}(X_{i+1})$, $i = 1, ..., n-1$. Therefore

$$\mathrm{E}(X_{i+1}|X_1 = x_1, ..., X_i = x_i) = \Sigma_{12}\Sigma_{\mathbf{X}_i}^{-1}\mathbf{x}_i,$$

where $\mathbf{x}_i = (x_1, ..., x_i)'$, see Johnson and Wichern (1988). On the other hand for $\mathbf{A} = [a_{jk}]_{j,k=1,...i}$, $n = i$, given by (2.4),

$$\mathbf{A}^{-1} = [a^{\#}_{jk}]_{j,k=1,\ldots,n}, a^{\#}_{ii} = 1, a^{\#}_{jk} = \alpha^{j-k}, \text{ for } j > k, a^{\#}_{jk} = 0, \text{ for } j < k.$$

Consequently, it follows from (2.3) that

$$\Sigma_{\mathbf{X}_i} = \mathbf{A}^{-1}\Sigma_{\mathbf{Z}_i}\left(\mathbf{A}^{-1}\right)', \ \Sigma^{-1}_{\mathbf{X}_i} = \mathbf{A}'\Sigma^{-1}_{\mathbf{Z}_i}\mathbf{A}$$

giving that

$$\Sigma^{-1}_{\mathbf{Z}_i} = \begin{bmatrix} \tau^2 & \mathbf{L}'_i \\ \mathbf{L}_i & \sigma^2\mathbf{I}_{i-1} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\sigma^2}{h_i} & -\frac{\mathbf{L}'_i}{h_i} \\ -\frac{\mathbf{L}_i}{h_i} & \frac{1}{\sigma^2}\mathbf{I}_{i-1} + \frac{\mathbf{L}_i\mathbf{L}'_i}{h_i\sigma^2} \end{bmatrix},$$

where $h_i = \tau^2\sigma^2 - \sum_{k=2}^{i}\zeta_k^2$ and $\mathbf{L}_i\mathbf{L}'_i = (\zeta_2\mathbf{L}_i, ...,\zeta_i\mathbf{L}_i)'$, $i = 2, ..., n$.
We have prepared the ingredients for proving the main theorem of this section.

Theorem 3.1. *Let* $\mathbf{X}_n = (X_1, \cdots, X_n)' \sim ARG(\mathbf{0}, \alpha, \Sigma)$. *Then the best (linear) predictor for* X_{i+1} *based on* $X_1, \cdots, X_i$ *is given by*

$$\hat{X}(i+1; 1, ..., i) = \sum_{j=1}^{i} d_i(j)Z_j, \tag{3.1}$$

with the mean-square prediction error

$$\begin{aligned} e(i+1; 1, ..., i) &= \sigma^2_{i+1} + d_i^2(1)\tau^2 + \sigma^2\sum_{j=2}^{i} d_i^2(j) \\ &\quad -2\sum_{j=1}^{i} d_i(j)(\alpha^{i-j+1}\tau^2 \\ &\quad +\alpha^i\zeta_j) + 2d_i(1)\sum_{j=2}^{i} d_i(j)\zeta_j, \end{aligned} \tag{3.2}$$

where

$$d_i(1) = \alpha^i + \frac{\sigma^2\zeta_{i+1}}{h_i} \text{ and } d_i(j) = \alpha^{i-j+1} - \frac{\zeta_j\zeta_{i+1}}{h_i}, \quad j = 2, ..., i. \tag{3.3}$$

Proof. By (3.1) we have

$$\begin{aligned}
\hat{X}\left(i+1;1,...,i\right) &= \mathrm{E}(X_{i+1}|X_1,...,X_i) = \mathrm{E}(X_{i+1}|Z_1,...,Z_i) \\
&= \mathrm{E}(\sum_{j=1}^{i+1}\alpha^{i-j+1}Z_j|Z_1,...,Z_i) \\
&= \sum_{j=1}^{i}\alpha^{i-j+1}Z_j + \mathrm{E}(Z_{i+1}|Z_1,...,Z_i) \\
&= \sum_{j=1}^{i}\alpha^{i-j+1}Z_j + (\zeta_{i+1},0,...,0)\Sigma_{\mathbf{Z}_i}^{-1}\mathbf{Z}_i \\
&= \sum_{j=1}^{i}\alpha^{i-j+1}Z_j + \frac{\zeta_{i+1}}{h_i}\sigma^2 Z_1 - \frac{\zeta_{i+1}}{h_i}\sum_{j=2}^{i}\zeta_j Z_j \\
&= \sum_{j=1}^{i}d_i(j)Z_{j,}
\end{aligned}$$

and the mean-square prediction error is also equal to

$$\begin{aligned}
e\left(i+1;1,...,i\right) &= E\left(X_{i+1}-\hat{X}\left(i+1;1,...,i\right)\right)^2 \\
&= \sigma_{i+1}^2 + d_i^2(1)\tau^2 + \sigma^2\sum_{j=2}^{i}d_i^2(j) \\
&\quad -2\sum_{j=1}^{i}d_i(j)(\alpha^{i-j+1}\tau^2+\alpha^i\zeta_j) + 2d_i(1)\sum_{j=2}^{i}d_i(j)\zeta_j,
\end{aligned}$$

where $d_i(1)$ and $d_i(j)$ are given by (3.3). □

Remark 3.1. It is clear that

$$\begin{aligned}
\hat{X}\left(i+1;1,...,i\right) &= \frac{\zeta_{i+1}}{h_i}\left(\sigma^2-\zeta_2\right)X_1 + \frac{\zeta_{i+1}}{h_i}\sum_{j=2}^{i-1}\left(\alpha\zeta_{i+1}-\zeta_j\right)X_j \\
&\quad + \left(\alpha - \frac{\zeta_i\zeta_{i+1}}{h_i}\right)X_i.
\end{aligned}$$

Example 3.1. In Example 2.1; it is easy to show that

$$d_i(1) = \alpha^i - \sqrt{\frac{i}{i+1}},$$

and

$$d_i(j) = \alpha^{i-j+1} - \sqrt{\frac{i}{(i+1)j(j-1)}}, \quad j = 2, ..., i.$$

Example 3.2. In Example 2.2, it is easy to show that

$$d_i(1) = \alpha^i + \theta\sqrt{\frac{1+(i-2)\theta}{(1-\theta)\left(1+(i-1)\theta\right)^3}},$$

and

$$d_i(j) \quad = \quad \alpha^{i-j+1} - \theta^2\sqrt{\frac{1+(i-2)\theta}{(1+(j-3)\theta)\left(1+(j-2)\theta\right)\left(1+(i-1)\theta\right)^3}},$$

for $j = 2, ..., i$.

4. Stationary ARG random vectors

Let $\mathbf{X}_n \sim \mathrm{ARG}\left(\mathbf{0}, \alpha, \Sigma\right)$. Then $\mathbf{X}_n = (X_1, X_2, ..., X_n)'$ is said to be *stationary* if

$$\mathrm{E}\left(X_i X_j\right) = \gamma\left(|i-j|\right), \quad i, j = 1, 2, .., n.$$

The following theorem provides necessary and sufficient conditions in terms of $\gamma\left(k\right)$, for an ARG random vector to be stationary.

Theorem 4.1. *Let* $\mathbf{X} \sim ARG\left(\mathbf{0}, \alpha, \Sigma\right)$. *Then* $\mathbf{X}$ *is stationary if and only if*

$$\begin{aligned} \tau^2 &= \gamma\left(0\right), \ \sigma^2 = \left(1+\alpha^2\right)\gamma\left(0\right) - 2\alpha\gamma\left(1\right), \\ \zeta_i &= \alpha^{-(i-2)}\left(\gamma\left(1\right) - \alpha\gamma\left(0\right)\right), \quad i = 1, ..., n, \end{aligned}$$

subject to

$$(\rho-\alpha)^2\left(\frac{\alpha^{-2(n-2)}-1}{\alpha^2-1}\right) + 1 - \rho^2 \geq 0, \tag{4.1}$$

where $\rho = \rho(1) = \frac{\gamma(1)}{\gamma(0)}$ *and* $|\alpha| \neq 1$.

Proof. *Necessity*: If $\mathbf{X} \sim \mathrm{ARG}\left(\mathbf{0}, \alpha, \Sigma\right)$ is stationary, then $\mathrm{E}(X_1)^2 = \gamma\left(0\right)$ and $\mathrm{E}(X_i - \alpha X_{i-1})^2 = \sigma^2$. Hence

$$\tau^2 = \gamma\left(0\right), (1+\alpha^2)\gamma(0) - 2\alpha\gamma(1) = \sigma^2.$$

By taking $i = j$ in (2.8), we conclude that

$$\begin{aligned}\gamma(0) - \alpha\gamma(1) &= \sigma^2 + \alpha^{i-1}\zeta_i, \\ \gamma(0) - \alpha\gamma(1) &= \left(1+\alpha^2\right)\gamma(0) - 2\alpha\gamma(1) + \alpha^{i-1}\zeta_i,\end{aligned}$$

and hence

$$\zeta_i = \alpha^{-(i-2)}\left(\gamma(1) - \alpha\gamma(0)\right).$$

The condition (2.2) reduces to

$$\gamma(0)\left(\left(1+\alpha^2\right)\gamma(0) - 2\alpha\gamma(1)\right) \geq \sum_{i=2}^{n} \alpha^{-2(i-2)}\left(\gamma(1) - \alpha\gamma(0)\right)^2,$$

or

$$\left(1+\alpha^2\right)\gamma^2(0) - 2\alpha\gamma(1)\gamma(0) \geq \left(\frac{\alpha^{-2(n-1)} - 1}{\alpha^{-2} - 1}\right)\left(\left(\gamma(1) - \alpha\gamma(0)\right)^2,\right.$$

which is equivalent to any of

$$\begin{aligned}&(1+\alpha^2) - 2\alpha\rho \geq \left(\tfrac{\alpha^{-2(n-1)}-1}{\alpha^{-2}-1}\right)(\rho-\alpha)^2, \\ &1-\rho^2 + (\rho-\alpha)^2 \geq \left(\tfrac{\alpha^{-2(n-1)}-1}{\alpha^{-2}-1}\right)(\rho-\alpha)^2, \\ &1-\rho^2 + (\rho-\alpha)^2 - \left(\tfrac{\alpha^{-2(n-1)}-1}{\alpha^{-2}-1}\right)(\rho-\alpha)^2 \geq 0, \\ &1-\rho^2 + (\rho-\alpha)^2\,\tfrac{\alpha^{-2(n-2)}-1}{\alpha^2-1} \geq 0\ .\end{aligned}$$

Sufficiency: Let $\zeta_i = \alpha^{-(i-2)}\left(\gamma(1) - \alpha\gamma(0)\right)$, $\tau^2 = \gamma(0)$ and $\sigma^2 = (1+\alpha^2)\gamma(0) - 2\alpha\gamma(1)$. We let $i = j+k$ in (2.7). Then for $|\alpha| \neq 1$,

$$\begin{aligned}\sigma_{(j+k)j} &= \alpha^{2j+k-2}\tau^2 + \left(\frac{\alpha^{2j+k-2} - \alpha^k}{\alpha^2 - 1}\right)\sigma^2 + 2\sum_{l=2}^{j}\alpha^{2j+k-l-1}\zeta_l \\ &\quad + \sum_{l=j+1}^{i}\alpha^{2j+k-l-1}\zeta_l \\ &= \left(\alpha^{2j+k-2} + \left(\frac{\alpha^{2j+k-2} - \alpha^k}{\alpha^2-1}\right) - \frac{\alpha^{2j+k} - \alpha^{k+2}}{\alpha^2-1} - \left(\frac{\alpha^{k+2} - \alpha^{-k+2}}{\alpha^2-1}\right)\right)\gamma(0) \\ &\quad + \left(\left(\frac{\alpha^{k-1} - \alpha^{-k+1}}{\alpha^2-1}\right)\right)\gamma(1) \\ &= \left(\frac{\alpha^{-k+2} - \alpha^k}{\alpha^2-1}\right)\gamma(0) + \left(\frac{\alpha^{k+1} - \alpha^{-k+1}}{\alpha^2-1}\right)\gamma(1) \\ &= \left(\frac{\alpha\gamma(1) - \gamma(0)}{\alpha^2-1}\right)\alpha^k + \left(\frac{\alpha^2\gamma(0) - \alpha\gamma(1)}{\alpha^2-1}\right)\alpha^{-k},\end{aligned}$$

which is independent of i for each k. Thus $\mathbf{X}$ is stationary. □

Remark 4.1. The autocovariance function of a stationary ARGRV(1) for $|\alpha| \neq 1$ is given by

$$\gamma(k) = \left(\frac{\alpha\gamma(1) - \gamma(0)}{\alpha^2 - 1}\right)\alpha^k + \left(\frac{\alpha^2\gamma(0) - \alpha\gamma(1)}{\alpha^2 - 1}\right)\alpha^{-k}, \quad k > 0, \tag{4.2}$$

and in the matrix form,

$$\Sigma_{\mathbf{X}} = \left(\frac{\alpha\gamma(1) - \gamma(0)}{\alpha^2 - 1}\right)\Sigma_1 + \left(\frac{\alpha^2\gamma(0) - \alpha\gamma(1)}{\alpha^2 - 1}\right)\Sigma_2, \tag{4.3}$$

where $\Sigma_1 = \left[\alpha^{|i-j|}\right]_{i,j=1,\ldots,n}$ and $\Sigma_2 = \left[\alpha^{-|i-j|}\right]_{i,j=1,\ldots,n}$.

Let us present the following properties of a stationary ARG random vector of order 1. Let $\mathbf{X}_n$ be a stationary ARGRV(1) with parameters α, $\gamma(0)$ and $\gamma(1)$. Then we have

(i) $X_i \sim \mathrm{N}(0, \gamma(0))$,

(ii) $(X_i, X_{i+1}) \sim \mathrm{N}_2(0, \begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix})$ and $P(\frac{X_{i+1}}{X_i} < \rho) = \frac{1}{2}$,

(iii) $(X_{i+1}|X_i = x_i) \sim \mathrm{N}(\rho x_i, \gamma(0)(1-\rho^2))$,

(iv) $(X_i, X_{i+k}) \sim \mathrm{N}_2(0, \begin{bmatrix} \gamma(0) & \gamma(k) \\ \gamma(k) & \gamma(0) \end{bmatrix})$ and $P(\frac{X_{i+k}}{X_i} < \rho(k)) = \frac{1}{2}$,

(v) $(X_{i+k}|X_i = x_i) \sim \mathrm{N}(\rho(k)x_i, \gamma(0)(1-\rho(k)^2))$,

(vi) $E(X_{i+1}|X_i = x_i, \ldots, X_1 = x_1) = \frac{\zeta_{i+1}}{h}\left(\sigma^2 - \zeta_2\right)x_1 + \left(\alpha - \frac{\zeta_{i+1}\zeta_i}{h}\right)x_i$.

The effect of x_1 on the best predictor for X_{i+1} is apparent from (vi), this is in contrast to the case of classical AR(1) in which

$$\mathrm{E}(X_{i+1}|X_i = x_i, \ldots, X_1 = x_1) = \alpha x_i.$$

Remark 4.2. It is easy to verify that if $\mathbf{X}_n$ is stationary with parameter α and ρ, then

(i) $\rho(k) = \frac{\gamma(k)}{\gamma(0)} = \left(\frac{\alpha\rho - 1}{\alpha^2 - 1}\right)\alpha^k + \left(\frac{\alpha^2 - \alpha\rho}{\alpha^2 - 1}\right)\alpha^{-k} = \rho\left(\frac{\alpha^k - \alpha^{-k}}{\alpha - \alpha^{-1}}\right) + \left(\frac{\alpha^{-k+1} - \alpha^{k-1}}{\alpha - \alpha^{-1}}\right)$,

(ii) $\frac{\rho(k+1) + \rho(k-1)}{\rho(k)} = \alpha + \alpha^{-1}$,

(iii) $\rho(k) - \alpha\rho(k-1) = \alpha^{-k+1}(\alpha - \rho)$.

Therefore unlike the classical AR(1) models, the correlation $\{\rho(k)\}$ are specified with two parameters, α and $\rho = \rho(1)$.

Throughout the rest of this section we consider the classical case of $|\alpha| < 1$.

Remark 4.3. If in our definition for ARGRV(1), we impose the condition that $X_j - \alpha X_{j-1}$ is independent from X_i, $i \leq j-1$, then we will encounter to an ordinary AR(1) with finite length. Indeed, in this case $\zeta_i = \alpha^{-(i-2)}(\gamma(1) - \alpha\gamma(0)) = 0$, $\gamma(1) - \alpha\gamma(0) = 0$, or $\alpha = \rho$; and hence

$$\gamma(k) = \gamma(-k) = \alpha^k \ \gamma(0), \quad k > 0.$$

Interestingly, $\alpha = \rho$ if and only if $X_j - \alpha X_{j-1}$ is independent from X_i, $i \leq j-1$. In this case $\mathbf{X}_n = (X_1, X_2, ..., X_n)'$ is a classical AR(1).

Remark 4.4. It follows from Theorem 4.1 that the law of a stationary ARG random vector is uniquely specified by parameters α and ρ; subject to (4.1). The condition (4.1) is restrictive and narrows the class of stationary ARG random vectors. For a given α, $\gamma(1)$ and $\gamma(0)$, or ρ, (4.1) is fulfilled if and only if,

$$\alpha - U(n,\alpha) < \rho < \alpha + V(n,\alpha),$$

where $U(n,\alpha) = \frac{1-\alpha^2}{\alpha+|\alpha^{-(n-2)}|}$, and $V(n,\alpha) = \frac{\alpha^2-1}{\alpha-|\alpha^{-(n-2)}|}$. Consequently,

$$V(n,\alpha) - U(n,\alpha) = \frac{2\alpha^{2n-3}(1-\alpha^2)}{1-\alpha^{2(n-1)}} = o(\alpha^n),$$

which indicates that the deviation of ρ from α rather fast approaches zero as n increases .

Table 1 indicates the lower and upper bounds for ρ, for given $\alpha = 0.5$ and some different values of n.

n	3	4	5	6	7	8	9	10	11	12
L	.0000	.2857	.4000	.4516	.4761	.4881	.4941	.4970	.4985	.4992
U	.8000	.6666	.5882	.5454	.5230	.5116	.5058	.5029	.5014	.5007

TABLE 1. n: series size; L: lower bound for ρ; U: Upper bound for ρ; $\alpha = 0.5$.

5. Discussion and further research

In the previous sections, we introduced and characterized autoregressive Gaussian random vectors of order one. In contrast to the classical AR models, the law of an ARGRV is specified by more than one parameter. This will alow the correlation between the first component and other components to be more resistance and does not die out exponentially, as the number of components increases. To the best of our knowledge, this is the first time that ARGRV are introduced. This study is indeed the beginning which initiates a new line of research and, we believe, will give rise to a new class of time series models, say ARMAGRV (parallel to ARMA). Such models are expected to be promising in modeling short time series in which the early values will have dominant affect on the entire series. The existence and stationarity of such Gaussian models are not straightforward. Indeed, we expected them to be challenging problems. In fact, based on our primary studies on ARGRV(p), we can say at this stage that elaborated matrix algebra is needed to provide necessary and sufficient conditions for the existence and stationarity. We leave this for future research. However, we would like to stress that the method presented here for the case $p = 1$ is very confined to this case, and will fail for $p > 1$.

The derivations on ARGRV in this work are also valid for AR(SO)RV. In the section for the prediction, the term "the best prediction" should be replaced by "the best linear prediction". There are issues that are not discussed in this work, and are postponed for later circumstances. Those are estimation and simulation. It is more traditional to consider the Gaussian random vectors. The Gaussian assumption will be useful in estimating the parameters of the model, through the maximum likelihood procedure, and will ease the work for the simulation.

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