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IRREDUCIBLE CHARACTERS OF SYLOW p-SUBGROUPS OF THE STEINBERG TRIALITY GROUPS ${}^3D_4(p^{3m})$

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(Communicated by Ali Reza Ashrafi)

Abstract. Here we construct and count all ordinary irreducible characters of Sylow p-subgroups of the Steinberg triality groups ${}^3D_4(p^{3m})$.
Keywords: Irreducible character, root system, Sylow subgroup, Steinberg triality.
MSC(2010): Primary 20C15. Secondary 20C33, 20D15, 20D20 .

1. Introduction

Let $\mathbb{F}_q$ be a finite field of order q where q is a power of a prime p. In 1960, a conjecture by Higman [5] on the number $k(U_n(q))$ of conjugacy classes of the unitriangular group $U_n(q)$ of degree n over $\mathbb{F}_q$ was that $k(U_n(q))$ is a polynomial in q with integral coefficients. From the relation between conjugacy classes and ordinary irreducible characters, Higman's Conjecture is also studied from the point of view of character theory. Isaacs [8] showed that the character degrees of $U_n(q)$ are powers of q.

The unitriangular group $U_n(q)$ is also known as a maximal unipotent subgroup of the special linear group $\mathrm{SL}_n(q)$. A generalization of Higman's Conjecture on the maximal unipotent subgroups $U(q)$ of other finite groups $G(q)$ of Lie type is that $k(U(q))$ is a polynomial in q with integral coefficients.

By α_0 we denote the highest root of the root system Φ of $G(q)$. It is well known that α_0 is a positive integral linear combination of the fundamental roots of Φ. So without loss $\alpha_0 = \sum_{i=1}^{r} a_i\alpha_i$ where the α_i's are fundamental roots of Φ. Recall that a prime s is bad to the Lie type of $G(q)$ if s is a divisor of some a_i. A prime s is good to the Lie type of $G(q)$ if it is not bad. Except for type A, the other Lie types have their own bad primes.

Many results have been obtained on the conjugacy classes and irreducible characters of maximal unipotent subgroups $U(q)$ of finite groups $G(q)$ of Lie type, see [11, 4, 12, 3]. The common behavior of small Lie ranks comes up

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as follows. For all good primes p, $k(U(q))$ is a polynomial in q with integral coefficients. If p is bad, then $k(U(q))$ is still a polynomial in q with integral coefficients but different from the one of good primes. Furthermore, in the bad prime cases, some character degrees of $U(q)$ are not powers of q, see [11], [7], [9]. In [7], all irreducible characters of Sylow p-subgroups $U(q)$ of the Chevalley groups $D_4(q) \cong P\Omega_8^+(q)$ have been computed. In this paper, we construct all irreducible characters of Sylow p-subgroups of the Steinberg triality groups ${}^3D_4(q^3)$. Denote by U a Sylow p-subgroup of ${}^3D_4(q^3)$ and let $\mathbb{F}_-^\times := \mathbb{F}_- - \{0\}$. We prove the following result.

Theorem 1.1. *The irreducible characters of U are classified into five families as listed in Table 1.*

TABLE 1. Irreducible characters of Sylow p-subgroups U of ${}^3D_4(q^3)$

Family	Notation	Parameter set	Number	Degree
$\mathfrak{F}_6$	$\chi_{6,q^4}^{a,b}$	$\mathbb{F}_q^\times \times \mathbb{F}_{q^3}$	$(q-1)q^3$	q^4
$\mathfrak{F}_5$	$\chi_{5,q^3}^{a_1,a_2,b}$	$\mathbb{F}_q^\times \times \mathbb{F}_q \times \mathbb{F}_{q^3}$	$(q-1)q^4$	q^3
$\mathfrak{F}_4^{odd}$	$\chi_{4,q^3}^{b,a}$	$\mathbb{F}_{q^3}^\times \times \mathbb{F}_q$	$(q^3-1)q$	q^3
$\mathfrak{F}_4^{even}$	χ_{4,q^3}^{b} $\chi_{4,q^3/2}^{b,a,c_1,c_2}$	$\mathbb{F}_{q^3}^\times$ $\mathbb{F}_{q^3}^\times \times \mathbb{F}_q^\times \times \mathbb{F}_2 \times \mathbb{F}_2$	q^3-1 $4(q^3-1)(q-1)$	q^3 $q^3/2$
$\mathfrak{F}_3$	$\chi_{3,q}^{b_1,b_2}$	$\mathbb{F}_{q^3}^\times \times (\mathbb{F}_{q^3}/\mathbb{F}_q)$	$(q^3-1)q^2$	q
$\mathfrak{F}_{lin}$	$\chi_{lin}^{b,a}$	$\mathbb{F}_{q^3} \times \mathbb{F}_q$	q^4	1

Here are some explanations of Table 1. There are five families of irreducible characters of U and each row represents one of them. The first column gives a name for each family. Notice that the family $\mathfrak{F}_4^{odd}$ exists only for q odd, while $\mathfrak{F}_4^{even}$ exists only if q is even. The index j of this notation describes the j-th positive root of maximal height such that Y_j is not contained in the kernel of any character of this family. (The positive root set of ${}^3D_4$ is presented in Subsection 2.3.) The family $\mathfrak{F}_{lin}$ contains all linear characters of U. The second column gives the notation of irreducible characters in each family. The lower indices indicate their family and degree. The upper indices are the parameters to confirm their uniqueness in the family where $a, a_i \in \mathbb{F}_q$, $b, b_i \in \mathbb{F}_{q^3}$ and $c_i \in \mathbb{F}_2$. These parameters take values from the set in the third column. The fourth column lists the cardinality of each family and the last column gives their degrees. More details will be given during the constructions of each family in Section 3.

Corollary 1.2. *If q is odd then $k(U) = 2q^5 + 2q^4 - q^3 - q^2 - q$. Otherwise, if q is even, then $k(U) = 2q^5 + 5q^4 - 4q^3 - q^2 - 4q + 3$.*

It is well known that the primes 2 and 3 are bad for the Chevalley groups of type G_2. Since the Steinberg triality groups ${}^3D_4(q^3)$ also has the Dynkin diagram of type G_2, we are curious that if the primes 2 and 3 show up as the bad primes of ${}^3D_4(q^3)$ in terms of the representation theory of the Sylow p-subgroup. Theorem 1.1 and Corollary 1.2 point out that only the prime 2 affects on the structure of U and $k(U)$, which is compatible to the global computation on character degrees of ${}^3D_4(q^3)$, see [10].

We approach Sylow p-subgroups U of ${}^3D_4(q^3)$ by its root system. The method to construct all irreducible characters of U is quite elementary, mainly using Clifford theory. In addition, we study the actions of $\mathbb{F}_{q^3}^\times$ on its $\mathbb{F}_q$-hyperplane set and on $\mathbb{F}_p$-hyperplane set. We obtain the structures of U and its factor groups U/Z where Z is generated by some root subgroups.

Let B be a Borel subgroup of ${}^3D_4(q^3)$, i.e. the normalizer of U. Notice that the conjugacy classes of B have been computed by Geck [2], and the irreducible characters of B have been computed by Himstedt [6]. Using the character table of B, one can also obtain all irreducible characters of U by Clifford theory. However, the parameterizations may be different from the ones in Table 1.

This paper is organized as follows. First we introduce some relevant finite field properties and some character theory notations which are used later for the proof of Theorem 1.1. Next we construct Sylow p-subgroups of the Steinberg triality groups ${}^3D_4(q^3)$ from its root system. Finally, we prove Theorem 1.1 in Section 3.

2. Basic set-up and notations

In this section we present some finite field properties, some fundamental notations of character theory and a construction of Sylow p-subgroups of the Steinberg triality ${}^3D_4(q^3)$.

2.1. **Some fundamental field results.** Throughout this paper, for each prime power q, we consider the field extension $\mathbb{F}_{q^3}/\mathbb{F}_q$. Fix nontrivial linear characters $\phi : \mathbb{F}_q \to \mathbb{C}^\times$, and $\varphi : \mathbb{F}_{q^3} \to \mathbb{C}^\times$. For each $a \in \mathbb{F}_q$, $b \in \mathbb{F}_{q^3}$, we define ϕ_a and φ_b by $\phi_a(x) := \phi(ax)$ for all $x \in \mathbb{F}_q$, and $\varphi_b(y) := \varphi(by)$ for all $y \in \mathbb{F}_{q^3}$. Hence, $\mathrm{Irr}(\mathbb{F}_q) = \{\phi_a : a \in \mathbb{F}_q\}$ and $\mathrm{Irr}(\mathbb{F}_{q^3}) = \{\varphi_b : b \in \mathbb{F}_{q^3}\}$. Recall the Frobenius map $\rho : \mathbb{F}_{q^3} \to \mathbb{F}_{q^3},\ t \mapsto t^q$.

Definition 2.1. For each $t \in \mathbb{F}_{q^3}$, we define

(i) $\mathbb{A}_t := \{tu^q + t^q u^{q^2} + t^{q^2} u : u \in \mathbb{F}_{q^3}\}$,
(ii) $\mathbb{B}_t := \{t^q u + t u^q : u \in \mathbb{F}_{q^3}\}$.

Notice that $\mathbb{A}_0 = \{0\} = \mathbb{B}_0$. Now we observe a few important properties of $\mathbb{A}_t$, $\mathbb{B}_t$.

Lemma 2.2. *We have $\mathbb{A}_t = \mathbb{F}_q$ for all $t \in \mathbb{F}_{q^3}^\times$.*

Proof. It is clear that $\mathbb{A}_t \subset \mathbb{F}_q$ since its elements are ρ-invariant. So $|\mathbb{A}_t| \leq q$. It suffices to show that $|\mathbb{A}_t| \geq q$.

For each $a \in \mathbb{F}_q$, the equation $t^{q^2}u + tu^q + t^q u^{q^2} = a$ has at most q^2 solutions for u in $\mathbb{F}_{q^3}$. Therefore, when u runs all over $\mathbb{F}_{q^3}$, $\mathbb{A}_t$ has at least $|\mathbb{F}_{q^3}|/q^2 = q$ elements. This completes the proof. □

For each $t \in \mathbb{F}_{q^3}^\times$, we define an $\mathbb{F}_q$-homomorphism $f_t : \mathbb{F}_{q^3} \to \mathbb{F}_{q^3}$ given by $f_t(u) := t^q u + tu^q$ for all $u \in \mathbb{F}_{q^3}$. By Definition 2.1 (ii), $\mathbb{B}_t = \mathrm{im}(f_t)$.

Lemma 2.3. *For all $t \in \mathbb{F}_{q^3}^\times$, the following hold.*

(i) *$\mathbb{B}_t$ is an $\mathbb{F}_q$-vector space and $\mathbb{B}_{tx} = \mathbb{B}_t = x\mathbb{B}_t$ for all $x \in \mathbb{F}_q^\times$.*
(ii) *If q is odd, then $\mathbb{B}_t = \mathbb{F}_{q^3}$.*
(iii) *If q is even, then $\ker(f_t) = t\,\mathbb{F}_q$ and $\mathbb{B}_t \leq (\mathbb{F}_{q^3}, +)$ of order q^2.*

Proof. Part (i) is clear by the $\mathbb{F}_q$-homomorphism property of f_t. Thus, it suffices to show parts (ii) and (iii) by finding the kernel of f_t.

We have $t^q u + tu^q = t^q u(1 + (t^{-1}u)^{q-1})$. So $f_t(u) = 0$ if and only if $u = 0$ or $(t^{-1}u)^{q-1} = -1$. Now we have two cases.

Case 1: If q is odd: Since $\mathbb{F}_{q^3}^\times$ is cyclic of order $q^3 - 1 = (q-1)(q^2+q+1)$ and (q^2+q+1) is odd, there is no $z \in \mathbb{F}_{q^3}^\times$ such that z^{q-1} has order 2. So the equation $(t^{-1}u)^{q-1} = -1$ has no solution for $u \in \mathbb{F}_{q^3}$. This shows that $\ker(f_t) = \{0\}$ and $\mathrm{im}(f_t) = \mathbb{F}_{q^3}$.

Case 2: If q is even: The equation $(t^{-1}u)^{q-1} = 1$ implies that $t^{-1}u \in \mathbb{F}_q^\times$, i.e. $u \in t\mathbb{F}_q^\times$. It is clear that $t\mathbb{F}_q \subset \ker(f_t)$. So $\ker(f_t) = t\mathbb{F}_q$ and $|\mathbb{B}_t| = q^2$. □

Lemma 2.4. *If q is even, i.e. $q = 2^a$, then the following hold.*

(i) *$\ker(f_1) = \mathbb{F}_q$ and $\mathbb{F}_{q^3} = \ker(f_1) \oplus \mathbb{B}_1$.*
(ii) *$\mathbb{B}_t = t^{q+1}\mathbb{B}_1$ for all $t \in \mathbb{F}_{q^3}^\times$.*
(iii) *$\{\mathbb{B}_t : t \in \mathbb{F}_{q^3}^\times\}$ is the $\mathbb{F}_q$-hyperplane set of $\mathbb{F}_{q^3}$. Moreover, $\mathbb{B}_t = \mathbb{B}_r$ if and only if $t \in r\mathbb{F}_q$.*
(iv) *There is a unique $t \in \mathbb{F}_{q^3}^\times$, up to a scalar of $\mathbb{F}_q^\times$, such that $\mathbb{B}_t \subset \ker(\varphi)$.*

Proof. (i) By Lemma 2.3 (iii), $\ker(f_1) = \mathbb{F}_q$ and $\mathbb{B}_1 = \{u + u^q : u \in \mathbb{F}_{q^3}\}$ is a 2-dimensional $\mathbb{F}_q$-vector space. It suffices to show that $\ker(f_1) \cap \mathbb{B}_1$ is trivial. Suppose that $\ker(f_1) \cap \mathbb{B}_1$ is nontrivial, i.e. there exist $x \in \mathbb{F}_q^\times$ and $u \in \mathbb{F}_{q^3}^\times$ such that $x = u + u^q$.

If $u \in \mathbb{F}_q$, then $x = u + u^q = 2u = 0$ since $\mathrm{char}(\mathbb{F}_q) = 2$, contrary to $x \in \mathbb{F}_q^\times$. Notice that $\mathbb{F}_{q^2} \not\subset \mathbb{F}_{q^3}$. If $u \notin \mathbb{F}_q$, from $u^{q^2} + u^q + u \in \mathbb{F}_q$, we have $u + u^q \notin \mathbb{F}_q$, contrary to $x = u + u^q \in \mathbb{F}_q$. So $\ker(f_1) \cap \mathbb{B}_1 = \{0\}$.

(ii) From $t^q u + tu^q = t^{q+1}(ut^{-1} + (ut^{-1})^q) \in t^{q+1}\mathbb{B}_1$, the claim is clear.

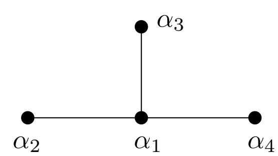
(iii) Since q is even and $q^3 - 1 = (q-1)(q^2+q+1)$, $\gcd(q+1, q^3-1) = 1$ and $\{t^{q+1} : t \in \mathbb{F}_{q^3}^\times\} = \mathbb{F}_{q^3}^\times$. By part (ii), $\mathbb{F}_{q^3}^\times$ acts transitively on the set of all $\mathbb{B}_t$'s. Since the left multiplication action of $\mathbb{F}_{q^3}^\times$ is also transitive on the $\mathbb{F}_q$-hyperplane set of $\mathbb{F}_{q^3}$, $\{\mathbb{B}_t : t \in \mathbb{F}_{q^3}^\times\}$ is the $\mathbb{F}_q$-hyperplane set of $\mathbb{F}_{q^3}$.

Here the cardinality of the $\mathbb{F}_q$-hyperplane set of $\mathbb{F}_{q^3}$ is $(q^3-1)/(q-1)$. By the transitive action of $\mathbb{F}_{q^3}^\times$ on $\{\mathbb{B}_t : t \in \mathbb{F}_{q^3}^\times\}$, we have $|\mathrm{Stab}_{\mathbb{F}_{q^3}^\times}(\mathbb{B}_1)| = q-1$. By Lemma 2.3 (i), $\mathrm{Stab}_{\mathbb{F}_{q^3}^\times}(\mathbb{B}_1) = \mathbb{F}_q^\times$. Thus, by part (ii), $\mathbb{B}_t = \mathbb{B}_r$ if and only if $(tr^{-1})^{q+1} \in \mathrm{Stab}_{\mathbb{F}_{q^3}^\times}(\mathbb{B}_1) = \mathbb{F}_q^\times$, if and only if $t \in r\mathbb{F}_q^\times$.

(iv) The uniqueness follows from part (iii) since $\mathbb{B}_t + \mathbb{B}_s = \mathbb{F}_{q^3} \supsetneq \ker(\varphi)$ for any $\mathbb{B}_t \neq \mathbb{B}_s$. Since $\{\ker(\varphi_r) : \varphi_r \in \mathrm{Irr}(\mathbb{F}_{q^3})^\times\}$ is the $\mathbb{F}_2$-hyperplane set of $\mathbb{F}_{q^3}$, $\mathbb{B}_1 \subset \ker(\varphi_b)$ for some $\varphi_b \in \mathrm{Irr}(\mathbb{F}_{q^3})^\times$. The existence follows from the transitive action of $\mathbb{F}_{q^3}^\times$ on the $\mathbb{F}_2$-hyperplane set of $\mathbb{F}_{q^3}$. □

2.2. **Character theory.** Let G be a group. Denote $G^\times := G - \{1\}$, $\mathrm{Irr}(G)$ the set of all complex irreducible characters of G, and $\mathrm{Irr}(G)^\times := \mathrm{Irr}(G) - \{1_G\}$. Let χ be a character of G and λ be a character of a subgroup H of G. We write λ^G for the induced character of λ to G and $\chi|_H$ for the restriction of χ to H. We denote $\mathrm{Irr}(G, \lambda) := \{\chi \in \mathrm{Irr}(G) : (\chi, \lambda^G) > 0\}$ the irreducible constituent set of λ^G, $\ker(\chi) := \{g \in G : \chi(g) = \chi(1)\}$ the kernel of χ, and $\mathrm{Z}(\chi) := \{g \in G : |\chi(g)| = \chi(1)\}$ the center of χ. Furthermore, for $N \trianglelefteq G$, let $\mathrm{Irr}(G/N)$ be the set of all irreducible characters of G with N in the kernel. For the others, our notations will be quite standard.

2.3. **Root systems of $^3D_4$ and Sylow p-subgroups of $^3D_4(q^3)$.** Let α_1, α_2, α_3, α_4 be fundamental roots of the root system Φ of type D_4. Here is the Dynkin diagram of Φ, see Carter [1, Chapter 3].



The positive roots are those which can be written as linear combinations of the fundamental roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with nonnegative coefficients and we write Φ_+ for the set of positive roots. We use the notation $\begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix}$ for the root $2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ and we use a similar notation for the remaining positive roots. The 12 positive roots of Φ are given in Table 2.

Fix a power p^f. Let $X_\alpha := \{x_\alpha(t) : t \in \mathbb{F}_{p^f}\}$ be the root subgroup corresponding to $\alpha \in \Phi$. The group generated by all X_α, $\alpha \in \Phi$ is known as the Chevalley group $D_4(p^f) \cong P\Omega_8^+(p^f)$. Let $UD_4(p^f)$ be the group generated by all X_α, $\alpha \in \Phi_+$. So $UD_4(p^f)$ is a Sylow p-subgroup of $D_4(p^f)$.

For positive roots, we use the abbreviation $x_i(t) := x_{\alpha_i}(t)$, $i \in [1, 12]$. Let γ be the permutation $(2,3,4)(5,6,7)(8,9,10)$. By [1, Proposition 12.2.3] the map

TABLE 2. Positive roots of the root system Φ of type D_4.

Height	Roots			
5	$\alpha_{12} := \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix}$			
4	$\alpha_{11} := \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix}$			
3	$\alpha_8 := \begin{smallmatrix} & 1 & \\ 1 & 1 & 0 \end{smallmatrix}$	$\alpha_9 := \begin{smallmatrix} & 1 & \\ 0 & 1 & 1 \end{smallmatrix}$	$\alpha_{10} := \begin{smallmatrix} & 0 & \\ 1 & 1 & 1 \end{smallmatrix}$	
2	$\alpha_5 := \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \end{smallmatrix}$	$\alpha_6 := \begin{smallmatrix} & 1 & \\ 0 & 1 & 0 \end{smallmatrix}$	$\alpha_7 := \begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \end{smallmatrix}$	
1	$\alpha_1 := \begin{smallmatrix} & 0 & \\ 0 & 1 & 0 \end{smallmatrix}$	$\alpha_2 := \begin{smallmatrix} & 0 & \\ 1 & 0 & 0 \end{smallmatrix}$	$\alpha_3 := \begin{smallmatrix} & 1 & \\ 0 & 0 & 0 \end{smallmatrix}$	$\alpha_4 := \begin{smallmatrix} & 0 & \\ 0 & 0 & 1 \end{smallmatrix}$

$\tau : UD_4(p^f) \to UD_4(p^f)$ defined by $\tau(x_i(t)) = x_{i\gamma}(t)$ induces an automorphism of $UD_4(p^f)$ of order 3.

The commutators $[x_i(t), x_j(u)] = x_i(t)^{-1}x_j(u)^{-1}x_i(t)x_j(u)$ are given in Table 3. All $[x_i(t), x_j(u)]$ not listed in this table are equal to 1.

TABLE 3. Commutator relations for type D_4.

$$\begin{array}{lcllcl}
[x_1(t), x_2(u)] & = & x_5(tu), & [x_1(t), x_3(u)] & = & x_6(tu), \\
[x_1(t), x_4(u)] & = & x_7(tu), & [x_1(t), x_{11}(u)] & = & x_{12}(tu), \\
[x_2(t), x_6(u)] & = & x_8(tu), & [x_2(t), x_7(u)] & = & x_{10}(tu), \\
[x_2(t), x_9(u)] & = & x_{11}(tu), & [x_3(t), x_5(u)] & = & x_8(tu), \\
[x_3(t), x_7(u)] & = & x_9(tu), & [x_3(t), x_{10}(u)] & = & x_{11}(tu), \\
[x_4(t), x_5(u)] & = & x_{10}(tu), & [x_4(t), x_6(u)] & = & x_9(tu), \\
[x_4(t), x_8(u)] & = & x_{11}(tu), & [x_5(t), x_9(u)] & = & x_{12}(tu), \\
[x_6(t), x_{10}(u)] & = & x_{12}(tu), & [x_7(t), x_8(u)] & = & x_{12}(tu).
\end{array}$$

Notice that the signs in Table 3 are chosen to satisfy this choice of τ, and τ is also known as the restriction of a triality automorphism of the Chevalley group $D_4(p^f)$ to $UD_4(p^f)$.

Under the action of the permutation γ on the positive roots of Φ, there are six orbits of roots as follows.

$$S_1 := \{\alpha_1 = \begin{smallmatrix} & 0 & \\ 0 & 1 & 0 \end{smallmatrix}\},$$

$$S_2 := \{\alpha_2 = \begin{smallmatrix} & 0 & \\ 1 & 0 & 0 \end{smallmatrix}, \alpha_3 = \begin{smallmatrix} & 1 & \\ 0 & 0 & 0 \end{smallmatrix}, \alpha_4 = \begin{smallmatrix} & 0 & \\ 0 & 0 & 1 \end{smallmatrix}\},$$

$$S_3 := \{\alpha_5 = \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \end{smallmatrix}, \alpha_6 = \begin{smallmatrix} & 1 & \\ 0 & 1 & 0 \end{smallmatrix}, \alpha_7 = \begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \end{smallmatrix}\},$$

$$S_4 := \{\alpha_8 = \begin{smallmatrix} & 1 & \\ 1 & 1 & 0 \end{smallmatrix}, \alpha_9 = \begin{smallmatrix} & 1 & \\ 0 & 1 & 1 \end{smallmatrix}, \alpha_{10} = \begin{smallmatrix} & 0 & \\ 1 & 1 & 1 \end{smallmatrix}\},$$

$$S_5 := \{\alpha_{11} = \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \end{smallmatrix}\},$$

$$S_6 := \{\alpha_{12} = \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \end{smallmatrix}\}.$$

For the construction of ${}^3D_4$, it requires an automorphism $\sigma = \tau\rho$ of $D_4(p^f)$ where ρ is a nontrivial field automorphism of $\mathbb{F}_{p^f}$ such that $\rho^3 = 1$, see [1, Section 13.4]. The existence of an order 3 field automorphism ρ of $\mathbb{F}_{p^f}$ forces $f = 3m$ for some $m \in \mathbb{N}$. So from now on we consider the field $\mathbb{F}_{q^3}$. By [1, Proposition 13.6.3], the σ-fixed points of $UD_4(q^3)$ corresponding to each root orbit are as follows.

$$\begin{aligned}
&Y_1 := \{y_1(t) := x_1(t) : t = t^q \in \mathbb{F}_{q^3}\} = \{y_1(t) : t \in \mathbb{F}_q\},\\
&Y_2 := \{y_2(t) := x_2(t)x_3(t^q)x_4(t^{q^2}) : t \in \mathbb{F}_{q^3}\},\\
&Y_3 := \{y_3(t) := x_5(t)x_6(t^q)x_7(t^{q^2}) : t \in \mathbb{F}_{q^3}\},\\
&Y_4 := \{y_4(t) := x_8(t)x_9(t^q)x_{10}(t^{q^2}) : t \in \mathbb{F}_{q^3}\},\\
&Y_5 := \{y_5(t) := x_{11}(t) : t = t^q \in \mathbb{F}_{q^3}\} = \{y_5(t) : t \in \mathbb{F}_q\},\\
&Y_6 := \{y_6(t) := x_{12}(t) : t = t^q \in \mathbb{F}_{q^3}\} = \{y_6(t) : t \in \mathbb{F}_q\}.
\end{aligned}$$

Let $\{p_i\}_i$ be Euclidian coordinates of the roots $\{\alpha_i\}_i$, see [1, Section 3.6]. By [1, Section 13.3.4], set $P_1 := p_1$, $P_2 := \frac{1}{3}(p_2+p_3+p_4)$, $P_3 := \frac{1}{3}(p_5+p_6+p_7)$, $P_4 := \frac{1}{3}(p_8+p_9+p_{10})$, $P_5 := p_{11}$, $P_6 := p_{12}$. It is easy to check that $P_3 = P_1+P_2$, $P_4 = P_1 + 2P_2$, $P_5 = P_1 + 3P_2$, $P_6 = 2P_1 + 3P_2$. So $\{P_i : i \in [1, 6]\}$ is the positive root set of ${}^3D_4$ of type G_2, where P_2 is short and P_1 is long.

We call each Y_i a *root subgroup* in the σ-fixed-point group ${}^3D_4(q^3)$. It is clear that each Y_i is abelian, the subgroup generated by all Y_i's is a maximal unipotent subgroup and a Sylow p-subgroup of ${}^3D_4(q^3)$, denote it by U. Since Y_1, Y_5, Y_6 have order q and Y_2, Y_3, Y_4 have order q^3, the order of U is $q^{3\cdot3+3} = q^{12}$. Using Table 3, we compute commutator relations among root subgroups Y_i as below. All $[y_i(t), y_j(u)]$ not listed are equal to 1. For all t, u in the appropriate fields to root subgroups Y_i, we have

$$\begin{aligned}
&[y_1(t), y_2(u)] = y_3(tu)y_4(-tu^{q+1})y_5(tu^{q^2+q+1})y_6(t^2u^{q^2+q+1}),\\
&[y_2(t), y_3(u)] = y_4(tu^q + t^qu)y_5(-t^{q+1}u^{q^2} - t^{q^2+q}u - t^{q^2+1}u^q)\\
&\qquad\qquad\qquad\qquad y_6(-tu^{q^2+q} - t^qu^{q^2+1} - t^{q^2}u^{q+1}),\\
&[y_2(t), y_4(u)] = y_5(tu^q + t^qu^{q^2} + t^{q^2}u),\\
&[y_3(t), y_4(u)] = y_6(tu^q + t^qu^{q^2} + t^{q^2}u),\\
&[y_1(t), y_5(u)] = y_6(tu).
\end{aligned}$$

From commutator relations, it is clear that $\mathrm{Z}(U) = Y_6$, $\mathrm{Z}(U/Y_6) = Y_5Y_6$, $\mathrm{Z}(U/Y_5Y_6) = Y_4Y_5Y_6$, $\mathrm{Z}(U/Y_4Y_5Y_6) = Y_3Y_4Y_5Y_6$, and $U/Y_3Y_4Y_5Y_6$ is abelian of order q^4. To classify all irreducible characters of U, we come up with the definition of almost faithful irreducible characters.

Definition 2.5. An irreducible character χ of a group G is said to be *almost faithful* if $\mathrm{Z}(G) \not\subseteq \ker(\chi)$.

Due to the center series of U and the inflations of irreducible characters from quotient groups, $\mathrm{Irr}(U)$ are classified as follows.

$\mathfrak{F}_6 := \{\chi \in \mathrm{Irr}(U) : Y_6 \not\subset \ker(\chi)\}$, the set of all almost faithful irreducible characters of U.

$\mathfrak{F}_5 := \{\chi \in \mathrm{Irr}(U) : Y_5 \not\subset \ker(\chi)$ and $Y_6 \subset \ker(\chi)\}$, the set of all almost faithful irreducible characters of U/Y_6.

$\mathfrak{F}_4 := \{\chi \in \mathrm{Irr}(U) : Y_4 \not\subset \ker(\chi)$ and $Y_5Y_6 \subset \ker(\chi)\}$, the set of all almost faithful irreducible characters of U/Y_5Y_6.

$\mathfrak{F}_3 := \{\chi \in \mathrm{Irr}(U) : Y_3 \not\subset \ker(\chi)$ and $Y_4Y_5Y_6 \subset \ker(\chi)\}$, the set of all almost faithful irreducible characters of $U/Y_4Y_5Y_6$.

$\mathfrak{F}_{lin} := \{\chi \in \mathrm{Irr}(U) : Y_3Y_4Y_5Y_6 \subset \ker(\chi)\}$, the linear character set of U.

Remark 2.6. For $i \geq 3$, each $\chi \in \mathfrak{F}_i$ satisfies $Y_i \leq \mathrm{Z}(\chi)$ and $Y_i \not\subset \ker(\chi)$. Thus, $\chi|_{Y_i} = \chi(1)\,\mu$ for some $\mu \in \mathrm{Irr}(Y_i)^\times$.

The definition of $\mathfrak{F}_i$'s provides an algorithm to construct their characters as follows (except $\mathfrak{F}_6$). We shall work with the corresponding quotient group, e.g. with $\mathfrak{F}_5$ we study U/Y_6. Let $\bar{U}$ be the quotient group and μ be a nontrivial linear character of $\mathrm{Z}(\bar{U})$. We extend μ to some maximal normal abelian subgroup of $\bar{U}$. For each extension λ of μ, we compute the inertia subgroup $I_{\bar{U}}(\lambda)$ and use Clifford theory to count all constituents of $\lambda^{\bar{U}}$, as well as the ones of $\mu^{\bar{U}}$.

3. Characters of Sylow p-subgroups of the triality groups ${}^3D_4(q^3)$

Let $\chi \in \mathrm{Irr}(U)$. In this section we shall prove Theorem 1.1 by constructing χ through each family from $\mathfrak{F}_6$ to $\mathfrak{F}_5, ..., \mathfrak{F}_{lin}$ as discussed above. Recall that for the uniform of parameters of irreducible characters in Table 1, we use $a, a_i \in \mathbb{F}_q$, $b, b_i \in \mathbb{F}_{q^3}$ and $c_i \in \mathbb{F}_2$.

3.1. **Family $\mathfrak{F}_6$ where χ is almost faithful.** First we show that every character in this family has degree q^4.

Let $T := Y_1Y_3Y_4Y_5Y_6$ and $V := Y_4Y_5Y_6$. It is easy to check that V is abelian, $V \lhd T \lhd U$ and $\mathrm{Z}(T) = Y_6$. By Lemma 2.2 and Clifford theory with the transversal Y_1Y_3 of V in T, all $\lambda \in \mathrm{Irr}(V)$ such that $\lambda|_{Y_6} \neq 1_{Y_6}$ satisfy $\lambda^T \in \mathrm{Irr}(T)$ of degree q^4. Thus all almost faithful irreducible characters of T have degree q^4. Since $\chi|_T$ decomposes into sum of almost faithful irreducible characters, $\chi(1) \geq q^4$. Since $\lambda^T|_{Y_4Y_5}$ is the regular character of the abelian group Y_4Y_5, $\chi|_V$ has a linear constituent θ such that $\theta|_{Y_5} = 1_{Y_5}$.

Let $H := Y_6Y_5Y_4Y_2$. Clearly $H \leq U$ and $\mathrm{Z}(H) = Y_5Y_6$. By the existence of θ, let ξ be an irreducible constituent of $\chi|_H$ such that $\xi|_{Y_5} = \xi(1)1_{Y_5}$. Since $Y_5 \leq \ker(\xi)$, ξ can be considered as a character of $H/Y_5 \cong Y_6Y_4Y_2$, which is abelian. So ξ is linear. We shall show that $\xi^U \in \mathrm{Irr}(U)$, which implies that χ is the unique constituent in $\mathrm{Irr}(U, \xi)$, i.e. $\chi = \xi^U$. Since $H \cap T = V$ and $U = HT$, by Mackey formula for the double coset $H\backslash U/T$ we have $\xi^U|_T = \xi|_{H\cap T}{}^T \in \mathrm{Irr}(T)$. Thus $\xi^U \in \mathrm{Irr}(U)$ as claimed. So all almost faithful irreducible characters of U have degree q^4.

By the extension of each λ^T to U through Y_2, we obtain $(q-1)q^3$ almost faithful irreducible characters, parameterized by $(Y_6^\times, Y_2) \cong (\mathbb{F}_q^\times, \mathbb{F}_{q^3})$ and denoted by $\chi_{6,q^4}^{a,b}$ where $(a,b) \in (\mathbb{F}_q^\times, \mathbb{F}_{q^3})$.

3.2. **Family $\mathfrak{F}_5$ where $Y_6 \subset \ker(\chi)$ and $Y_5 \not\subset \ker(\chi)$.** We study the quotient group $\bar{U} := U/Y_6$. Abusing terminology slightly we call the image under the natural projection of a root group of U, a root group of $\bar{U}$.

Let $H := Y_5Y_4Y_3Y_1$. Clearly H is an abelian normal subgroup of $\bar{U}$. Let $\lambda \in \mathrm{Irr}(H)$ such that $\lambda|_{Y_5} \neq 1_{Y_5}$. By Lemma 2.2, for each $y_2(t) \in Y_2^\times$, there exists $y_4(u) \in Y_4$ such that $[y_4(u), y_2(t)] = y_5(tu^q + t^q u^{q^2} + t^{q^2} u) \notin \ker(\lambda)$, i.e. ${}^{y_2(t)}\lambda(y_4(u)) \neq \lambda(y_4(u))$. So the inertia group $I_{\bar{U}}(\lambda) = H$. By Clifford theory, $\lambda^{\bar{U}} \in \mathrm{Irr}(\bar{U})$.

Since H has $(q-1)q^7$ linear characters λ such that $\lambda|_{Y_5} \neq 1_{Y_5}$, by the above argument we obtain $(q-1)q^4$ irreducible characters of degree q^3 in this family, parameterized by $(Y_5^\times, Y_1, Y_3) \cong (\mathbb{F}_q^\times, \mathbb{F}_q, \mathbb{F}_{q^3})$ and denoted by $\chi_{5,q^3}^{a_1,a_2,b}$ where $(a_1, a_2, b) \in (\mathbb{F}_q^\times, \mathbb{F}_q, \mathbb{F}_{q^3})$.

3.3. **Family $\mathfrak{F}_4$ where $Y_5Y_6 \subset \ker(\chi)$ and $Y_4 \not\subset \ker(\chi)$.** We study the quotient group $\bar{U} := U/Y_6Y_5$. Recall that $\mathrm{Z}(\bar{U}) = Y_4$ and the commutator relations in $\bar{U}$ are as follows.

$$\begin{aligned}
&[y_1(t), y_2(u)] = y_3(tu)y_4(-tu^{q+1}),\\
&[y_2(t), y_3(u)] = y_4(tu^q + t^q u).
\end{aligned}$$

Let $H := Y_4Y_3Y_1$. Clearly H is an abelian normal subgroup of $\bar{U}$ and Y_2 is a transversal of H in $\bar{U}$. Let $\lambda \in \mathrm{Irr}(H)$ such that $\lambda|_{Y_4} = \varphi_{b_4}$ where $b_4 \in \mathbb{F}_{q^3}^\times$. First we compute the stabilizer $\mathrm{Stab}_{Y_2}(\lambda|_{Y_4Y_3})$. For $y_2(t) \in Y_2$ and $y_3(u) \in Y_3$, we have

$$\begin{aligned}
{}^{y_2(t)}\lambda(y_3(u)) &= \lambda(y_3(u))\lambda([y_3(u), y_2(t)])\\
&= \lambda(y_3(u))\lambda(y_4(-(tu^q + t^q u)))\\
&= \lambda(y_3(u))\varphi_{b_4}(-(tu^q + t^q u)).
\end{aligned}$$

So $y_2(t) \in \mathrm{Stab}_{Y_2}(\lambda|_{Y_4Y_3})$ if and only if $\varphi_{b_4}(-(tu^q + t^q u)) = 1$ for all $u \in \mathbb{F}_{q^3}$.

3.3.1. *The case where q is odd.* We have $\mathrm{Stab}_{Y_2}(\lambda) \leq \mathrm{Stab}_{Y_2}(\lambda|_{Y_4Y_3}) = \{1\}$ by Lemma 2.3 (ii). Thus, the inertia group $I_{\bar{U}}(\lambda) = H$ and $\lambda^{\bar{U}} \in \mathrm{Irr}(\bar{U})$ by Clifford theory. So there are $(q^3-1)q$ irreducible characters of degree q^3 in this family, parameterized by $(Y_4^\times, Y_1) \cong (\mathbb{F}_{q^3}^\times, \mathbb{F}_q)$ and denoted by $\chi_{4,q^3}^{b,a} \in \mathfrak{F}_4^{odd}$ where $(b,a) \in (\mathbb{F}_{q^3}^\times, \mathbb{F}_q)$.

3.3.2. *The case where q is even.* Recall the function $f_t(u) := t^q u + tu^q$ defined right before Lemma 2.3, and $\mathbb{B}_t = \mathrm{im}(f_t)$. From

$${}^{y_2(t)}\lambda(y_3(u)) = \lambda(y_3(u))\lambda(y_4(tu^q + t^q u)) = \lambda(y_3(u))\,\varphi_{b_4}(f_t(u)),$$

we have ${}^{y_2(t)}\lambda|_{Y_4Y_3} = \lambda|_{Y_4Y_3}$ if and only if $\varphi_{b_4}(f_t(u)) = 1$ for all $u \in \mathbb{F}_{q^3}$, i.e. $\mathrm{im}(f_t) = \mathbb{B}_t \subset \ker(\varphi_{b_4})$. By Lemma 2.4 (iv), there exists uniquely $t_0 \in \mathbb{F}_{q^3}^\times$, up to a scalar of $\mathbb{F}_q^\times$, such that $\mathbb{B}_{t_0} \subset \ker(\varphi_{b_4})$. So we have $St_2 := \mathrm{Stab}_{Y_2}(\lambda|_{Y_4Y_3}) = \{y_2(rt_0) : r \in \mathbb{F}_q\}$.

Let $\lambda|_{Y_3} = \varphi_{b_3}$, $b_3 \in \mathbb{F}_{q^3}$. Now we compute $St := \mathrm{Stab}_{Y_2}(\lambda) = \mathrm{Stab}_{St_2}(\lambda)$. For $y_1(v) \in Y_1$ and $y_2(rt_0) \in St_2$, we have

$$\begin{aligned} {}^{y_2(rt_0)}\lambda(y_1(v)) &= \lambda(y_1(v))\lambda([y_1(v), y_2(rt_0)]) \\ &= \lambda(y_1(v))\lambda(y_3(vrt_0)y_4(-v(rt_0)^{q+1})) \\ &= \lambda(y_1(v))\varphi_{b_4}(vrt_0(b_3b_4^{-1} + rt_0^q)). \end{aligned}$$

So ${}^{y_2(rt_0)}\lambda = \lambda$ if and only if $\varphi_{b_4}(vrt_0(b_3b_4^{-1} + rt_0^q)) = 1$ for all $v \in \mathbb{F}_q$. Thus we find $r \in \mathbb{F}_q^\times$ such that $vrt_0(b_3b_4^{-1} + rt_0^q) \in \ker(\varphi_{b_4})$ for all $v \in \mathbb{F}_q$, where $b_3 \in \mathbb{F}_{q^3}$ is a conditional parameter.

If $b_3b_4^{-1}t_0^{-q} \in \mathbb{F}_q^\times$ then it is a nontrivial solution for r. Later we shall see that this is the unique nontrivial solution for r and $St = \{1, y_2(rt_0)\}$.

Now we suppose that $b_3b_4^{-1}t_0^{-q} \notin \mathbb{F}_q^\times$. Thus $b_3b_4^{-1} + rt_0^q \neq 0$ for all $r \in \mathbb{F}_q^\times$. For each $r \in \mathbb{F}_q^\times$, let $T(r, b_3) := \{vrt_0(b_3b_4^{-1} + rt_0^q) : v \in \mathbb{F}_q\}$ be a one-dimensional $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^3}$.

We claim that $T(r, b_3) \subset \ker(\varphi_{b_4})$ if and only if $rt_0(b_3b_4^{-1} + rt_0^q) \in \mathbb{B}_{t_0}$. If $T(r, b_3) \cap \mathbb{B}_{t_0} = \{0\}$, then $\mathbb{F}_{q^3} = T(r, b_3) \oplus \mathbb{B}_{t_0} \subset \ker(\varphi_{b_4})$, a contradiction. So $T(r, b_3) \cap \mathbb{B}_{t_0}$ is nontrivial. By Lemma 2.3 (i), we have $T(r, b_3) \subset \mathbb{B}_{t_0}$. So $rt_0(b_3b_4^{-1} + rt_0^q) \in \mathbb{B}_{t_0}$ as claimed.

By Lemma 2.4 (ii) with $\mathbb{B}_{t_0} = t_0^{q+1}\mathbb{B}_1$, if $T(r, b_3) \subset \ker(\varphi_{b_4})$, then there exists $y \in \mathbb{B}_1$ such that $rt_0(b_3b_4^{-1} + rt_0^q) = t_0^{q+1}y$. Solve this equation for b_3 with parameter $r \in \mathbb{F}_q^\times$, we have $b_3 \in \{b_4t_0^q(r^{-1}y + r) : r \in \mathbb{F}_q^\times, y \in \mathbb{B}_1\} =: I_3$. Here it is clear that $St \neq \{1\}$ if and only if $b_3 \in I_3$.

We claim $|I_3| = (q-1)q^2$ by proving that if there are $r, s \in \mathbb{F}_q^\times$ and $y, z \in \mathbb{B}_1$ such that $r^{-1}y + r = s^{-1}z + s$ then $r = s$ and $y = z$. Since $\mathbb{B}_1$ is an $\mathbb{F}_q$-vector space, $r^{-1}y + s^{-1}z \in \mathbb{B}_1$. From $r^{-1}y + r + s^{-1}z + s = 0 \in \mathbb{B}_1$, we have $r + s \in \mathbb{B}_1$. By Lemma 2.4 (i) with $\mathbb{F}_{q^3} = \mathbb{F}_q \oplus \mathbb{B}_1$, we have $r + s \in \mathbb{F}_q \cap \mathbb{B}_1 = \{0\}$, i.e. $r = s$.

This shows that each $b_3 \in I_3$ determines uniquely $r_0 \in \mathbb{F}_q^\times$ and $y \in \mathbb{B}_1$ such that $b_3 = b_4t_0^q(r_0^{-1}y + r_0)$. Thus, $St = \{1, y_2(r_0t_0)\}$.

Remark 3.1. For $r \in \mathbb{F}_q^\times$ and $y \in \mathbb{B}_1$, we have $r^{-1}y + r \in \mathbb{F}_q^\times$ if and only if $y = 0$. This confirms our above claim when $b_3b_4^{-1}t_0^{-q} \in \mathbb{F}_q^\times$, the solution $r_0 = b_3b_4^{-1}t_0^{-q} = r \in \mathbb{F}_q^\times$ is unique corresponding to $b_3 = b_4t_0^q r \in I_3$.

The $|I_3 \times Y_1| = (q-1)q^3$ linear characters λ of H with $\lambda|_{Y_3} = \varphi_{b_3}$, $b_3 \in I_3$ satisfy $St = \mathrm{Stab}_{Y_2}(\lambda) = \{1, y_2(r_0t_0)\}$ and $I_{\bar{U}}(\lambda) = HSt$. This set is partitioned into $q-1$ orbits under the action of Y_2.

It is easy to check that $[HSt, HSt] \leq \ker(\lambda)$. Thus, these $(q-1)q^3$ linear characters extend to $2(q-1)q^3$ linear characters (of their inertia groups) and induce irreducibly to $\bar{U}$, which gives $4(q-1)$ irreducible characters of degree

$q^3/2$. Together with all $b_4 \in \mathbb{F}_{q^3}^{\times}$, we obtain $4(q^3-1)(q-1)$ irreducible characters of degree $q^3/2$, parameterized by $(Y_4^{\times}, I_3^*, Y_2^*, Y_1^*) \cong (\mathbb{F}_{q^3}^{\times}, \mathbb{F}_q^{\times}, \mathbb{F}_2, \mathbb{F}_2)$ and denoted by $\chi_{4,q^3/2}^{b,a,c_1,c_2} \in \mathfrak{F}_4^{even}$ where $(b,a,c_1,c_2) \in (\mathbb{F}_{q^3}^{\times}, \mathbb{F}_q^{\times}, \mathbb{F}_2, \mathbb{F}_2)$, which is proven below in details.

The other q^3 linear characters λ of H with $\lambda|_{Y_3} = \varphi_{b_3}$, $b_3 \in \mathbb{F}_{q^3} - I_3$ have $St = \mathrm{Stab}_{Y_2}(\lambda) = \{1\}$. By Clifford theory and the fact that $b_4 \in \mathbb{F}_{q^3}^{\times}$, we obtain q^3-1 irreducible characters of $\bar{U}$ of degree q^3, parameterized by $(Y_4^{\times}) \cong (\mathbb{F}_{q^3}^{\times})$ and denoted by $\chi_{4,q^3}^{b} \in \mathfrak{F}_4^{even}$ where $b \in \mathbb{F}_{q^3}^{\times}$.

3.3.3. The parametrization $(Y_4^{\times}, I_3^*, Y_2^*, Y_1^*) \cong (\mathbb{F}_{q^3}^{\times}, \mathbb{F}_q^{\times}, \mathbb{F}_2, \mathbb{F}_2)$ of $\chi_{4,q^3/2}^{b,a,c_1,c_2}$. Here $b = b_4 \in \mathbb{F}_{q^3}^{\times}$ and $a = b_3 \in I_3^*$ which is a representative set of $q-1$ orbits of $\mathrm{Irr}(H, \lambda|_{Y_4})$ with $b_3 \in I_3$ under the action of Y_2 as above. It suffices to show the parametrization $(Y_2^*, Y_1^*) \cong (\mathbb{F}_2, \mathbb{F}_2)$.

Let $\mu := \lambda|_{Y_4Y_3}$. So μ extends to $M := I_{\bar{U}}(\lambda) = HSt$ where $H = Y_4Y_3Y_1$ and $St = \{1, y_2(r_0t_0)\}$. Let η_1, η_2 be two extensions of μ to M. Here μ also extends to $K := Y_4Y_3St_2$ where $St_2 = \{y_2(rt_0) : r \in \mathbb{F}_q\}$ since $[K,K] \le \ker(\mu)$. Let θ be an extension of μ to K and $S_1 := \mathrm{Stab}_{Y_1}(\theta)$.

We prove the following statements.

(i) $|S_1| = 2$ and θ extends to $N := I_{\bar{U}}(\theta) = KS_1$.
Let γ be an extension of θ to N.

(ii) $(\eta_i^{\bar{U}}, \gamma^{\bar{U}}) = 1$ if and only if $\eta_i|_{St} = \gamma|_{St}$ and $\eta_i|_{S_1} = \gamma|_{S_1}$ for $i = 1, 2$.

(iii) $(\eta_1^{\bar{U}}, \eta_2^{\bar{U}}) = 1$ if and only if $\eta_1|_{St} = \eta_2|_{St}$ and $\eta_1|_{S_1} = \eta_2|_{S_1}$.

An easy way to see these is described in the following diagram.

$$\begin{array}{rccccccl}
Y_4Y_3: & & & \mu & & & & \\
& & \swarrow & & \searrow & & & \\
H = Y_4Y_3Y_1: & \lambda & & & & \theta & & : Y_4Y_3St_2 = K \\
& \downarrow & & & & \downarrow & & \\
M = HSt: & \eta_i & & & & \gamma & & : KS_1 = N \\
& & \searrow & & \swarrow & & & \\
\bar{U}: & & & \eta_i^{\bar{U}}, \gamma^{\bar{U}} & & \in & & \mathrm{Irr}(\bar{U})
\end{array}$$

Proof. (i) From the proof of $\theta|_{Y_3} = \mu|_{Y_3} = \varphi_{b_3}$ where $b_3 \in I_3$, all elements in $\mathrm{Irr}(\bar{U}, \theta)$ have degree $q^3/2$. By Clifford theory, the inertia group $I_{\bar{U}}(\theta)$ has order $2|K| = 2q^7$ and θ extends to $I_{\bar{U}}(\theta)$.

Let T_2 be a transversal of St_2 in Y_2. It is clear that T_2Y_1 is a transversal of K in $\bar{U}$. There is no $y_2(t)y_1(s) \in \mathrm{Stab}_{\bar{U}}(\theta) \le \mathrm{Stab}_{\bar{U}}(\mu)$ where $0 \ne t \ne r_0t_0$ since otherwise $y_2(t) \ne y_2(r_0t_0) \in \mathrm{Stab}_{Y_2}(\lambda) = St$. Since $Y_4Y_3Y_1$ is abelian, we have $\mathrm{Stab}_{T_2Y_1}(\theta) = \mathrm{Stab}_{Y_1}(\theta)$. Thus $S_1 = \mathrm{Stab}_{Y_1}(\theta)$ has order 2.

(ii) Set $\eta = \eta_i$. We show that $(\eta^{\bar{U}}, \gamma^{\bar{U}}) = 1$ if and only if $\eta|_{St} = \gamma|_{St}$ and $\eta|_{S_1} = \gamma|_{S_1}$. Notice that both $\eta^{\bar{U}}, \gamma^{\bar{U}} \in \mathrm{Irr}(\bar{U})$ by Clifford theory.

Since $M \lhd \bar{U} = MK_2$, by Mackey formula for the double coset $M\backslash\bar{U}/N$ and Frobenius reciprocity we have

$$\begin{aligned}(\eta^{\bar{U}}, \gamma^{\bar{U}}) &= (\eta^{\bar{U}}|_N, \gamma)\\ &= (\eta|_{M\cap N}{}^N, \gamma)\\ &= (\eta|_{M\cap N}, \gamma|_{M\cap N}).\end{aligned}$$

Since $M \cap N = Y_4Y_3StS_1$ and both η, γ are linear, the claim holds.

(iii) Choosing $\gamma \in \mathrm{Irr}(N)$ such that $\gamma|_{M\cap N} = \eta_1|_{M\cap N}$, by part (ii) we have $\eta_1^{\bar{U}} = \gamma^{\bar{U}}$. Again by part (ii) we have $(\eta_2^{\bar{U}}, \gamma^{\bar{U}}) = 1$ if and only if $\eta_2|_{M\cap N} = \gamma|_{M\cap N}$, which completes the proof. □

Notice that the parameterizations of Y_2^* and Y_1^* correspond to $St \leq Y_2$ and $S_1 \leq Y_1$ respectively, which both are cyclic of order 2.

3.4. **Family $\mathfrak{F}_3$ where $Y_4Y_5Y_6 \subset \ker(\chi)$ and $Y_3 \not\subset \ker(\chi)$.** We study $\bar{U} := U/Y_6Y_5Y_4$. The commutator relation in $\bar{U}$ is $[y_1(t), y_2(u)] = y_3(tu)$. Let $H := Y_3Y_2$. Clearly H is an abelian normal subgroup $\bar{U}$ and Y_1 is a transversal of H in $\bar{U}$. Let $\lambda \in \mathrm{Irr}(H)$ such that $\lambda|_{Y_3} = \varphi_{b_3}$ where $b_3 \in \mathbb{F}_{q^3}^\times$.

Now we compute $\mathrm{Stab}_{Y_1}(\lambda)$. For $y_1(t) \in Y_1$ and $y_2(u) \in Y_2$, we have

$$\begin{aligned}{}^{y_1(t)}\lambda(y_2(u)) &= \lambda(y_2(u))\lambda([y_2(u), y_1(t)])\\ &= \lambda(y_2(u))\lambda(y_3(tu))\\ &= \lambda(y_2(u))\varphi_{b_3}(tu).\end{aligned}$$

So $y_1(t) \in \mathrm{Stab}_{Y_1}(\lambda)$ if and only if $\varphi_{b_3}(tu) = 1$ for all $u \in \mathbb{F}_{q^3}$, if and only if $t = 0$. Thus, $I_{\bar{U}}(\lambda) = H$ and $\lambda^{\bar{U}} \in \mathrm{Irr}(\bar{U})$ of degree q by Clifford theory.

For each $b_3 \in \mathbb{F}_{q^3}^\times$, let Y_2^* be a representative set of q^2 orbits of $\mathrm{Irr}(H, \lambda|_{Y_3})$ under the action of Y_1. So $\mathfrak{F}_3$ contains $(q^3-1)q^2$ irreducible characters of degree q, parameterized by $(Y_3^\times, Y_2^*) \cong (\mathbb{F}_{q^3}^\times, \mathbb{F}_{q^3}/\mathbb{F}_q)$ and denoted by $\chi_{3,q}^{b_1,b_2}$ where $(b_1, b_2) \in (\mathbb{F}_{q^3}^\times, \mathbb{F}_{q^3}/\mathbb{F}_q)$.

3.5. **Family $\mathfrak{F}_{lin}$ where $Y_3Y_4Y_5Y_6 \subset \ker(\chi)$.** We study $\bar{U} := U/Y_6Y_5Y_4Y_3$. Since $\bar{U}$ is abelian, this family is the set of all linear characters of $\bar{U}$. Here we obtain q^4 linear characters parameterized by $(Y_2, Y_1) \cong (\mathbb{F}_{q^3}, \mathbb{F}_q)$ and denoted by $\chi_{lin}^{b,a}$ where $(b, a) \in (\mathbb{F}_{q^3}, \mathbb{F}_q)$.

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