ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Special Issue of the

Bulletin of the Iranian Mathematical Society

Special Issue Dedicated to ORO2013 Conference

Vol. 42 (2016), No. 7, pp. 5–23

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 7, pp. 5–23
Online ISSN: 1735-8515

OPTIMALITY CONDITIONS FOR APPROXIMATE SOLUTIONS OF VECTOR OPTIMIZATION PROBLEMS WITH VARIABLE ORDERING STRUCTURES

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(Communicated by Majid Soleimani-damaneh)

Abstract. We consider nonconvex vector optimization problems with variable ordering structures in Banach spaces. Under certain boundedness and continuity properties we present necessary conditions for approximate solutions of these problems. Using a generic approach to subdifferentials we derive necessary conditions for approximate minimizers and approximately minimal solutions of vector optimization problems with variable ordering structures applying nonlinear separating functionals and Ekeland's variational principle.
Keywords: Nonconvex vector optimization, variable ordering structure, Ekeland's variational principle, optimality conditions.
MSC(2010): Primary: 90C29; Secondary: 90C26, 90C30, 90C46.

1. Introduction and preliminaries

Our aim here is to derive new necessary conditions for approximate minimizers and approximately minimal solutions of vector optimization problems with variable ordering structures by using nonlinear separating functionals and their subdifferentials. Bao and Mordukhovich [3, 4] showed necessary conditions for nondominated points of sets and nondominated solutions of vector optimization problems with variable ordering structures and general geometric constraints, applying methods of variational analysis and generalized differentiation (see Mordukhovich [22], and Mordukhovich and Shao [23]). Furthermore, Bao et al. [2] gave necessary conditions for approximately nondominated solutions of vector optimization problems with variable ordering structures in Asplund spaces using a vector-valued variant of Ekeland's variational principle. Here, we introduce a generic approach to subdifferentials which includes many well-known subdifferentials. In the next section, we recall definitions of approximately

Article electronically published on December 18, 2016.
Received: 14 October 2014, Accepted: 13 January 2015.
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minimal, approximately nondominated solutions and approximate minimizers of vector optimization problems with respect to variable ordering structures. In the case of exact solutions of a vector optimization problem, specially in the variable ordering case, authors use a cone or a pointed convex cone-valued map in order to describe the solution concepts but here, we use a set-valued map and this map is not necessarily a (pointed convex) cone-valued map. In the third section, we will give necessary conditions for approximately minimal solutions of vector optimization problems with variable ordering structures. For this purpose, we will use a generalization of nonlinear separating functionals studied by Gerth and Weidner in [14]. Moreover, we give necessary conditions for approximate minimizers. In order to derive necessary conditions for approximate minimizers of vector optimization problems with variable ordering structures, a modification of the nonlinear separating functionals by Chen and Yang [6] and the special case of scalarization functional defined by Chen et al. [7] are used.

Let X and Y be real Banach spaces and C be a nonempty set in Y. The notations $\operatorname{int} C$, $\operatorname{cl} C$, and $\operatorname{bd} C$ respectively stand for the topological interior, the topological closure, and the topological boundary of the set C. For a nonconvex set C, the convex hull of C is denoted by $\operatorname{conv} C$. The set C is said to be *solid* if and only if $\operatorname{int} C \neq \emptyset$, *proper* if and only if $C \neq \emptyset$ and $C \neq Y$, *pointed* if and only if $C \cap (-C) \subseteq \{\mathbf{0}\}$, and a *cone* if and only if $\lambda c \in C$, for all $c \in C$ and $\lambda \geq 0$; see [15, 19] for basic definitions and solution concepts of vector optimization problems, and [14, 24] for some scalarization methods and their properties.

As usual, for a set $S \subset X$, we denote by I_S the indicator function of S ($I_S(x) = 0$, if $x \in S$ and $I_S(x) = +\infty$, if $x \notin S$).

Here, we derive necessary conditions for approximate minimizers and minimal solutions of vector optimization problems with variable ordering structures using the following generic approach to subdifferentials; e.g., see [8, 9] and [18].

Let $\mathcal{X}$ be a class of Banach spaces which contains the class of finite dimensional normed vector spaces. By an abstract subdifferential ∂ we mean a map which associates to every lower semicontinuous (lsc) function $h : X \in \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ and to every $x \in X$ a (possible empty) subset $\partial h(x) \subset X^*$. We use the notation $\operatorname{Dom} h := \{x \in X \mid h(x) \neq +\infty\}$. Let $X, Y \in \mathcal{X}$ and denote by $\mathcal{F}(X, Y)$ a class of functions acting between X and Y having the property that by composition at left with an lsc function from Y to $\overline{\mathbb{R}}$, the resulting function is still lsc.

In the following we work with some properties of the abstract subdifferential ∂:

(H1) If h is convex, then $\partial h(x)$ coincides with the Fenchel subdifferential.

(H2) If x is a local minimal point for h, then $0 \in \partial h(x)$; $\partial h(u) = \emptyset$ if $u \notin \operatorname{Dom} h$.

(Note that (H1) and (H2) are quite natural requirements for any subdifferential.)

(H3) If $\varphi : Y \to \mathbb{R} \cup \{+\infty\}$ is convex and $\psi \in \mathcal{F}(X, Y)$, then for every x,

$$\partial(\varphi \circ \psi)(x) \subseteq \bigcup_{y^* \in \partial\varphi(\psi(x))} \partial(y^* \circ \psi)(x).$$

(H4) If $\varphi : Y \to \mathbb{R} \cup \{+\infty\}$ is convex, $\psi \in \mathcal{F}(X, Y)$, and $S \subset X$ is a closed set containing x, then

$$\partial(\varphi \circ \psi + I_S)(x) \subseteq \partial(\varphi \circ \psi)(x) + \partial I_S(x).$$

(H5) If h is convex and $g : X \to \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz, then for every $x \in \operatorname{Dom} h \cap \operatorname{Dom} g$,

$$\partial(h + g)(x) \subseteq \partial h(x) + \partial g(x).$$

As usual, for a closed set $S \subset X$ the set $\partial I_S(x)$ is denoted by $N_\partial(S, x)$ and is called the set of normal directions to S at $x \in S$ with respect to ∂.

The properties (H3), (H4) and (H5) are "exact calculus rules" for sums and for composition, and as examples of subdifferentials with these properties we can mention the followings:

- the limiting (or Mordukhovich) subdifferential, when $\mathcal{X}$ is the class of Asplund spaces, Y is finite dimensional and $\mathcal{F}(X, Y)$ is the class of Lipschitz functions from X into Y (see [23]);
- the approximate (or Ioffe) subdifferential, when $\mathcal{X}$ is the class of Banach spaces and $\mathcal{F}(X, Y)$ is the class of strongly compactly Lipschitz functions from X into Y (see [18]).

A significant result in nonlinear analysis is Ekeland's variational principle [13], which shows the existence of an exact solution of a perturbed problem in a neighborhood of an approximate solution of the original problem without convexity and compactness assumptions.

Theorem 1.1. *Let X be a real Banach space, $\varepsilon > 0$ and $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous functional bounded from below on a closed set $\Omega \subset X$. Suppose $\overline{x} \in \Omega$ such that $\varphi(\overline{x}) \leq \inf_{x \in \Omega} \varphi(x) + \varepsilon$. Then, there exists $x_\varepsilon \in \operatorname{Dom} \varphi \cap \Omega$ such that*

(a) $\varphi(x_\varepsilon) \leq \varphi(\overline{x}) \leq \inf_{x \in \Omega} \varphi(x) + \varepsilon$,
(b) $\|x_\varepsilon - \overline{x}\| \leq \sqrt{\varepsilon}$,
(c) $\varphi(x_\varepsilon) + \sqrt{\varepsilon}\, \|\overline{x} - x_\varepsilon\| \leq \varphi(\overline{x})$.

2. Different concepts of approximate solutions of vector optimization problems with variable ordering structures

Vector optimization with variable ordering structures is a growing area of research (see [12] for a recent overview). In this section, we recall definitions of ϵk^0-minimizers, ϵk^0-nondominated and ϵk^0-minimal solutions of vector optimization problems with respect to variable ordering structures. For sure there is no difference between ϵk^0-minimizers, ϵk^0-nondominated and ϵk^0-minimal solutions in vector optimization problems with fixed ordering structures. This statement is also true for weakly and strongly ϵk^0-optimal solutions. Here, we show that this statement is not true for vector optimization problems with variable ordering structures and all these three definitions define different elements. This will be shown by several examples. For more details, properties and characterization of these solution concepts, see [26, 27].

We will use following assumptions.

(A) X, Y are Banach spaces, $\Omega \subset X$ is a closed set in X, $f \in \mathcal{F}(X, Y)$ is a function with $\operatorname{Dom} f \neq \emptyset$ and $\varepsilon \geq 0$.

(B) The set-valued mapping $C : Y \rightrightarrows Y$ satisfies $\mathbf{0} \in \operatorname{bd} C(y)$, $C(y)$ is closed, solid and pointed for all $y \in Y$, and the nonzero vector $k^0 \in Y \setminus \{\mathbf{0}\}$ satisfies $C(y) + [0, +\infty)k^0 \subset C(y)$, for all $y \in Y$.

Under assumptions (A) and (B), we consider the following vector optimization problem with respect to a variable ordering structure:

$$\text{(VVOP)} \qquad \text{minimize } \ f(x) \text{ subject to } \ x \in \Omega \text{ with respect to } \ C.$$

In order to introduce the different concepts for approximate solutions of (VVOP) we suppose that the assumptions (A) and (B) are fulfilled, $x^1, x^2, x^3 \in X$ and define the following three different domination relations:

$$f(x^1) \leq_1 f(x^2) \ \text{ if } \ f(x^2) \in f(x^1) + (C(f(x^2)) \backslash \{0\}), \tag{2.1}$$

$$f(x^1) \leq_2 f(x^2) \ \text{ if } \ f(x^2) \in f(x^1) + (C(f(x^1)) \backslash \{0\}), \tag{2.2}$$

$$f(x^1) \leq_3 f(x^2) \ \text{ if for all } x^3 \in X, \ \ f(x^2) \in f(x^1) + (C(f(x^3)) \backslash \{0\}). \tag{2.3}$$

If $C(f(x^1)) = C(f(x^2)) = C(f(x^3))$, for all $x^1, x^2, x^3 \in X$, then these three domination relations are the same and the problem reduces to the optimization with standard domination structure.

The first concept of the approximate solution is based on the domination relation (2.1), called approximately minimal solutions of (VVOP); see [27] for more details and properties of approximately minimal solutions of problem (VVOP).

Definition 2.1. Let assumptions (A) and (B) be fulfilled, $\varepsilon \geq 0$ and consider (VVOP).

(a) An element $\overline{x} \in \Omega$ is said to be an εk^0-minimal solution of (VVOP) with respect to the variable ordering structure $C(\cdot)$ if and only if there is no element $y \in f(\Omega) := \cup_{x\in\Omega}\{f(x)\}$ such that $y + \varepsilon k^0 \leq_1 f(\overline{x})$; i.e.,

$$\big(f(\overline{x}) - \varepsilon k^0 - (C(f(\overline{x})) \setminus \{\mathbf{0}\})\big) \cap f(\Omega) = \emptyset.$$

(b) Suppose that $\operatorname{int} C(f(\overline{x})) \neq \emptyset$. An element $\overline{x} \in \Omega$ is said to be a weakly εk^0-minimal solution of (VVOP) with respect to $C(\cdot)$ if and only if

$$\big(f(\overline{x}) - \varepsilon k^0 - \operatorname{int}\, C(f(\overline{x}))\big) \cap f(\Omega) = \emptyset.$$

(c) $\overline{x} \in \Omega$ is said to be a strongly εk^0-minimal solution of (VVOP) with respect to $C(\cdot)$ if and only if

$$\forall x \in \Omega\backslash\{\overline{x}\}, \qquad f(\overline{x}) - \varepsilon k^0 \in f(x) - (C(f(\overline{x})) \setminus \{\mathbf{0}\}).$$

The case $\varepsilon = 0$, it coincides with the usual definition of (weakly) minimal solutions; e.g., see [10, 17]. We denote the set of εk^0-minimal, weakly εk^0-minimal and strongly εk^0-minimal solutions by εk^0-M(Ω, f, C), εk^0-WM(Ω, f, C) and εk^0-SM(Ω, f, C), respectively. For $\varepsilon = 0$, we also write M(Ω, f, C), WM(Ω, f, C) and SM(Ω, f, C).

We now introduce a second concept of approximate solutions based on the domination relation (2.2) called approximately nondominated solutions to (VVOP). More details and properties of approximately nondominated solutions can be found in [27].

Definition 2.2. Let assumptions (A) and (B) be fulfilled, $\varepsilon \geq 0$ and consider (VVOP).

(a) $\overline{x} \in \Omega$ is said to be an εk^0-nondominated solution of the problem (VVOP) with respect to the ordering map $C : Y \rightrightarrows Y$ if and only if

$$\forall x \in \Omega, \qquad (f(\overline{x}) - \varepsilon k^0 - (C(f(x))\backslash\{0\})) \cap \{f(x)\} = \emptyset.$$

(b) Suppose that $\operatorname{int} C(f(x)) \neq \emptyset$, for all $x \in \Omega$. $\overline{x} \in \Omega$ is said to be a weakly εk^0-nondominated solution of (VVOP) with respect to the ordering map $C : Y \rightrightarrows Y$ if and only if

$$\forall x \in \Omega, \qquad (f(\overline{x}) - \varepsilon k^0 - \operatorname{int} C(f(x))) \cap \{f(x)\} = \emptyset.$$

(c) $\overline{x} \in \Omega$ is said to be a strongly εk^0-nondominate solution of (VVOP) with respect to the ordering map $C : Y \rightrightarrows Y$ if and only if

$$\forall x \in \Omega\backslash\{\overline{x}\}, \qquad f(\overline{x}) - \varepsilon k^0 \in f(x) - (C(f(x))\backslash\{0\}).$$

We denote the set of εk^0-nondominated, weakly εk^0-nondominated and strongly εk^0-nondominated solutions by εk^0-$N(\Omega, f, C)$, εk^0-$WN(\Omega, f, C)$ and εk^0-$SN(\Omega, f, C)$, respectively. For $\varepsilon = 0$, we write $\mathrm{N}(\Omega, f, C)$, $\mathrm{WN}(\Omega, f, C)$ and $\mathrm{SN}(\Omega, f, C)$; see also [10, 29] for definition of exact nondominated solution of vector optimization problems with variable ordering structures.

A third concept of approximate solutions based on the domination relation (2.3) is as follows; see [27] for more details and properties of approximate minimizers of (VVOP).

Definition 2.3. Let assumptions (A) and (B) be fulfilled, $\varepsilon \geq 0$ and consider (VVOP).

(a) $\overline{x} \in \Omega$ is said to be an εk^0-minimizer of the problem (VVOP) with respect to the ordering map $C : Y \rightrightarrows Y$ if and only if

$$\forall x, x^1 \in \Omega, \qquad (f(\overline{x}) - \varepsilon k^0 - (C(f(x))\backslash\{0\})) \cap \{f(x^1)\} = \emptyset.$$

Equivalently, $\overline{x}$ is an εk^0-minimizer if and only if

$$\forall x \in \Omega, \qquad (f(\overline{x}) - \varepsilon k^0 - (C(f(x))\backslash\{0\})) \cap \{f(\Omega)\} = \emptyset.$$

(b) Suppose that $\operatorname{int} C(f(x)) \neq \emptyset$ for all $x \in \Omega$. $\overline{x} \in \Omega$ is said to be a weakly εk^0- minimizer of (VVOP) with respect to the ordering map $C : Y \rightrightarrows Y$ if and only if

$$\forall x, x^1 \in \Omega, \qquad (f(\overline{x}) - \varepsilon k^0 - \operatorname{int} C(f(x))) \cap \{f(x^1)\} = \emptyset.$$

Equivalently, $\overline{x}$ is an εk^0-minimizer if and only if

$$\forall x \in \Omega, \qquad (f(\overline{x}) - \varepsilon k^0 - \operatorname{int} C(f(x))) \cap \{f(\Omega)\} = \emptyset.$$

(c) $\overline{x} \in \Omega$ is said to be a strongly εk^0-minimizer of (VVOP) with respect to the ordering map $C : Y \rightrightarrows Y$ iff

$$\forall x, x^1 \in \Omega\backslash\{\overline{x}\}, \qquad f(\overline{x}) - \varepsilon k^0 \in f(x) - (C(f(x^1))\backslash\{0\}).$$

We denote the set of εk^0-minimizers, weak εk^0-minimizers and strong εk^0-minimizers by εk^0-$MZ(\Omega, f, C)$, εk^0-$WMZ(\Omega, f, C)$ and εk^0-$SMZ(\Omega, f, C)$ respectively. For $\varepsilon = 0$, we also write $\mathrm{MZ}(\Omega, f, C)$, $\mathrm{WMZ}(\Omega, f, C)$ and $\mathrm{SMZ}(\Omega, f, C)$; see also [5] for the definition of minimizers under a different name.

Obviously, by definitions 2.1, 2.2 and 2.3, sets of ϵk^0-minimizers, ϵk^0-nondominated and ϵk^0-minimal solutions of vector optimization problems with fixed ordering structures coincide. This statement is also true for weakly and strongly ϵk^0-optimal solutions. Now, by several examples, we show that this statement cannot be true for vector optimization problems with variable ordering structures. For reader's convenience, in the following examples, we suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the identity function and $f(\Omega) = \Omega$.

Example 2.4. Let $\epsilon = \frac{1}{100}$ and $k^0 = (1,0)^T$. Also, suppose that

$$\Omega = \big\{(y_1, y_2) \mid \{(y_1 + y_2 \geq 1)\} \cap \{0 \leq y_1, y_2 \leq 1\}\big\},$$

and

$$C(y_1, y_2) = \begin{cases} \mathbb{R}^2_+, & \text{if } y_1 = 0, \\ \text{cone conv}\{(1,0)^T, (y_1, y_2)\}, & \text{otherwise.} \end{cases}$$

Obviously, $C(y) + [0, +\infty)k^0 \subseteq C(y)$, for all $y \in \Omega$, and elements of

$$\{(y_1, y_2) \in \Omega \mid y_1 + y_2 \leq 1 + \frac{1}{100}\},$$

are ϵk^0-minimizer, ϵk^0-nondominated and ϵk^0-minimal solutions and the sets of all these points coincide (see Figure 1).

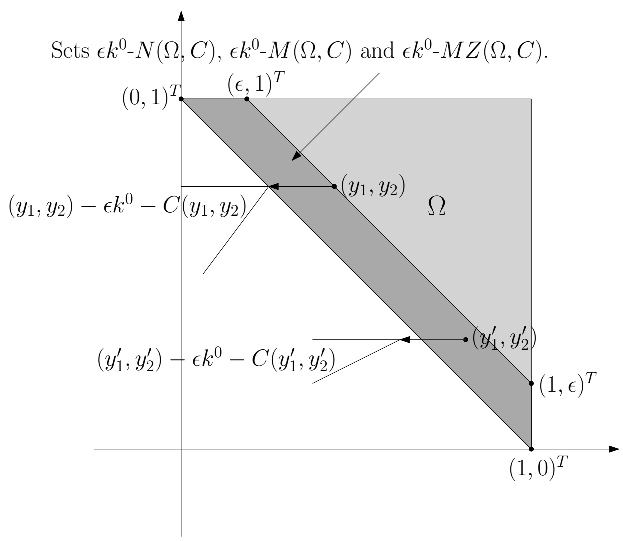

FIGURE 1. Example 2.4, where sets of ϵk^0-$N(\Omega, f, C)$, ϵk^0-$MZ(\Omega, f, C)$ and ϵk^0-$M(\Omega, f, C)$ of Ω coincide.

In the following example, we show that there exists an approximately minimal solution of vector optimization problems with variable ordering structures which is neither an approximate minimizer nor an approximately nondominated solution.

Example 2.5. Let $\epsilon = \frac{1}{100}$ and $k^0 = (1,0)^T$. Consider

$$\Omega = \{(y_1, y_2) \mid 0 \leq y_1, y_2 \leq 1\},$$

and

$$C(y_1, y_2) = \begin{cases} \{(d_1, d_2) \in \mathbb{R}^2 \mid d_1 \geq 0,\ d_2 \leq 0\}, & \text{if } y_1 = 0, \\ \text{cone conv}\{(1,0)^T, (y_1, y_2)\}, & \text{otherwise.} \end{cases}$$

It is obvious to see that $C(y) + [0, +\infty)k^0 \subseteq C(y)$, for all $y \in \Omega$, and $\{(y_1, y_2) \in \Omega \mid y_1 \leq \epsilon\}$ is the set of ϵk^0-minimal solutions. But, only the elements of the set $\{(y_1, y_2) \in \Omega \mid y_1 < \epsilon\} \bigcup \{(\epsilon, 1)^T\}$ are ϵk^0-minimizers and ϵk^0-nondominated solutions (see Figure 2).

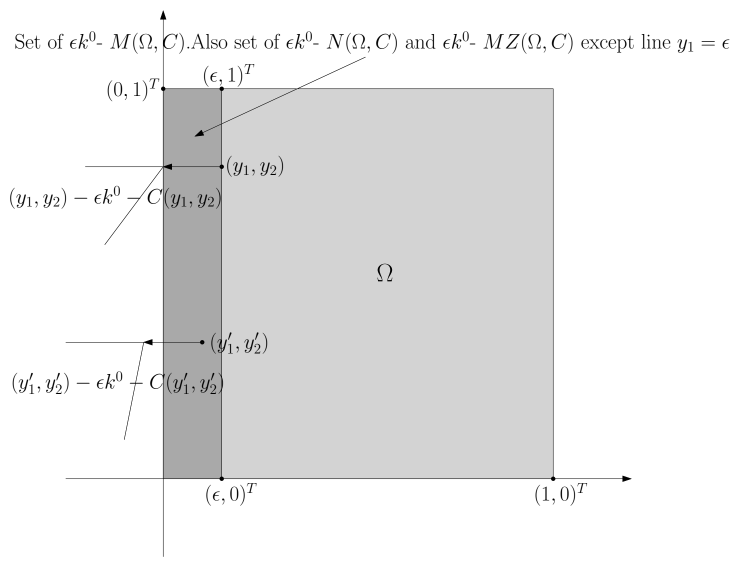

Figure 2. Example 2.5, where there exists an ϵk^0-minimal solution of the set Ω which is neither ϵk^0-minimizer nor ϵk^0-nondominated solution.

In the following example, we show that there exists an approximately nondominated solution of vector optimization problems with variable ordering structures which is neither an approximately nondominated solution nor an approximate minimizer.

Example 2.6. Assume that $\epsilon = \frac{1}{100}$ and $k^0 = (1,1)^T$. Furthermore, suppose that

$$\Omega = \big\{(y_1, y_2) \in \mathbb{R}^2 \mid \{y_1 + y_2 \geq -1\} \cap \{y_1 \leq 0,\ y_2 \leq 0\}\big\},$$

and

$$C(y_1, y_2) = \begin{cases} \{(d_1, d_2) \in \mathbb{R}^2 \mid d_2 \geq 0,\ d_1 + d_2 \geq -1\}, & \text{for } (y_1, y_2) = (-1, 0)^T, \\ \mathbb{R}^2_+, & \text{otherwise.} \end{cases}$$

Obviously, $C(y) + [0, +\infty)k^0 \subseteq C(y)$, for all $y \in \Omega$, and $(-1, 0)^T$ is not an ϵk^0-minimal solution. In fact, $\{(y_1, y_2) \in \Omega \mid y_1 + y_2 \leq -\frac{98}{100},\ y_1 \neq -1\}$ is the set of ϵk^0-minimal solutions. However, $(-1, 0)^T$ belongs to the set of ϵk^0-nondominated solutions which is $\{(y_1, y_2) \in \Omega \mid y_1 + y_2 \leq -\frac{98}{100}\}$. Obviously, $(-1, 0)^T$ is not an ϵk^0-minimizer and $\{(y_1, y_2) \in \Omega \mid -1 < y_2 \leq -1 + \epsilon\}$ is the set of ϵk^0-minimizers (see Figure 3).

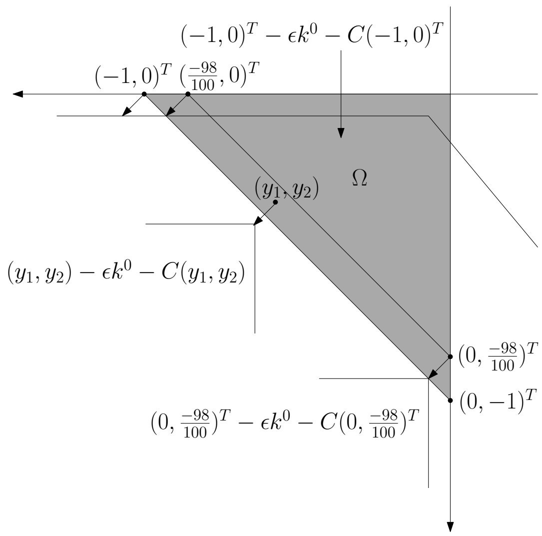

FIGURE 3. Example 2.6, where $(-1, 0)^T$ is an ϵk^0-nondominated solution of the set Ω, but it is neither ϵk^0-minimizer nor ϵk^0-minimal solution.

In the following example, we show that there exists an approximately optimal solution which is both ϵk^0-nondominated and ϵk^0-minimal solution but it is not an ϵk^0-minimizer.

Example 2.7. Let $\epsilon = \frac{1}{100}$ and $k^0 = (0, 1)^T$. Consider

$$\Omega = \big\{(y_1, y_2) \in \mathbb{R}^2_+ \mid \{y_1 + y_2 \geq 2\} \cap \{y_1 \geq 0,\ 0 \leq y_2 \leq 2\}\big\},$$

and

$$C(y_1, y_2) = \begin{cases} \{(d_1, d_2) \in \mathbb{R}^2 \mid d_1 \leq 0,\ d_2 \geq 0\}, & \text{if } (y_1, y_2) = (4, 2)^T, \\ \mathbb{R}^2_+, & \text{otherwise.} \end{cases}$$

Then, $C(y) + [0, +\infty)k^0 \subseteq C(y)$, for all $y \in \Omega$, and the set of ϵk^0-minimal and ϵk^0-nondominated solutions is $\{(y_1, y_2) \in f(\Omega) \mid y_1 + y_2 \leq 2 + \epsilon\}$. But, only the elements of the set $\big\{(y_1, y_2) \in \Omega \mid y_2 < \epsilon = \frac{1}{100}\big\}$ are ϵk^0-minimizer.

This shows that there exists an approximately optimal solution which is both ϵk^0-nondominated and ϵk^0-minimal but it is not an ϵk^0-minimizer (see Figure 4).

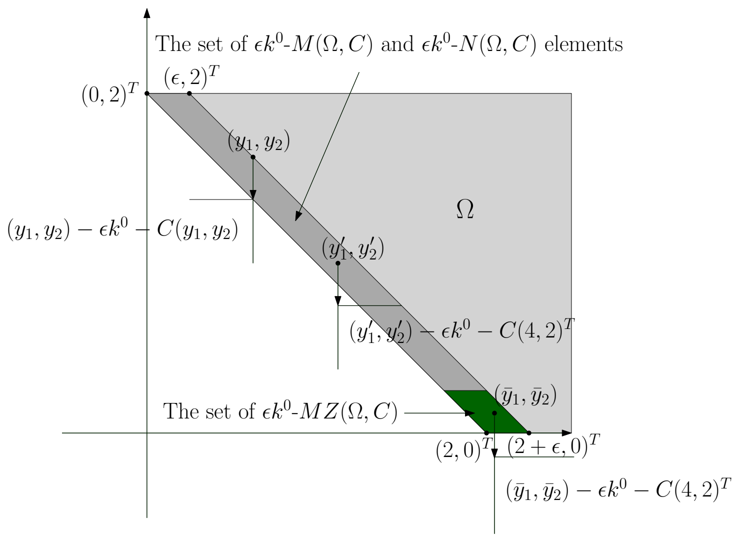

FIGURE 4. Example 2.7, where there exists an element which is both ϵk^0-nondominated and ϵk^0-minimal but it is not an ϵk^0-minimizer.

3. Optimality conditions for εk^0-minimal solutions of (VVOP)

In this section, with the help of nonlinear separating functionals and their properties [14], we will characterize approximately minimal solutions of vector optimization problems with variable ordering structures, and using this characterization in the main theorem of this section, we will show necessary conditions for approximately minimal solutions of vector optimization problems with variable ordering structures.

Here, we suppose that (A) and (B) hold and consider $\overline{x} \in X$. In order to derive necessary optimality conditions for approximately minimal solutions of (VVOP), we use the scalarization functional $\theta_{\overline{x}} : Y \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, defined by

$$\theta_{\overline{x}}(y) := \inf\{t \in \mathbb{R} \mid y \in tk^0 + f(\overline{x}) - C(f(\overline{x}))\}. \tag{3.1}$$

The following results provide some properties of this nonlinear separating functional.

Theorem 3.1. [15, Theorem 2.3.1] *Let assumptions* (A) *and* (B) *hold and* $\overline{x} \in X$*. The functional* $\theta_{\overline{x}} : Y \to \overline{\mathbb{R}}$*, defined by* (3.1)*, has the following properties:*

(a) $\theta_{\overline{x}}$ *is proper if and only if* $C(f(\overline{x}))$ *does not contain lines parallel to* k^0*; i.e.,*

$$\forall y \in Y, \exists t \in \mathbb{R} \ : \ y + tk^0 \notin C(f(\overline{x})).$$

(b) $\theta_{\overline{x}}(\lambda y) = \lambda \theta_{\overline{x}}(y)$*, for all* $\lambda > 0$ *and* $y \in Y$*, if and only if* $C(f(\overline{x}))$ *is a cone.*

(c) $\theta_{\overline{x}}$ *is finite-valued if and only if* $C(f(\overline{x}))$ *does not contain lines parallel to* k^0 *and* $\mathbb{R}k^0 - C(f(\overline{x})) = Y$*.*

(d) *The domain of* $\theta_{\overline{x}}$ *is* $\mathbb{R}k^0 - C(f(\overline{x}))$ *and*

$$\theta_{\overline{x}}(y + \lambda k^0) = \theta_{\overline{x}}(y) + \lambda, \quad \forall y \in Y, \ \forall \lambda \in \mathbb{R}.$$

(e) *Let* $D \subset Y$*;* $\theta_{\overline{x}}$ *be* D*-monotone; i.e.,* $y^2 - y^1 \in D \Rightarrow \theta_{\overline{x}}(y^1) \leq \theta_{\overline{x}}(y^2)$ *if and only if* $C(f(\overline{x})) + D \subseteq C(f(\overline{x}))$*.*

(f) $\theta_{\overline{x}}$ *is convex if and only if* $C(f(\overline{x}))$ *is convex.*

(g) $\theta_{\overline{x}}$ *is subadditive if and only if* $C(f(\overline{x})) + C(f(\overline{x})) \subseteq C(f(\overline{x}))$*.*

Theorem 3.2. [15, Theorem 2.3.1] *Suppose that assumptions* (A) *and* (B) *hold and* $\overline{x} \in X$*. Then,*

(a) $\theta_{\overline{x}} : Y \to \overline{\mathbb{R}}$ *defined by* (3.1) *is lower semicontinuous.*

(b) *Furthermore, if* $C(f(\overline{x})) + (0, +\infty)k^0 \subset \operatorname{int} C(f(\overline{x}))$*, then* $\theta_{\overline{x}}$ *is continuous and*

$$\begin{aligned}
\{y \in Y | \theta_{\overline{x}}(y) < \lambda\} &= \lambda k^0 - \operatorname{int} C(f(\overline{x})), \ \forall \lambda \in \mathbb{R}, \\
\{y \in Y | \theta_{\overline{x}}(y) = \lambda\} &= \lambda k^0 - \operatorname{bd} C(f(\overline{x})), \ \forall \lambda \in \mathbb{R}, \\
\{y \in Y | \theta_{\overline{x}}(y) \leq \lambda\} &= \lambda k^0 - C(f(\overline{x})), \ \forall \lambda \in \mathbb{R}.
\end{aligned}$$

If the functional $\theta_{\overline{x}}$ is proper and convex we get the following result concerning the classical (Fenchel) subdifferential ∂ of $\theta_{\overline{x}}$.

Theorem 3.3. [9, Theorem 2.2] *Let* $\overline{x} \in X$ *and* $C(f(\overline{x})) \subset Y$ *be a closed convex proper set and* $k^0 \in Y \backslash \{0\}$ *such that* $C(f(\overline{x})) + [0, +\infty)k^0 \subset C(f(\overline{x}))$ *holds and for every* $y \in Y$ *there exists* $t \in \mathbb{R}$ *such that* $y + tk^0 \notin C(f(\overline{x})) - f(\overline{x})$*. Consider the function* $\theta_{\overline{x}}$ *given by (3.1) and let* $\hat{y} \in \operatorname{Dom} \theta_{\overline{x}}$*. Then,*

$$\partial \theta_{\overline{x}}(\hat{y}) = \{v^* \in Y^* \mid \forall d \in D : \ v^*(k^0) = 1, v^*(d) + v^*(\hat{y}) - \theta_{\overline{x}}(\hat{y}) \geq 0\}, \tag{3.2}$$

where $D := C(f(\overline{x})) - f(\overline{x})$*.*

The following theorem gives a characterization of approximately minimal solutions of (VVOP) using a scalarization by means of the functional $\theta_{\overline{x}} : Y \to \overline{\mathbb{R}}$, defined by (3.1); for similar results and characterization of approximately nondominated solutions and approximate minimizers under different scalarizations, see [26].

Theorem 3.4. *Suppose that assumptions* (A) *and* (B) *hold. Let* $\overline{x} \in \Omega$ *be an* εk^0*-minimal solution of (VVOP). Consider the function* $\theta_{\overline{x}}$ *given by (3.1). Then,* $\theta_{\overline{x}}(f(\overline{x})) \leq \inf_{x\in\Omega} \theta_{\overline{x}}(f(x)) + \varepsilon$*, for all* $x \in \Omega$.

Proof. Set $\overline{y} = f(\overline{x})$ and suppose that $\theta_{\overline{x}}(\overline{y}) = \overline{t}$. First, we prove that $\overline{t} = 0$. By $\theta_{\overline{x}}(\overline{y}) = \overline{t}$ and Theorem 3.2(b), we get

$$\overline{t}k^0 + \overline{y} - \overline{y} \in C(\overline{y}) \implies \overline{t}k^0 \in C(\overline{y}).$$

By $0 \in \operatorname{bd} C(\overline{y})$, we get $\overline{t} \leq 0$. We claim that $\overline{t} = 0$. Suppose that $\overline{t} < 0$. Then, by $0 \in \operatorname{bd} C(\overline{y})$ and $C(f(\overline{x})) + [0, +\infty)k^0 \subset C(f(\overline{x}))$, we get $-\overline{t}k^0 \in C(\overline{y})$ and $\overline{t}k^0 \in C(\overline{y}) \cap (-C(\overline{y}))$. But, this is a contradiction to pointedness of $C(\overline{y})$ in assumption (B) and therefore $\overline{t} = 0$. Now, by the contrary, suppose that there exists an element $x \in \Omega$ such that $\theta_{\overline{x}}(f(x)) + \varepsilon < \theta_{\overline{x}}(\overline{y}) = 0$. This means that there exists $\gamma > 0$ such that $\theta_{\overline{x}}(f(x)) + \varepsilon + \gamma = 0$ and $\theta_{\overline{x}}(f(x)) = -\varepsilon - \gamma$. By part (b) of Theorem 3.2, we get

$$(-\varepsilon - \gamma)k^0 + \overline{y} - f(x) \in C(\overline{y}) \implies \overline{y} - \varepsilon k^0 - y \in C(\overline{y}) + \gamma k^0 \subset C(\overline{y}) \setminus \{0\}.$$

This means that $(\overline{y} - \varepsilon k^0 - (C(\overline{y}) \setminus \{0\})) \cap \Omega \neq \emptyset$. But, this is a contradiction to approximate minimality of $\overline{x}$ and therefore,

$$\theta_{\overline{x}}(f(\overline{x})) \leq \inf_{x\in\Omega} \theta_{\overline{x}}(f(x)) + \varepsilon,$$

for all $x \in \Omega$.

$\square$

Definition 3.5. Consider problem (VVOP). We say that the function $f : X \to Y$ is bounded from below over Ω with respect to $y \in Y$ and $\Theta \subset Y$ if and only if $f(\Omega) \subseteq y + \Theta$.

Lemma 3.6. *Let assumptions* (A) *and* (B) *hold. Consider the problem (VVOP),* $\overline{x} \in \Omega$ *and the functional* $\theta_{\overline{x}}$ *given by (3.1). Set* $\overline{y} := f(\overline{x})$*. Suppose that* $C(\overline{y}) + C(\overline{y}) \subseteq C(\overline{y})$*. If* $f : X \to Y$ *is bounded from below over* Ω *in the sense of Definition 3.5 with respect to an element* $y \in Y$ *with* $\theta_{\overline{x}}(y) > -\infty$ *and* $\Theta := C(\overline{y})$*, then the functional* $\theta_{\overline{x}} \circ f$ *is bounded from below.*

Proof. Under the assumption $C(\overline{y}) + C(\overline{y}) \subseteq C(\overline{y})$, the functional $\theta_{\overline{x}}$ is $C(\overline{y})$-monotone taking into account Theorem 3.1(e). The $C(\overline{y})$-monotonicity of $\theta_{\overline{x}}$ and $f(\Omega) \subseteq y + C(\overline{y})$ implies

$$\forall x \in \Omega, \qquad \theta_{\overline{x}}(f(x)) \geq \theta_{\overline{x}}(y),$$

that is, $\theta_{\overline{x}} \circ f$ is bounded from below. $\square$

In the next theorem, we show necessary conditions for approximately minimal solutions of vector optimization problems with variable ordering structures.

Theorem 3.7. *Let assumptions* (A) *and* (B) *hold. Consider problem* (VVOP), $\overline{x} \in \varepsilon k^0$-$\mathrm{M}(\Omega, f, C)$ *and the functional* $\theta_{\overline{x}}$ *given by (3.1), and set* $\overline{y} := f(\overline{x})$. *Suppose that* $C(\overline{y})$ *is a convex set,* $C(\overline{y})+C(\overline{y}) \subseteq C(\overline{y})$ *and* $C(\overline{y})+(0,+\infty)k^0 \subset \operatorname{int} C(\overline{y})$.

Assume that $f \in \mathcal{F}(X,Y)$ *is locally Lipschitz and bounded from below in the sense of Definition 3.5 with respect to an element* $y \in Y$ *with* $\theta_{\overline{x}}(y) > -\infty$ *and* $\Theta := C(\overline{y})$. *Consider an abstract subdifferential* ∂, *for which (H1) – (H5) hold. Then, there exists* $x_\varepsilon \in \operatorname{Dom} f \cap \Omega$ *and* $v^* \in \partial(\theta_{\overline{x}}(f(x_\varepsilon))$ *such that*

$$\mathbf{0} \in \partial(v^* \circ f)(x_\varepsilon) + N(x_\varepsilon;\Omega) + \sqrt{\varepsilon} B_{X^*}. \tag{3.3}$$

Proof. Let $\overline{x} \in \varepsilon k^0$-$\mathrm{M}(\Omega, \mathrm{f}, \mathrm{C})$. Applying Theorem 3.4, we get $\theta_{\overline{x}}(f(\overline{x})) \leq \inf\limits_{x\in\Omega} \theta_{\overline{x}}(f(x)) + \varepsilon$. Therefore, $\overline{x}$ is an approximate solution of the scalar problem with the objective functional $\theta_{\overline{x}} \circ f$. From Theorem 3.2 (a) we get that $(\theta_{\overline{x}} \circ f)$ is lower semicontinuous because of $f \in \mathcal{F}(X,Y)$. Furthermore, $(\theta_{\overline{x}} \circ f)$ is bounded from below because of Lemma 3.6. This ensures that the assumptions of the scalar Ekeland's variational principle (Theorem 1.1) hold.

By Theorem 1.1, there exists an element $x_\varepsilon \in \operatorname{Dom} f \cap \Omega$ satisfying Theorem 1.1(a)-(c) and being an exact solution of minimizing a functional $h : X \to \mathbb{R} \cup \{+\infty\}$ over Ω with

$$h(x) := (\theta_{\overline{x}} \circ f)(x) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\|\,, \text{ for all } x \in X.$$

Taking into account (H2) and (H4), we get

$$\mathbf{0} \in \partial h(x_\varepsilon) + N(x_\varepsilon;\Omega).$$

Under the given assumptions, the functional $\theta_{\overline{x}}$ is convex and continuous taking into account Theorem 3.1 (f) and Theorem 3.2 (b). Since f is locally Lipschitz and $\theta_{\overline{x}}$ is convex and continuous (and hence locally Lipschitz; see [25, Proposition 1.6]), it is clear that $\theta_{\overline{x}} \circ f$ is also locally Lipschitz. This, along with the convexity of $\|\cdot\|$ and (H5) imply

$$\partial h(x_\varepsilon) \quad \subseteq \quad \partial\big(\theta_{\overline{x}} \circ f\big)(x_\varepsilon) + \partial(\sqrt{\varepsilon}\,\|\cdot - x_\varepsilon\|)(x_\varepsilon).$$

From (H3), we get

$$\partial(\theta_{\overline{x}} \circ f)(x_\varepsilon) \subseteq \bigcup \big\{\partial(v^* \circ f)(x_\varepsilon) \mid v^* \in \partial\theta_{\overline{x}}(f(x_\varepsilon))\big\}.$$

Because of the convexity of the norm and (H1), we get $\partial\,\|\cdot - x_\varepsilon\|\,(x_\varepsilon) = B_{X^*}$, and by the last three inclusions, we can find $v^* \in \partial\theta_{\overline{x}}(f(x_\varepsilon))$ satisfying

$$\mathbf{0} \in \partial(v^* \circ f)(x_\varepsilon) + N(x_\varepsilon;\Omega) + \sqrt{\varepsilon} B_{X^*},$$

and the proof is complete. □

Remark 3.8. Taking into account Theorem 3.3, for Theorem 3.7, we get the existence of $v^* \in Y^*$ such that (3.2) holds.

The following corollary gives the necessary conditions for approximately nondominated solutions of (VVOP).

Corollary 3.9. *Let assumptions* (A) *and* (B) *hold. Consider problem* (VVOP), $\overline{x} \in \varepsilon k^0$-$\mathrm{N}(\Omega, f, C)$ *and the functional* $\theta_{\overline{x}}$ *given by (3.1), and set* $\overline{y} := f(\overline{x})$. *Suppose that* $C(\overline{y})$ *is a convex set,* $C(\overline{y}) + C(\overline{y}) \subseteq C(\overline{y})$, $C(f(\overline{x})) \subseteq C(f(x))$ *for all* $x \in \Omega$, *and* $C(\overline{y}) + (0, +\infty)k^0 \subset \operatorname{int} C(\overline{y})$.

Suppose that $f \in \mathcal{F}(X, Y)$ *is locally Lipschitz and bounded from below in the sense of Definition 3.5 with respect to an element* $y \in Y$ *with* $\theta_{\overline{x}}(y) > -\infty$ *and* $\Theta := C(\overline{y})$.

Consider an abstract subdifferential ∂ *for which (H1)–(H5) hold. Then, there exists* $x_\varepsilon \in \operatorname{Dom} f \cap \Omega$ *and* $v^* \in \partial(\theta_{\overline{x}}(f(x_\varepsilon))$ *such that*

$$\mathbf{0} \in \partial(v^* \circ f)(x_\varepsilon) + N(x_\varepsilon; \Omega) + \sqrt{\varepsilon} B_{X^*}. \tag{3.4}$$

Proof. Consider $\overline{x} \in \varepsilon k^0$-$\mathrm{N}(\Omega, f, C)$. By $C(f(\overline{x})) \subseteq C(f(x))$ and [27, Theorem 5.3], $\overline{x}$ is an approximately minimal solution of (VVOP); i.e., $\overline{x} \in \varepsilon k^0$-$\mathrm{M}(\Omega, f, C)$, and the proof is complete by applying Theorem 3.7. □

Remark 3.10. If $\overline{x}$ is an approximately minimal solution of (VVOP) and $C(f(x)) \subseteq C(f(\overline{x}))$, for all $x \in \Omega$, then by [27, Theorem 5.3], $\overline{x}$ is also an approximately nondominated solution of (VVOP) and all the results about optimality conditions for approximately nondominated solutions given by Bao et al. [2] can also be used for approximately minimal solutions.

4. Optimality condition for εk^0-minimizers

Here, we give necessary conditions for approximate minimizers of vector optimization problems with variable ordering structures. First, we recall a theorem (see [27, Theorems 5.1, 5.2]), which shows that each (approximate) minimizer of a vector optimization problem with variable ordering structures is both (approximate) minimal and nondominated solution of a vector optimization problem with variable ordering structures (VVOP). It worth to remember that all these solution concepts coincide in the case of vector optimization problems with fixed ordering structures.

Theorem 4.1. [27, Theorems 5.1, 5.2] *Let assumptions* (A) *and* (B) *hold.*

(a) *Every* ϵk^0*-minimizer of* (VVOP) *is also an* ϵk^0*-nondominated solution.*
(b) *Every* ϵk^0*-minimizer of* (VVOP) *is also an* ϵk^0*-minimal solution.*

Remark 4.2. By Theorem 4.1, all necessary conditions presented in the previous section and results in the paper by Bao et al. [2] also hold for approximate minimizers of (VVOP).

Let assumptions (A) and (B) hold and $\overline{x} \in X$. In order to derive necessary conditions for approximate minimizers of vector optimization problems with variable ordering structures, we use the following functional which is a slight modification of the functional defined by Chen and Yang [6], specially concerning the assumptions for the set-valued map C. We define $\xi_{\overline{x}}(z,y) : Y \times Y \to \overline{\mathbb{R}}$ as follows:

$$\xi_{\overline{x}}(z,y) := \inf\{t \in \mathbb{R} \mid z \in tk^0 + f(\overline{x}) - C(y)\}. \tag{4.1}$$

Lemma 4.3. [6, Lemma 2.3] *Let assumptions* (A) *and* (B) *hold and* $\overline{x} \in X$. *For each* $t \in \mathbb{R}$ *and* $y, z \in Y$, *the followings hold:*

$$\xi_{\overline{x}}(z,y) > t \Longleftrightarrow z \notin tk^0 + f(\overline{x}) - C(y),$$

$$\xi_{\overline{x}}(z,y) \geqq t \Longleftrightarrow z \notin tk^0 + f(\overline{x}) - \operatorname{int} C(y),$$

$$\xi_{\overline{x}}(z,y) = t \Longleftrightarrow z \in tk^0 + f(\overline{x}) - \operatorname{bd} C(y),$$

$$\xi_{\overline{x}}(z,y) \leqq t \Longleftrightarrow z \in tk^0 + f(\overline{x}) - C(y),$$

$$\xi_{\overline{x}}(z,y) < t \Longleftrightarrow z \in tk^0 + f(\overline{x}) - \operatorname{int} C(y).$$

Theorem 4.4. *Suppose assumptions* (A) *and* (B) *hold and additionally,* $C(y) + (0,+\infty)k^0 \subset \operatorname{int} C(y)$, *for all* $y \in Y$. *Then, for each arbitrary fixed* $y \in Y$, $\xi_{\overline{x}}(\cdot, y)$ *is continuous.*

Proof. Let $y \in Y$ be an arbitrary but fixed element. We prove that for any $t \in \mathbb{R}$, the set

$$S_t := \{z \in Y \mid \xi_{\overline{x}}(z,y) \leq t\},$$

is closed. For this, we suppose that $z^n \to z^0$ is a sequence and $z^n \in S_t$. We show that the limit point of this sequence belongs to the set S_t and this proves that S_t is a closed set. Since $z^n \in S_t$, $\xi_{\overline{x}}(z^n, y) \leq t$. By Lemma 4.3, we have

$$z^n \in tk^0 + f(\overline{x}) - C(y) \Rightarrow \; tk^0 + f(\overline{x}) - z^n \in C(y).$$

Taking into account that $C(y)$ is a closed set, the limit point of the sequence $tk^0 + f(\overline{x}) - z^n \to tk^0 + f(\overline{x}) - z^0$ also belongs to $C(y)$ and $z^0 \in tk^0 + f(\overline{x}) - C(y)$ and by Lemma 4.3, we get $\xi_{\overline{x}}(z^0, y) \leq t$. This means that S_t is a closed set for any $t \in \mathbb{R}$ and $\xi_{\overline{x}}(\cdot, y)$ is lower semicontinuous for any $y \in Y$.

Now, we show that $\xi_{\overline{x}}(\cdot, y)$ is upper semicontinuous and for any $t \in \mathbb{R}$, the set

$$\overline{S}_t := \{z^1 \in Y \mid \xi_{\overline{x}}(z^1, y) \geq t\},$$

is a closed set. For this, we suppose that $z^n \to z^0$ is a sequence and $z^n \in \overline{S}_t$. Since $z^n \in \overline{S}_t$, $\xi_{\overline{x}}(z^n, y) \geq t$. By Lemma 4.3, we get

$$z^n \notin tk^0 + f(\overline{x}) - \operatorname{int} C(y) \Rightarrow tk^0 + f(\overline{x}) - z^n \notin \operatorname{int} C(y).$$

This implies $tk^0 + f(\overline{x}) - z^n \in (\operatorname{int} C(y))^c$. Since $\operatorname{int} C(y)$ is an open set, its complement $(\operatorname{int} C(y))^c$ is a closed set and includes all the limit points. Therefore, $tk^0 + f(\overline{x}) - z^0 \in (\operatorname{int} C(y))^c$ and this means

$$tk^0 + f(\overline{x}) - z^0 \notin \operatorname{int} C(y) \Rightarrow z^0 \notin tk^0 + f(\overline{x}) - \operatorname{int} C(y).$$

Again, by Lemma 4.3, we have $\xi_{\overline{x}}(z^0, y) \geq t$, and this implies that $\overline{S}_t$ is a closed set and $\xi_{\overline{x}}(\cdot, y)$ is upper semicontinuous. Since $\xi_{\overline{x}}(\cdot, y)$ is also lower semicontinuous, $\xi_{\overline{x}}(\cdot, y)$ is continuous. □

Theorem 4.5. *Suppose that assumptions* (A) *and* (B) *hold,* $\overline{x} \in X$ *and additionally,* $C(y)$ *is a convex cone, for all* $y \in Y$*. Then,* $\xi_{\overline{x}}(\cdot, y)$ *is convex, for all* $y \in Y$*.*

Proof. Let $y \in Y$ be an arbitrary but fixed element. Assume that $\lambda \in [0, 1]$ and $z^1, z^2 \in Y$ such that $\xi_{\overline{x}}(z^1, y) = t_1$ and $\xi_{\overline{x}}(z^2, y) = t_2$. By Lemma 4.3, we have the followings:

$$\xi_{\overline{x}}(z^1, y) = t_1 \Longrightarrow y^1 \in t_1 + f(\overline{x}) - C(y),$$

$$\xi_{\overline{x}}(z^1, y) = t_2 \Longrightarrow y^2 \in t_2 + f(\overline{x}) - C(y).$$

This means that there exists $c, d \in C(y)$ such that $z^1 = t_1k^0 + f(\overline{x}) - c$, $z^2 = t_2k^0 + f(\overline{x}) - d$ and

$$\begin{aligned}
&\lambda z^1 + (1-\lambda)z^2 \\
&\quad = \lambda t_1 k^0 + \lambda f(\overline{x}) - \lambda c + (1-\lambda)t_2 k^0 + (1-\lambda) f(\overline{x}) - (1-\lambda)d \\
&\quad = \big(\lambda t_1 + (1-\lambda)t_2\big)k^0 + f(\overline{x}) - \big(\lambda c + (1-\lambda)d\big).
\end{aligned}$$

By $c, d \in C(y)$ and convexity of $C(y)$, we get $\lambda c + (1-\lambda)d \in C(y)$ and therefore,

$$\lambda z^1 + (1-\lambda)z^2 \in (\lambda t_1 + (1-\lambda)t_2)k^0 + f(\overline{x}) - C(y).$$

Again, by Lemma 4.3, $\xi_{\overline{x}}(\lambda z^1 + (1-\lambda)z^2, y) \leq \lambda t_1 + (1-\lambda)t_2$,

$$\xi_{\overline{x}}(\lambda z^1 + (1-\lambda)z^2, y) \leq \lambda \xi_{\overline{x}}(z^1, y) + (1-\lambda)\xi_{\overline{x}}(z^2, y),$$

and $\xi_{\overline{x}}(\cdot, y)$ is convex, completing the proof.

□

Definition 4.6. Consider $\overline{x} \in X$ and the functional $\xi_{\overline{x}} : Y \times Y \to \overline{\mathbb{R}}$, given by (4.1). $f : X \to Y$ is called bounded from below over Ω with respect to $\xi_{\overline{x}}$ if and only if for all $\omega \in \Omega$, there exists a real number $\alpha > -\infty$ such that

$$\inf_{x \in \Omega} \xi_{\overline{x}}(f(x), f(\omega)) > \alpha.$$

The following theorem gives a characterization of approximate minimizers of (VVOP) using a scalarization by means of the functional $\xi_{\overline{x}} : Y \times Y \to \overline{\mathbb{R}}$, defined by (4.1).

Theorem 4.7. *Suppose that assumptions* (A) *and* (B) *hold and let* $\overline{x} \in \Omega$ *be an* ϵk^0*-minimizer of (VVOP). Then, for all* $\omega \in \Omega$,

$$\xi_{\overline{x}}(f(\overline{x}), f(\omega)) \leq \inf_{x \in \Omega} \xi_{\overline{x}}(f(x), f(\omega)) + \epsilon. \tag{4.2}$$

Proof. Let ω be an arbitrary but fixed element of Ω and set $\overline{y} = f(\overline{x})$. We prove that $\xi_{\overline{x}}(f(\overline{x}), f(\omega)) = 0$. Suppose $\xi_{\overline{x}}(f(\overline{x}), f(\omega)) = \overline{t}$. By Theorem 4.3, we get

$$tk^0 + \overline{y} - \overline{y} \in C(f(\omega)) \implies \overline{t}k^0 \in C(f(\omega)).$$

By $0 \in \operatorname{bd} C(f(\omega))$, we get $\overline{t} \leq 0$. If $\xi_{\overline{x}}(f(\overline{x}), f(\omega)) < 0$, then $\overline{t} < 0$ and $-\overline{t} > 0$. By $0 \in \operatorname{bd} C(f(\omega))$ and $C(f(\omega)) + [0, +\infty)k^0 \subset C(f(\omega))$, we get $-\overline{t}k^0 \in C(f(\omega))$, and $\overline{t}k^0 \in C(f(\omega)) \cap (-C(f(\omega)))$, arriving at a contradiction because $C(y)$ is a pointed set, for all $y \in Y$. This means that $\overline{t} = 0$. Now, we prove that $\xi_{\overline{x}}(f(\overline{x}), f(\omega)) \leq \inf_{x \in \Omega} \xi_{\overline{x}}(f(x), f(\omega)) + \epsilon$. Suppose, by the contrary, that (4.2) does not hold and there exists an element $x \in \Omega$ such that $\xi_{\overline{x}}(f(x), f(\omega)) + \epsilon < \xi_{\overline{x}}(f(\overline{x}), f(\omega)) = 0$. This means that there exists $\beta > 0$ such that $\xi_{\overline{x}}(f(x), f(\omega)) = -\varepsilon - \beta$. By Theorem 4.3, we get

$$\begin{aligned} & (-\varepsilon - \beta)k^0 + \overline{y} - f(x) \in C(f(\omega)) \\ \implies \quad & \overline{y} - \varepsilon k^0 - f(x) \in C(f(\omega)) + \beta k^0 \subset C(f(\omega)) \setminus \{0\}. \end{aligned}$$

This means that there exists $\omega \in \Omega$ such that

$$(\overline{y} - \varepsilon k^0 - (C(f(\omega)) \setminus \{0\}) \cap \Omega \neq \emptyset$$

and $\overline{y} \notin \varepsilon k^0$-MZ$(\Omega, f, C)$. But this is a contradiction, because $\overline{x}$ is an ϵk^0-minimizer of (VVOP). Therefore,

$$\xi_{\overline{x}}(f(\overline{x}), f(\omega)) \leq \inf_{x \in \Omega} \xi_{\overline{x}}(f(x), f(\omega)) + \epsilon,$$

for all $x, \omega \in \Omega$. □

Now, we use Theorem 4.7 and Ekeland's variational principle (Theorem 1.1) in order to derive necessary conditions for minimizers of vector optimization problems with variable ordering structures.

Theorem 4.8. *Let assumptions* (A) *and* (B) *hold. Consider problem* ($VVOP$), $\overline{x} \in \varepsilon k^0$-MZ$(\Omega, \mathrm{f}, \mathrm{C})$, *the functional* $\xi_{\overline{x}}$ *given by* (4.1) *and set* $\overline{y} = f(\overline{x})$.

Suppose that $f \in \mathcal{F}(X, Y)$ *is locally Lipschitz and bounded from below in the sense of Definition 4.6 over* Ω *with respect to* $\xi_{\overline{x}}$. *Assume that* $C(y)$ *is a convex set and* $C(y) + (0, +\infty)k^0 \subset \operatorname{int} C(\overline{y})$, *for all* $y \in Y$. *Consider the abstract subdifferential* ∂ *for which (H1)–(H5) hold. Then, for all* $\omega \in \Omega$, *there exists* $x_\omega \in \operatorname{Dom} f \cap \Omega$ *and* $v^*_\omega \in \partial(\xi_{\overline{x}}(f(x_\omega), f(\omega))$ *such that*

$$\mathbf{0} \in \partial(v^*_\omega \circ f)(x_\omega) + N(x_\omega; \Omega) + \sqrt{\varepsilon} B_{X^*}. \tag{4.3}$$

Proof. Let $\overline{x} \in \varepsilon k^0$-MZ$(\Omega, \mathrm{f}, \mathrm{C})$. Applying Theorem 4.7, we get, for all $\omega \in \Omega$,

$$\xi_{\overline{x}}(f(\overline{x}), f(\omega)) \leq \inf_{x \in \Omega} \xi_{\overline{x}}(f(x), f(\omega)) + \epsilon.$$

Therefore, $\overline{x}$ is an approximate minimizer of the scalar problem with the objective functionals $\xi_{\overline{x}}(\cdot, f(\omega))$, for all $\omega \in \Omega$. Taking into account Theorem 4.4 and $f \in \mathcal{F}(X, Y)$, we get $\xi_{\overline{x}}(\cdot, f(\omega))$ to be lower semicontinuous, for all $\omega \in \Omega$.

Furthermore, $\xi_{\overline{x}}(\cdot, f(\omega))$ is bounded from below, for all $\omega \in \Omega$. By the scalar Ekeland's variational principle (Theorem 1.1), for all $\omega \in \Omega$, there exists an element $x_\omega \in \operatorname{Dom} f \cap \Omega$ satisfying parts (a), (b) and (c) of Theorem 1.1 and being an exact solution of an optimization problem with the objective function $\overline{h} : X \to \mathbb{R} \cup \{+\infty\}$ over Ω with

$$\overline{h}(x) := \xi_{\overline{x}}(f(x), f(\omega)) + \sqrt{\varepsilon}\,\|x - x_\omega\|\,, \quad \text{for all } x \in X.$$

By (H2) and (H4), we get

$$\mathbf{0} \in \partial\overline{h}(x_\omega) + N(x_\omega; \Omega).$$

Under the given assumptions, the functional $\xi_{\overline{x}}(\cdot, f(\omega))$ is convex (Theorem 4.5) and continuous (Theorem 4.4), taking into account Theorem 4.4 and Theorem 4.5. Since f is locally Lipschitz and $\xi_{\overline{x}}(\cdot, f(\omega))$ is convex and continuous and hence locally Lipschitz, the composition $\xi_{\overline{x}}(f(\cdot), f(\omega))$ is also locally Lipschitz. This together with the convexity of the norm $\|\cdot\|$ and (H5) imply

$$\partial\overline{h}(x_\omega) \quad \subseteq \quad \partial\big(\xi_{\overline{x}}(f(\cdot), f(\omega))\big)(x_\omega) + \partial(\sqrt{\varepsilon}\,\|\cdot - x_\omega\|)(x_\omega).$$

By (H3), we get

$$\partial(\xi_{\overline{x}}(f(\cdot), f(\omega)))(x_\omega) \subseteq \bigcup \big\{\partial(v_\omega^* \circ f)(x_\omega) \mid v_\omega^* \in \partial\xi_{\overline{x}}(f(x_\omega), f(\omega))\big\}.$$

Because of the convexity of the norm and (H1), we get $\partial\,\|\cdot - x_\omega\|\,(x_\omega) = B_{X^*}$, and by the last three inclusions, we can find $v_\omega^* \in \partial\xi_{\overline{x}}(f(x_\omega), f(\omega))$ satisfying

$$\mathbf{0} \in \partial(v_\omega^* \circ f)(x_\omega) + N(x_\omega; \Omega) + \sqrt{\varepsilon} B_{X^*},$$

and the proof is complete. □

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