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FIRST STEP IMMERSION IN INTERVAL LINEAR PROGRAMMING WITH LINEAR DEPENDENCIES

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Abstract. We consider a linear programming problem in a general form and suppose that all coefficients may vary in some prescribed intervals. Contrary to classical models, where parameters can attain any value from the interval domains independently, we study problems with linear dependencies between the parameters. We present a class of problems that are easily solved by reduction to the classical case. In contrast, we also show a class of problems with very simple dependencies, which appear to be hard to deal with. We also point out some interesting open problems.
Keywords: Linear programming, interval analysis, linear dependencies.
MSC(2010): Primary: 90C05; Secondary: 90C31, 65G40.

1. Introduction

Here, we are concerned with linear programming (LP) problems with interval data. Interval data naturally appear in many situations because of errors in measurement, missing or imperfect information, or other types of uncertainty. By interval parameters we can also check stability, sensitivity and robustness of our model or solution.

The core principle of the interval (or possibilistic) approach is to take all possible realizations of interval data into account. In the context of Linear Programming (LP), we get a family of linear programs such that their data are guaranteed to lie in given intervals. The most important question is to compute rigorous bounds on the optimal value and optimal solutions.

Before we state the problem more precisely, we introduce some notations first.

1.1. Notation.

An interval matrix $\boldsymbol{A}$ is defined as

$$\boldsymbol{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n};\ \underline{A} \le A \le \overline{A}\},$$

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where $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$, $\underline{A} \le \overline{A}$, are given, and the inequality is to hold entrywise. The midpoint and the radius of $\boldsymbol{A}$ are defined respectively as

$$A^c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A^\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

The set of all m-by-n interval matrices is denoted by $\mathbb{IR}^{m\times n}$. Interval vectors are considered as one-column interval matrices. For interval arithmetic and more on interval computating, we refer to [1,19,21].

Let $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$, $\boldsymbol{b} \in \mathbb{IR}^m$ and consider the feasible set

$$\mathcal{M}(A, b) \tag{1.1}$$

of a general linear system with the constraint matrix A and the right-hand side b, which can (depending on the context) consist of equations, inequalities, or both, and each variable may be nonnegative, nonpositive or free of sign. Thus, $\mathcal{M}(A, b)$ can take, for instance, the form of

$$\begin{aligned} \mathcal{M}(A, b) &= \{x \in \mathbb{R}^n : Ax = b,\ x \ge 0\}, \text{ or} \\ \mathcal{M}(A, b) &= \{x \in \mathbb{R}^n : Ax \le b\}, \text{ or} \\ \mathcal{M}(A, b) &= \{x \in \mathbb{R}^n : Ax \le b,\ x \le 0\}, \text{ or} \\ \mathcal{M}(A = (A_1; A_2), b = (b_1; b_2)) &= \{x \in \mathbb{R}^n : A_1x \le b_1, A_2x = b_2\}. \end{aligned}$$

A family of systems (1.1) with $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$ is called an *interval linear system* and its solution set is defined as

$$\mathcal{M} := \bigcup_{A\in\boldsymbol{A},\, b\in\boldsymbol{b}} \mathcal{M}(A, b).$$

We say that (1.1) is *strongly feasible* if it is feasible (i.e., $\mathcal{M}(A, b) \ne \emptyset$) for each $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$.

1.2. **Dependencies and relaxation.** In classical interval computation, we assume that each entry in the linear system corresponds to a unique interval parameter. For example, the interval linear equations are usually defined as

$$Ax = b, \quad A \in \boldsymbol{A},\ b \in \boldsymbol{b},$$

and the solution set is:

$$\{x \in \mathbb{R}^n : (\exists A \in \boldsymbol{A})(\exists b \in \boldsymbol{b})\ Ax = b\}. \tag{1.2}$$

However, in many practical situations, parameters are somehow related or are subject to additional constraints. For example, we might be interested in the "dependent" system,

$$\{x \in \mathbb{R}^n : (\exists A \in \boldsymbol{A})(\exists b \in \boldsymbol{b})\ \varphi(A, b) \text{ and } Ax = b\}, \tag{1.3}$$

where $\varphi(A, b)$ is a given predicate. For example, we can take $\varphi(A, b) \equiv [A_{11} = A_{12}]$ or

$$\varphi(A = (A_1; A_2), b = (b_1; b_2)) \equiv [A_2 = -A_1 \text{ and } b_2 = -b_1]. \tag{1.4}$$

The predicate φ imposes additional conditions on parameters and causes their *dependency*. Replacement of (1.3) by (1.2) is called *relaxation*.

Remark 1.1 (A tempting problem)**.** Currently, available theory of dependent systems is able to work with only simple predicates φ and there is a big research potential.

Example 1.2. Consider the system

$$ax + 0y = 2,\ 0x + ay = 2, \quad a \in \boldsymbol{a} = [1, 2].$$

This system contains a dependency of the form of a "correlation" between parameters, here caused by double occurrence of a. The solution set (1.3) is the line segment from $(1, 1)^T$ to $(2, 2)^T$. Relaxation (1.2) leads to the interval system

$$ax + 0y = 2,\ 0x + a'y = 2, \quad a, a' \in \boldsymbol{a} = [1, 2],$$

the solution of which expands the line segment to the square $1 \leq x \leq 2$, $1 \leq y \leq 2$. This property is fundamental: relaxation implies that the solution set never loses any solution, only becomes larger. Therefore, for verified bounds on the solutions, relaxation is still applicable, but it may be more conservative.

1.3. **Interval linear programming.** Let $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$, $\boldsymbol{b} \in \mathbb{IR}^m$ and $\boldsymbol{c} \in \mathbb{IR}^n$ be given. By an interval LP problem [2–4, 7, 9, 15, 18, 23], we understand a family of LP problems,

$$\min c^T x \ \text{ subject to } \ x \in \mathcal{M}(A, b), \tag{1.5}$$

where $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $c \in \boldsymbol{c}$, and $\mathcal{M}(A, b)$ is the feasible set described by linear constraints such as equations, inequalities or both. In traditional LP, usually one formulation is taken as a basic one and alternative formulations can be transformed to the basic one. For example, taking the form

$$\min c^T x \ \text{ subject to } \ Ax \leq b$$

as the basic one, then the form

$$\min c^T x \ \text{ subject to } \ Ax = b, x \geq 0 \tag{1.6}$$

can be rewritten into the basic form

$$\min c^T x \ \text{ subject to } \ Ax \leq b, -Ax \leq -b, -x \leq 0. \tag{1.7}$$

In interval LP, however, transformations among different formulations are not possible, in general, since they may cause dependencies; indeed, the independent form (1.6) has been transformed into the dependent form (1.7) with double occurrences of A, b; cf. (1.4).

For any type of LP problem (1.5), its dual problem has the form

$$\max b^T u \ \text{ subject to } \ u \in \mathcal{N}(A^T, c), \tag{1.8}$$

where $\mathcal{N}(A^T, c)$ is the dual feasible set described by linear constraints, as well.

Example 1.3. The primal problem (1.5) can be of the form

$$\min c^T x \ \text{ subject to } \ Ax \leq b,\ x \geq 0.$$

The corresponding dual is:

$$\max b^T u \ \text{ subject to } \ \mathcal{N}(A^T, c) = \{u : A^T u \leq c,\ u \leq 0\}.$$

As another example, the primal problem

$$\min c^T x \ \text{ subject to } \ Ax \leq b$$

has the dual counterpart

$$\max b^T u \ \text{ subject to } \ \mathcal{N}(A^T, c) = \{u : A^T u = c,\ u \leq 0\}.$$

A recent survey on interval linear programming can be found in [9], where three basic formulations were considered, fundamental problems discussed, useful techniques presented, and some open problems stated.

1.4. **Two major problems: Problem 1.** Denote the optimal value function by

$$f(A, b, c) := \min c^T x \ \text{ subject to } \ x \in \mathcal{M}(A, b).$$

Notice that infinite values are possible: $+\infty$ corresponds to infeasibility and $-\infty$ corresponds to unboundedness. There are two major problems in interval LP. The first one is to compute the optimal value range $\boldsymbol{f} = [\underline{f}, \overline{f}]$, where

$$\underline{f} := \inf f(A, b, c) \ \text{ subject to } \ A \in \boldsymbol{A},\ b \in \boldsymbol{b},\ c \in \boldsymbol{c},$$
$$\overline{f} := \sup f(A, b, c) \ \text{ subject to } \ A \in \boldsymbol{A},\ b \in \boldsymbol{b},\ c \in \boldsymbol{c}.$$

In some special cases, the problem is easily solvable, but, in general, it is NP-hard [4,5,8]. A general framework for determining the optimal value range was presented in [7]. We state this in Algorithm 1, where

$$\mathcal{N} := \bigcup_{A \in \boldsymbol{A},\, c \in \boldsymbol{c}} \mathcal{N}(A^T, c)$$

is the dual solution set. Solvability of a general form of interval linear constraints was characterized in [11]. Particular cases were investigated by many researchers; see, e.g., [3, 4, 20], and an approximation of the computationally hard instances in [13].

Algorithm 1.

1: Compute

$$\underline{f} := \inf (c^c)^T x - (c^\Delta)^T |x| \ \text{ subject to } \ x \in \mathcal{M}. \tag{1.9}$$

2: If $\underline{f} = \infty$ then set $\overline{f} := \infty$ and stop.

3: Compute

$$\overline{\varphi} := \sup (b^c)^T u + (b^\Delta)^T |u| \ \text{ subject to } \ u \in \mathcal{N}. \tag{1.10}$$

4: If $\overline{\varphi} = \infty$ then set $\overline{f} := \infty$ and stop.
5: If the primal problem is strongly feasible then set $\overline{f} := \overline{\varphi}$ else set $\overline{f} := \infty$.

Algorithm 1 has an exponential-time, in general, due to the presence of $|x|$ and $|u|$. To compute (1.9), in general, one has to solve 2^n linear programs to get

$$\underline{f} = \inf_{s \in \{-1,1\}^n} \inf (c^c)^T x - (c^\Delta)^T D_s x \ \text{ subject to } \ x \in \mathcal{M},$$

where D_s is the diagonal matrix with entries $s_1, \ldots, s_n$. A similar idea holds for (1.10). However, in Section 2, we will take advantage of the following observation.

Lemma 1.4. *Let $\mathbb{R}^k_s$ denote the orthant $\{x \in \mathbb{R}^k : D_s x \geq 0\}$. If two sign vectors $s \in \{-1,1\}^n$ and $t \in \{-1,1\}^m$ exist (and are known in advance) such that*

(a) $\mathcal{M} \subseteq \mathbb{R}^n_s$,
(b) $\mathcal{N} \subseteq \mathbb{R}^m_t$,

and if, in addition,

(c) *strong feasibility of (1.5) can be tested in a polynomial time,*

then Algorithm 1 runs in a polynomial time.

Proof. Nonlinear conditions (1.9) and (1.10) reduce to the linear programs

$$\underline{f} = \inf (c^c)^T x - (c^\Delta)^T D_s x \ \text{ subject to } \ x \in \mathcal{M}, \tag{1.11}$$

$$\overline{\varphi} = \sup (b^c)^T u + (b^\Delta)^T D_t u \ \text{ subject to } \ u \in \mathcal{N}. \tag{1.12}$$

Due to [11], the sets $\mathcal{M}$ and $\mathcal{N}$ are convex polyhedral sets. □

Obviously, the above result can be directly extended to the case when $\mathcal{M}$ and $\mathcal{N}$ intersect a polynomial number of orthants, and we are able to enumerate them in a polynomial time.

1.5. **Two major problems: Problem 2.** The second problem is even much more challenging. Therein, we have to determine the set of all optimal solutions,

$$\mathcal{S} := \bigcup_{A \in \boldsymbol{A},\, b \in \boldsymbol{b},\, c \in \boldsymbol{c}} \mathcal{S}(A, b, c),$$

where $\mathcal{S}(A, b, c)$ is the optimal solution set to (1.5). The very few of results in this direction can be seen in [2, 10], for example.

A special property of basis stability was investigated in [12, 15, 22]. If basis stability is fulfilled, then both main problems are solvable more easily.

Recently, the concept of a solution was extended to a quantified one, including both existential and universal quantifiers, and being more flexible in reflecting a particular robustness standpoint of a decision maker [16–18].

1.6. **Goal.** In the standard interval LP model, one assumes that interval parameters can attain any value from their interval domains independently of other parameters. This assumption is very restrictive; there are many examples and applications when dependencies must be taken into account. As an example we have seen that a transformation of one standard form of a linear program to another standard form may cause dependencies.

Even though relaxing such dependencies (correlations) still leads to rigorous bounds on optimal values and optimal solutions, the results may be highly overestimated. Thus, here we aim to handle at least a particular form of dependency among the interval parameters. Besides some notes in [7], we are aware of no other results. We present one class which is easily solvable (Section 2); however, we also show that some very simple dependencies can also cause the problem to become very difficult (Section 3).

2. One polynomially solvable sub-class

Consider an interval LP problem as

$$\begin{aligned} &\min c^T x + d^T y + e^T z \\ &\quad \text{subject to } \ Ax + Bz \le a,\ Ay + Cz \le b,\ x, y, z \ge 0, \end{aligned} \tag{2.1}$$

where A, B, C, a, b, c, d, e vary in given interval matrices and vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$, respectively. Due to the dependencies caused by double appearance of the matrix A in the constraints, this is *not* a standard form of an interval LP problem. Nevertheless, we will show that the dependency can be relaxed, and both appearances of A can be handled as independent matrices from $\boldsymbol{A}$ with no effect on the optimal value range $\boldsymbol{f}$.

2.1. **Optimal value range.** To apply Algorithm 1 and Lemma 1.4, we need to

(a) describe the primal feasible set $\mathcal{M}$ (observe that its presence in the nonnegative orthant of $\mathbb{R}^n$ is obvious from (2.1)),
(b) describe the dual feasible set $\mathcal{N}$ and check that it lies in a single orthant of $\mathbb{R}^m$, and
(c) describe a method for testing primal strong feasibility.

By [6, Corollary 3.2], the feasible solutions of (2.1) are described by the linear system

$$\underline{A}x + \underline{B}z \le \overline{a},\ \underline{A}y + \underline{C}z \le \overline{b},\ x, y, z \ge 0. \tag{2.2}$$

The dual to (2.1) is:

$$\begin{aligned} &\max a^T u + b^T v \\ &\quad \text{subject to } \ A^T u \le c,\ A^T v \le d,\ B^T u + C^T v \le e,\ u, v \le 0. \end{aligned}$$

By [6, Corollary 3.2] and [11], the feasible solutions of the interval family of the dual problems are described linearly as

$$\overline{A}^T u \leq \overline{c},\ \overline{A}^T v \leq \overline{d},\ \overline{B}^T u + \overline{C}^T v \leq \overline{e},\ u, v \leq 0. \tag{2.3}$$

Eventually, by [14, Theorem 7], strong feasible feasibility of (2.1) is equivalent to feasibility of the real system,

$$\overline{A}x + \overline{B}z \leq \underline{a},\ \overline{A}y + \overline{C}z \leq \underline{b},\ x, y, z \geq 0. \tag{2.4}$$

Remark 2.1 (A tempting problem)**.** The last result can be strengthened to to following form:

$$\text{(2.1) is strongly feasible if and only if } \bigcap_{A \in \boldsymbol{A}, b \in \boldsymbol{b}} \mathcal{M}(A, b) \neq \emptyset,$$

which can be read as follows: *our interval LP problem (either primal or dual) is strongly feasible if and only if all LP problems in the system share a common feasible point.* This (nontrivial) property is known for systems without dependencies; see [4]. Here, we have this property for a system with a particular form of dependency. It is tempting to restate this result for more general dependency structures in linear systems.

Surprisingly, we can even derive the following stronger result avoiding the checking for strong feasibility of $\mathcal{M}$.

Theorem 2.2. *We have*

$$\begin{aligned} \underline{f} := \min\ & \underline{c}^T x + \underline{d}^T y + \underline{e}^T z \\ \text{subject to }\ & \underline{A}x + \underline{B}z \leq \overline{a},\ \underline{A}y + \underline{C}z \leq \overline{b},\ x, y, z \geq 0, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \overline{f} := \min\ & \overline{c}^T x + \overline{d}^T y + \overline{e}^T z \\ \text{subject to }\ & \overline{A}x + \overline{B}z \leq \underline{a},\ \overline{A}y + \overline{C}z \leq \underline{b},\ x, y, z \geq 0. \end{aligned} \tag{2.6}$$

Proof. The lower bound $\underline{f}$ simply follows from (1.9) and (1.11). The problem (2.6) is dual to (1.12), and so they have the same optimal value as long as (2.4), or equivalently (2.6), is feasible; if not feasible, then $\overline{f} = \infty$, in accordance with Algorithm 1. □

In view of Theorem 2.2, the optimal value range is the same if we relax or if we do not relax the dependencies. Therefore, we have the following result.

Theorem 2.3. *The optimal value range of the interval LP problem* (2.1) *is the same as the one for the interval LP problem with no dependencies,*

$$\begin{aligned} \min\ & c^T x + d^T y + e^T z \\ \text{subject to }\ & Ax + Bz \leq a,\ A'y + Cz \leq b,\ x, y, z \geq 0, \end{aligned} \tag{2.7}$$

where $A, A' \in \boldsymbol{A}$*,* $B \in \boldsymbol{B}$*,* $C \in \boldsymbol{C}$*,* $a \in \boldsymbol{a}$*,* $b \in \boldsymbol{b}$*,* $c \in \boldsymbol{c}$*,* $d \in \boldsymbol{d}$*, and* $e \in \boldsymbol{e}$*.*

Proof. By Vajda's Theorem [9, 24], the optimal value range for (2.7) can be computed exactly by the same LP problems as given by (2.5) and (2.6). □

Notice that (2.7) is a standard interval linear programming problem as there are no dependencies. Contrary to the form (2.1), both matrices A and A' come from $\boldsymbol{A}$ independently of each other. Thus, we come up with a class of problems that can be relaxed (that is, the dependencies can be "forgotten") with no effect on the optimal value range.

2.2. **Optimal solution set.** Unfortunately, the property stated in Theorem 2.3 is no longer valid for the optimal solution set $\mathcal{S}$. As the following example illustrates, relaxation of dependencies leads to a blow-up of the solution set, even with respect to its dimension.

Example 2.4. Consider the problem

$$\min -x-y \ \text{ subject to } \ ax \leq 2,\ ay \leq 2,\ x, y \geq 0,$$

where $a \in \boldsymbol{a} = [1, 2]$. The optimal solution to a concrete setting of $a \in \boldsymbol{a}$ is the point $(2/a, 2/a)^T$, and so the overall optimal solution set $\mathcal{S}$ is the line segment joining the points $(1, 1)^T$ and $(2, 2)^T$.

In contrast, relaxing the dependencies yields the problem

$$\min -x-y \ \text{ subject to } \ ax \leq 2,\ a'y \leq 2,\ x, y \geq 0,$$

where $a, a' \in \boldsymbol{a} = [1, 2]$. Now, the optimal solution is $(2/a, 2/a')^T$, whence the optimal solution set $\mathcal{S}$ represents the square with vertices in $(1, 1)^T$, $(1, 2)^T$, $(2, 2)^T$ and $(2, 1)^T$; cf. Example 1.2.

This example shows a high overestimation caused by the relaxation, but on the other hand, the *spreads of the solution sets in each coordinate are the same.*

Remark 2.5. (A tempting problem). It is an open question whether the observation typeset in italic is exceptional or whether it is a rule.

3. One hard-to-solve sub-class with simple dependencies

Consider an interval LP problem

$$\begin{aligned} &\min c^T x + d^T y + e^T z \\ &\quad \text{subject to } \ Ax + Bz \leq a,\ Ay + Cz \leq b, \end{aligned} \tag{3.1}$$

where A, B, C, a, b, c, d, e comes from interval matrices/vectors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$, respectively. Contrary to the easy case, (2.1), nonnegativity of variables is not required.

Let us focus on the optimal value range problem. By [6, Theorem 3.1], the feasible solutions of (3.1) are described by the nonlinear nonsmooth system (involving not only inequalities, but also logical implications):

$$\begin{aligned}
&A^{\Delta}|x| + B^{\Delta}|z| + \overline{a} \geq A^c x + B^c z,\\
&A^{\Delta}|y| + C^{\Delta}|z| + \overline{b} \geq A^c y + C^c z,\\
&\forall k = 1,\dots,n:\ \text{if } x_k y_k < 0,\ \text{then } |y_k| \cdot (\overline{a} - A^c x - B^c z + B^{\Delta}|z|)\\
&\qquad + |x_k| \cdot (\overline{b} - A^c y - C^c z + C^{\Delta}|z|) + A^{\Delta}|y_k x - x_k y| \geq 0.
\end{aligned}$$

Remark 3.1. Observe that if $x, y, z \geq 0$, then the last system reduces to (2.2). This helped us in Section 2. But, now we treat x, y, z as unconstrained variables.

The dual problem to (3.1) is

$$\begin{aligned}
&\max a^T u + b^T v\\
&\ \text{subject to}\ \ A^T u = c,\ A^T v = d,\ B^T u + C^T v = e,\ u, v \leq 0.
\end{aligned}$$

By [6] and [11], the dual feasible set is

$$\begin{aligned}
-(A^{\Delta})^T u + c^{\Delta} &\geq |c^c - (A^c)^T u|,\\
-(A^{\Delta})^T v + d^{\Delta} &\geq |d^c - (A^c)^T v|,\\
-(B^{\Delta})^T u - (C^{\Delta})^T v + e^{\Delta} &\geq |e^c - (B^c)^T u - (C^c)^T v|,\\
-c^{\Delta} v^T - d^{\Delta} u^T + &\\
(A^{\Delta})^T |u v^T - v u^T| &\geq |(c^c - (A^c)^T u) v^T - (d^c - (A^c)^T v) u^T|.
\end{aligned}$$

Concerning strong feasibility of (3.1), a necessary and sufficient condition, by [14, Theorem 8], is to check infeasibility of the system

$$\begin{aligned}
\underline{A}^T p \leq 0 &\leq \overline{A}^T p,\\
\underline{A}^T q \leq 0 &\leq \overline{A}^T q,\\
\underline{B}^T p + \underline{C}^T q \leq 0 &\leq \overline{B}^T p + \overline{C}^T q,\\
|(A^c)^T (p q^T - q p^T)| &\leq (A^{\Delta})^T |p q^T - q p^T|,\\
\underline{b}^T p + \underline{c}^T q &\leq -1, \quad p, q \geq 0.
\end{aligned}$$

We see that all three conditions that are necessary to employ Algorithm 1, summarized in (a)–(c) of Lemma 1.4, to describe the primal and dual feasible sets, and to check strong feasibility of the primal one, seem to be very difficult to verify. Those inequality systems are nonlinear and even nonsmooth, and so it is a challenging problem to check their solvability. Yet, there might be simpler characterizations by other means, while the above results indicate otherwise.

4. Conclusion

Our aim was to show that even very simple dependencies in constraint coefficients may results in a computationally tough problem. This behavior, however, should not discourage us from further research. Seeking for polynomially solvable classes of problems, such as the one given in Section 2, or developing computationally cheap approximation methods, are possible directions for future research.

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