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AN IMPROVED INFEASIBLE INTERIOR-POINT METHOD FOR SYMMETRIC CONE LINEAR COMPLEMENTARITY PROBLEM

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Abstract. We present an improved version of a full Nesterov-Todd step infeasible interior-point method for linear complementarity problem over symmetric cone (Bull. Iranian Math. Soc., 40, (2014), no. 3, 541–564). In the earlier version, each iteration consisted of one so-called feasibility step and a few -at most three - centering steps. Here, each iteration consists of only a feasibility step. Thus, the new algorithm demands less work in each iteration and admits a simple analysis of complexity bound. The complexity result coincides with the best-known iteration bound for infeasible interior-point methods.
Keywords: Linear complementarity problem, infeasible interior-point method, symmetric cones, polynomial complexity.
MSC(2010): Primary: 90C33; Secondary: 90C51.

1. Introduction

Consider a Euclidean Jordan algebra $(\mathcal{J}, \circ, \langle \cdot, \cdot \rangle)$, where $\circ$ denotes the Jordan product and $\mathcal{J}$ is a finite-dimensional vector space over the real field R equipped with the inner product $\langle \cdot, \cdot \rangle$, and let $\mathcal{K}$ be a symmetric cone in $\mathcal{J}$. The monotone linear complementarity problem over symmetric cone (SCLCP) requires the computation of a vector pair $(x, s) \in \mathcal{K} \times \mathcal{K}$ satisfying

$$s = Mx + q, \; x \circ s = 0, \tag{1.1}$$

where $q \in R^n$ and $M \in R^{n \times n}$ such that $v = Mu$ implies that $\langle u, v \rangle \geq 0$. Although SCLCP is not an optimization problem, it is closely related to one, because optimality conditions of several important optimization problems can be written as an SCLCP; e.g., linear optimization (LO) problem over symmetric cone (SCO). Faybusovich was the first to analyze a short-step path-following

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interior-point method (IPM) for SCLCP [2, 3]. In addition to Faybusovich's results, Rangarajan [10] proposed the first infeasible IPM (IIPM) for SCLCP. The primal-dual full-Newton step feasible IPM for LO was first analyzed by Roos et al. [12] and was later extended to infeasible version by Roos [11]. Both versions of the method were extended by Kheirfam and Mahdavi-Amiri [8] to SCLCP by using the Nesterov-Todd (NT) direction as a search direction. The obtained iteration bounds coincide with the ones derived for LO, currently being best known iteration bounds for SCLCP. Subsequently, both versions were extended by Kheirfam and Mahdavi-Amiri [6, 7] to SCLCP based on modified NT-directions using Euclidean Jordan algebra. Recently, Roos [13] proposed a new method for LO by improving the full-Newton step IIPMs so that the centering steps not be needed, whereas the above-mentioned methods require a few (at most three) centering steps in each (main) iteration. Motivated by Roos' recent work, we present a new full-NT step IIPM for SCLCP which uses only a full step in each (main) iteration. The new algorithm starts from an infeasible point, located in a small neighborhood of the central path of a perturbed SCLCP. Then, after a full-NT step the new iterate is well-centered for the new perturbed SCLCP. This kind of strategy reduces the number of iterations and the resulting complexity coincides with the best known bound, while tendering a simple analysis.

In what follows, we briefly recall some concepts, properties, and results from Euclidean Jordan algebras as needed. A comprehensive treatment of Euclidean Jordan algebra can be found in [1].

A Euclidean Jordan algebra is a triple $(\mathcal{J}, \circ, \langle\cdot,\cdot\rangle)$, where $(\mathcal{J}, \langle\cdot,\cdot\rangle)$ is an n-dimensional inner product space over R and $(x, y) \to x \circ y$ on $\mathcal{J}$ is a bilinear mapping satisfying the following conditions for all $x, y, z \in \mathcal{J}$:

(i) $x \circ y = y \circ x$,
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, where $x^2 = x \circ x$,
(iii) $\langle x \circ y, z\rangle = \langle x, y \circ z\rangle$,

where the inner product $\langle\cdot,\cdot\rangle$ is defined by $\langle x, y\rangle := \mathrm{tr}(x \circ y)$ for any $x, y \in \mathcal{J}$.

Note that, by Theorem III.2.1 of [1], the symmetric cone $\mathcal{K}$ coincides with the set of squares $\{x^2 : x \in \mathcal{J}\}$ of some Euclidean Jordan algebra $\mathcal{J}$. We assume that there exists an element e such that $x \circ e = e \circ x = x$, for all $x \in \mathcal{J}$. The rank of $(\mathcal{J}, \circ, \langle\cdot,\cdot\rangle)$ is defined to be

$$r := \max\big\{\deg(x) : x \in \mathcal{J}\big\},$$

where $\deg(x)$ is the degree of $x \in \mathcal{J}$, given by

$$\deg(x) := \min\big\{k : \{e, x, \ldots, x^k\} \text{ are linearly dependent}\big\}.$$

For any $x \in \mathcal{J}$, the Lyapunov transformation $L(x) : \mathcal{J} \to \mathcal{J}$ is defined as $L(x)y = x \circ y$, for all $y \in \mathcal{J}$. It follows from (i) and (iii) above that the Lyapunov transformation is symmetric; i.e., $\langle L(x)y, z\rangle = \langle y, L(x)z\rangle$ holds for all $y, z \in \mathcal{J}$. Specially, $L(x)e = x$ and $L(x)x = x^2$, for $x \in \mathcal{J}$. Using the Lyapunov transformation, the quadratic representation of $x \in \mathcal{J}$ is defined as

$$P(x) := 2L(x)^2 - L(x^2),$$

where $L(x)^2 = L(x)L(x)$. For any $x \in \mathcal{K}$, $L(x)$ is positive semidefinite; i.e., $\langle L(x)x, x\rangle \geq 0$, and $\langle x, y\rangle = 0$ if and only if $x \circ y = 0$, for any $x, y \in \mathcal{K}$ (Lemma 2.2 of [2]). An element $c \in \mathcal{J}$ is idempotent if $c \circ c = c \neq 0$, which is also primitive if it cannot be written as a sum of two idempotents. We say that $\{c_1, c_2, \ldots, c_k\}$ is a Jordan frame, if each c_i is a primitive idempotent, $c_i \circ c_j = 0 (i \neq j)$ and $\sum_{i=1}^{k} c_i = e$. Let $(\mathcal{J}, \circ, \langle\cdot,\cdot\rangle)$ be a Euclidean Jordan algebra with $rank(\mathcal{J}) = r$. Then, for any $x \in \mathcal{J}$, there exist a Jordan frame $\{c_1, c_2, \ldots, c_r\}$ and real numbers $\lambda_1(x), \lambda_2(x), \ldots, \lambda_r(x)$ such that $x = \sum_{i=1}^{r} \lambda_i(x)c_i$ (spectral decomposition, Theorem III.1.2 of [1]). Every $\lambda_i(x)$ is called an eigenvalue of x. We denote the minimum and maximum eigenvalues of x by $\lambda_{\min}(x)$ and $\lambda_{\max}(x)$, respectively. If two elements x and y share the same Jordan frames in the spectral decompositions, then they operator commute; i.e., they satisfy $L(x)L(y) = L(y)L(x)$ (Theorem 27 of [14]). The trace of x is defined to be $\mathrm{tr}(x) := \sum_{i=1}^{r} \lambda_i(x)$ and the Frobenius norm of x is $\|x\|_F := \sqrt{\langle x, x\rangle} = \sqrt{\lambda_1(x)^2 + \ldots + \lambda_r(x)^2}$. Observe that $\|e\|_F = \sqrt{r}$. We also see that $x \in \mathcal{K}$ (respectively, $x \in \mathrm{int}\mathcal{K}$) if and only if $\lambda_i(x) \geq 0$ (respectively, $\lambda_i(x) > 0$), for all $i = 1, 2, \ldots, r$. For any $x \in \mathcal{J}$, having the spectral decomposition $x = \sum_{i=1}^{r} \lambda_i(x)c_i$, we denote

$$x^{\frac{1}{2}} := \sum_{i=1}^{r} \sqrt{\lambda_i(x)}c_i, \text{ if } \lambda_i(x) \geq 0 \text{ and } x^{-1} := \sum_{i=1}^{r} \lambda_i(x)^{-1}c_i, \text{ if } \lambda_i(x) \neq 0.$$

The remainder of our work is organized as follows. In the next section, we introduce the perturbed problem and describe our proposed algorithm. Section 3 gives an analysis of the algorithm. In subsection 3.1, we derive an upper bound for the proximity measure after a full step. Subsection 3.2 serves to derive an upper bound for $\omega(v)$. In subsection 3.3, we fix the values of the parameters θ and τ in the algorithm. Here, τ is a uniform upper bound for the values of the proximity measure, $\delta(x, s; \mu)$, occurring during the course of the algorithm, and θ determines the progress to feasibility and optimality of the iterates. As a result, we realize the algorithm to be well-defined for the chosen values of θ and τ. Finally, we derive the complexity of the algorithm coinciding with the best known iteration bound for IIPMs.

2. Infeasible full-NT step IPM

In the case of an infeasible method, we call the pair (x, s) an ϵ-solution of (1.1) if the Frobenius norm of the residual vector $s - Mx - q$ does not exceed ϵ, and also $\langle x, s\rangle := \mathrm{tr}(x \circ s) \leq \epsilon$.

2.1. **The perturbed problem.** In accordance with the available results on IIPMs (e.g., see [8]), it is assumed that there exists a solution (x^*, s^*) such that

$$\|x^*\|_\infty \le \rho_p,\ \max\{\|s^*\|_\infty, \|\rho_p Me + q\|_F\} \le \rho_d, \tag{2.1}$$

where ρ_p and ρ_d are positive numbers. Furthermore, we choose arbitrarily $(x^0, s^0) \in \mathrm{int}\mathcal{K}$ and $\mu^0 > 0$ such that

$$x^0 = \rho_p e,\ s^0 = \rho_d e,\ \text{and}\ \mu^0 = \rho_p \rho_d, \tag{2.2}$$

as the starting point of the algorithm, where ρ_p and ρ_d are defined as in (2.1). The initial value of the residual vector is denoted to be $r_q^0 = s^0 - Mx^0 - q$. In general, we have $r_q^0 \neq 0$. However, a sequence of perturbed problems is constructed below in a such a way that the initial iterate is strictly feasible for the first perturbed problem in the sequence.

For any ν, with $0 < \nu \le 1$, we consider the perturbed problem (SCLCP$_\nu$), defined by

$$s - Mx - q = \nu r_q^0,\ (x, s) \in \mathcal{K} \times \mathcal{K}\ (\mathrm{SCLCP}_\nu).$$

It is obvious that $(x, s) = (x^0, s^0)$ is a strictly feasible solution of (SCLCP$_\nu$), when $\nu = 1$. This means that the perturbed problem (SCLCP$_\nu$) satisfies the interior point condition (IPC), for $\nu = 1$; i.e., there exists $(x^0, s^0) \in \mathrm{int}\mathcal{K} \times \mathrm{int}\mathcal{K}$ such that $s^0 - Mx^0 - q = \nu r_q^0$, which then straightforwardly leads to the following result.

Theorem 2.1. (Theorem 3.1 of [8]) *Let* (1.1) *be feasible and* $0 < \nu \le 1$. *Then, the perturbed problem* (SCLCP$_\nu$) *satisfies the IPC.*

Let (1.1) be feasible and $0 < \nu \le 1$. Then, Theorem 2.1 implies that the problem (SCLCP$_\nu$) satisfies the IPC, for $0 < \nu \le 1$, and hence a corresponding central path exists. This means that the system

$$\begin{aligned} s - Mx - q = \nu r_q^0,\ (x, s) \in \mathcal{K} \times \mathcal{K}, \\ x \circ s = \mu e, \end{aligned} \tag{2.3}$$

has a unique solution for every $\mu > 0$, as the μ-center of the perturbed problem (SCLCP$_\nu$). The set of μ-centers is called the central path. In what follows, the parameters μ and ν always satisfy the relation $\mu = \nu\mu^0 = \nu\rho_p\rho_d$. It is also worth noting that, according to (2.2), $x^0 \circ s^0 = \mu^0 e$; hence, (x^0, s^0) is the μ^0-center of the perturbed problem (SCLCP$_\nu$) for $\nu = 1$. Therefore, the algorithm can easily be started since by construction we have the initial starting point lying exactly on the central path of (SCLCP$_\nu$) for $\nu = 1$.

Let (x, s) be a feasible solution of (SCLCP$_\nu$), and $\mu = \nu\mu^0$. Then, we measure proximity to the μ-center of the perturbed problem (SCLCP$_\nu$) by the quantity

$$\delta(x, s; \mu) := \delta(v) := \frac{1}{2}\left\|v^{-1} - v\right\|_F,\ \text{where}\ v := \frac{P(w^{-\frac{1}{2}})x}{\sqrt{\mu}}\Big[= \frac{P(w^{\frac{1}{2}})s}{\sqrt{\mu}}\Big],$$

and $w := P(x^{\frac{1}{2}})\big(P(x^{\frac{1}{2}})s\big)^{-\frac{1}{2}}[= P(s^{-\frac{1}{2}})\big(P(s^{\frac{1}{2}})x\big)^{\frac{1}{2}}]$ is called the scaling point of x and s (Lemma 3.2 of [4]). As a consequence, we have the following lemma.

Lemma 2.2. (Lemma 4.7 of [5]) *If* $\delta := \delta(v)$, *then*

$$\frac{1}{\rho(\delta)} \leq \lambda_{\min}(v) \leq \lambda_{\max}(v) \leq \rho(\delta), \text{ where } \rho(\delta) := \delta + \sqrt{1+\delta^2}.$$

2.2. **An iteration of our algorithm.** Suppose that for some $\mu \in (0, \mu^0]$ we have x and s satisfying the first equation in (2.3) for $\nu = \frac{\mu}{\mu^0}$ and such that $\delta(x, s; \mu) \leq \tau$. This certainly holds at the start of the first iteration, since $s^0 - Mx^0 - q = \nu r_q^0$, when $\nu = 1$ and $\delta(x^0, s^0; \mu^0) = 0$. We reduce μ to $\mu^+ := (1-\theta)\mu$ and ν to $\nu^+ := (1-\theta)\nu$ with $\theta \in (0,1)$, and find new iterates x^+ and s^+ satisfying the first equation in (2.3), with μ replaced by μ^+ and ν by $\nu^+ = \frac{\mu^+}{\mu^0}$, and such that $\delta(x^+, s^+; \mu^+) \leq \tau$. Note that $\nu^+ = (1-\theta)\nu$. So, the relation $\mu = \nu\mu^0$ is maintained in every iteration.

Suppose that we have strictly feasible iterates x and s for $(SCLCP_\nu)$. This means that (x, s) satisfies the first equation of (2.3) with $x \in \mathrm{int}\mathcal{K}$ and $s \in \mathrm{int}\mathcal{K}$. With ν replaced by $\nu^+ = (1-\theta)\nu$, we find displacements Δx and Δs such that

$$\begin{aligned} M\Delta x - \Delta s &= \theta\nu r_q^0, \\ s \circ \Delta x + x \circ \Delta s &= \mu e - x \circ s. \end{aligned} \tag{2.4}$$

Due to the fact that $L(x)L(s) \neq L(s)L(x)$, in general, the above system does not always have a unique solution. To overcome this difficultly, the second equation of the system (2.3) is replaced by the following equivalent scaled equation (see Lemma 28 of [14]):

$$P(w^{-\frac{1}{2}})x \circ P(w^{\frac{1}{2}})s = \mu e,$$

where w is the NT-scaling point of x and s. This scaling point was first proposed by Nesterov and Todd [9] for self-scaled cones and then adapted by Faybusovich [3] for symmetric cones. In this case, the system (2.4) becomes

$$\begin{aligned} M\Delta x - \Delta s &= \theta\nu r_q^0, \\ P(w^{\frac{1}{2}})s \circ P(w^{-\frac{1}{2}})\Delta x + P(w^{-\frac{1}{2}})x \circ P(w^{\frac{1}{2}})\Delta s &= \\ \mu e - P(w^{-\frac{1}{2}})x \circ P(w^{\frac{1}{2}})s. \end{aligned} \tag{2.5}$$

It is easily seen that $x^+ := x + \Delta x$ and $s^+ := s + \Delta s$ satisfy $s - Mx - q = \nu^+ r_q^0$. The main part of the analysis is to guarantee that $x^+ \in \mathrm{int}\mathcal{K}$ and $s^+ \in \mathrm{int}\mathcal{K}$ satisfy $\delta(x^+, s^+; \mu^+) \leq \tau$.

2.3. **The algorithm.** A formal description of the new algorithm is given in Figure 1.

Infeasible interior − point algorithm

Input :
accuracy parameter $\epsilon > 0$;
barraier update parameter θ, $0 < \theta < 1$;
begin
$x := \rho_p e$; $s := \rho_d e$; $\nu = 1$; $\mu = \rho_p \rho_d$;
while $\max\big(\mathrm{tr}(x \circ s), \|r_q\|_F\big) > \epsilon$ **do**
begin
$(x, s) := (x, s) + (\Delta x, \Delta s)$;
$\mu := (1-\theta)\mu$; $\nu := (1-\theta)\nu$;
end
end

Figure 1 : The algorithm

3. Analysis of the algorithm

Let x and s denote the iterates at the start of an iteration, and assume $\delta(x, s; \mu) \leq \tau$.

3.1. **Upper bound for $\delta(v^+)$.** As established in subsection 2.2, the full-NT step generates new iterates x^+ and s^+ that satisfy the feasibility condition for $(SCLCP_{\nu^+})$, except for possibly the constraints on the cone $\mathcal{K}$. A crucial element in the analysis is to show that, after the full-NT step, $\delta(x^+, s^+; \mu^+) \leq \tau$.

Defining

$$d_x := \frac{P(w^{-\frac{1}{2}})\Delta x}{\sqrt{\mu}}, \ d_s := \frac{P(w^{\frac{1}{2}})\Delta s}{\sqrt{\mu}}, \tag{3.1}$$

the second equation of (2.5) turns to

$$d_x + d_s = v^{-1} - v.$$

In this case, using the definition of v and (3.1), we get

$$x^+ = \sqrt{\mu} P(w^{\frac{1}{2}})(v + d_x), \ s^+ = \sqrt{\mu} P(w^{-\frac{1}{2}})(v + d_s). \tag{3.2}$$

Since $P(w^{\frac{1}{2}})$ and its inverse $P(w^{-\frac{1}{2}})$ are automorphisms of $\mathrm{int}\mathcal{K}$, x^+ and s^+ belong to $\mathrm{int}\mathcal{K}$ if and only if $v + d_x$ and $v + d_s$ belong to $\mathrm{int}\mathcal{K}$. We have

$$\begin{aligned} (v + d_x) \circ (v + d_s) &= v^2 + v \circ (d_x + d_s) + d_x \circ d_s \\ &= e + d_x \circ d_s. \end{aligned} \tag{3.3}$$

Lemma 3.1. *The iterate (x^+, s^+) is strictly feasible if $e + d_x \circ d_s \in \mathrm{int}\mathcal{K}$.*

Proof. The proof is similar to the proof of Lemma 4.2 in [5], and is therefore omitted. □

Corollary 3.2. *The iterate* (x^+, s^+) *is strictly feasible if* $\|\lambda(d_x \circ d_s)\|_\infty < 1$.

Proof. By Lemma 3.1, the iterate (x^+, s^+) is strictly feasible if $e+d_x\circ d_s \in \mathrm{int}\mathcal{K}$. If $\|\lambda(d_x \circ d_s)\|_\infty < 1$. Then we have $-1 < \lambda_i(d_x \circ d_s) < 1$, for all $i = 1, \ldots, r$. Therefore,

$$\lambda_i(e + d_x \circ d_s) = 1 + \lambda_i(d_x \circ d_s) > 0, \;\; i = 1, \ldots, r.$$

The last inequalities mean that $e + d_x \circ d_s \in \mathrm{int}\mathcal{K}$, and the proof follows. □

In the sequel, we use the notation

$$\omega(v) := \frac{1}{2}\big(\|d_x\|_F^2 + \|d_s\|_F^2\big). \tag{3.4}$$

It follows, from Lemmas 2.16 and 2.12 of [5], that

$$\begin{aligned} \|\lambda(d_x \circ d_s)\|_\infty \leq \|d_x \circ d_s\|_F &\leq \frac{1}{2}\|d_x^2 + d_s^2\|_F \\ &\leq \frac{1}{2}(\|d_x\|_F^2 + \|d_s\|_F^2) = \omega(v). \end{aligned} \tag{3.5}$$

Corollary 3.3. *If* $\omega(v) < 1$*, then the iterate* (x^+, s^+) *is strictly feasible.*

Proof. Due to (3.5), $\omega(v) < 1$ implies $\|\lambda(d_x \circ d_s)\|_\infty < 1$. By Corollary 3.2, the proof is complete. □

Assuming $\omega(v) < 1$, which guarantees strict feasibility of the iterate (x^+, s^+), we proceed by deriving an upper bound for $\delta(x^+, s^+; \mu^+)$. By definition, we have

$$\delta(x^+, s^+; \mu^+) := \frac{1}{2}\big\|v^+ - (v^+)^{-1}\big\|_F,$$

where $v^+ := \frac{P((w^+)^{-\frac{1}{2}})x^+}{\sqrt{\mu^+}}\Big[= \frac{P((w^+)^{\frac{1}{2}})s^+}{\sqrt{\mu^+}}\Big]$. In what follows, we denote $\delta(x^+, s^+; \mu^+)$ shortly by $\delta(v^+)$.

Lemma 3.4. *Let* $\omega(v) < 1$*. Then, we have*

$$\delta(v^+) \leq \frac{\theta\sqrt{r} + \omega(v)}{2\sqrt{(1-\theta)(1-\omega(v))}}.$$

Proof. Since $\omega(v) < 1$, using corollaries 3.3 and 3.2, Lemma 3.1, (3.3) and (3.2), it follows that $v + d_x, v + d_s$ and $(v + d_x) \circ (v + d_s)$ belong to $\mathrm{int}\mathcal{K}$. Similar to the proof of lemma 3.3 of [8], we have

$$\sqrt{1-\theta}v^+ \sim \big[P(v + d_x)^{\frac{1}{2}}(v + d_s)\big]^{\frac{1}{2}}.$$

Using Theorem 4 of [15] and (3.5), we have

$$\begin{aligned}\lambda_{\min}(v^+)^2 &= \lambda_{\min}\Big(P\Big(\frac{v+d_x}{\sqrt{1-\theta}}\Big)^{\frac{1}{2}}\Big(\frac{v+d_s}{\sqrt{1-\theta}}\Big)\Big)\\ &\geq \lambda_{\min}\Big(\Big(\frac{v+d_x}{\sqrt{1-\theta}}\Big)\circ\Big(\frac{v+d_s}{\sqrt{1-\theta}}\Big)\Big)\\ &= \frac{\lambda_{\min}(e+d_x\circ d_s)}{1-\theta}\\ &= \frac{1+\lambda_{\min}(d_x\circ d_s)}{1-\theta}\\ &\geq \frac{1-\|d_x\circ d_s\|_F}{1-\theta}\\ &\geq \frac{1-\omega(v)}{1-\theta}.\end{aligned}$$

Hence, using Lemma 2.9 of [10], the above inequality, Lemma 30 of [14], (3.3), the triangle inequality and (3.5), we may write

$$\begin{aligned}2\delta(v^+) &= \|v^+-(v^+)^{-1}\|_F\\ &\leq \frac{\|(v^+)^2-e\|_F}{\lambda_{\min}(v^+)}\\ &\leq \frac{\sqrt{1-\theta}}{\sqrt{1-\omega(v)}}\Big\|P\Big(\frac{v+d_x}{\sqrt{1-\theta}}\Big)^{\frac{1}{2}}\Big(\frac{v+d_s}{\sqrt{1-\theta}}\Big)-e\Big\|_F\\ &\leq \frac{\sqrt{1-\theta}}{\sqrt{1-\omega(v)}}\Big\|\Big(\frac{v+d_x}{\sqrt{1-\theta}}\Big)\circ\Big(\frac{v+d_s}{\sqrt{1-\theta}}\Big)-e\Big\|_F\\ &= \frac{\sqrt{1-\theta}}{\sqrt{1-\omega(v)}}\Big\|\frac{e+d_x\circ d_s}{1-\theta}-e\Big\|_F\\ &= \frac{\big\|\theta e+d_x\circ d_s\big\|_F}{\sqrt{(1-\theta)(1-\omega(v))}}\\ &\leq \frac{\theta\sqrt{r}+\omega(v)}{\sqrt{(1-\theta)(1-\omega(v))}}.\end{aligned}$$

This completes the proof. □

3.2. **Upper bound for $\omega(v)$.** We may easily check that the system (2.5), defining the search directions Δx and Δs, can be expressed in terms of the scaled search directions d_x and d_s as follows:

$$\begin{aligned}\overline{M}d_x-d_s &= \frac{P(w^{\frac{1}{2}})}{\sqrt{\mu}}\theta\nu r_q^0,\\ d_x+d_s &= v^{-1}-v,\end{aligned} \tag{3.6}$$

where $\overline{M} := P(w)^{\frac{1}{2}} M P(w)^{\frac{1}{2}}$ is positive semidefinite [8].

Lemma 3.5. *Let $a \in R^n$ and $\overline{M} \in R^{n\times n}$ be positive semidefinite. Then, the solution (u, v) of the linear system*

$$\begin{aligned} \overline{M}u - v &= 0, \\ u + v &= a, \end{aligned} \tag{3.7}$$

satisfies

$$\|u\|_F^2 + \|v\|_F^2 \leq \|a\|_F^2.$$

Proof. We have

$$\|u\|_F^2 + \|v\|_F^2 = \|u+v\|_F^2 - 2\langle u, v\rangle \leq \|u+v\|_F^2 = \|a\|_F^2.$$

The inequality follows from $\overline{M}$ being positive semidefinite to complete the proof. □

Lemma 3.6. *Let a and b be two n-dimensional vectors, and let $\overline{M} \in R^{n\times n}$ be positive semidefinite. Then, the solution (u, v) of the linear system*

$$\begin{aligned} \overline{M}u - v &= b, \\ u + v &= a, \end{aligned} \tag{3.8}$$

satisfies

$$\|u\|_F^2 + \|v\|_F^2 \leq 4\|a\|_F^2 + 6\|b\|_F^2.$$

Proof. It is easily seen that the system (3.8) can be written as

$$\begin{aligned} \overline{M}u - \tilde{v} &= 0, \\ u + \tilde{v} &= a + b, \end{aligned} \tag{3.9}$$

where $\tilde{v} = v + b$. Applying Lemma 3.5 to (3.9), it follows that

$$\|u\|_F^2 + \|\tilde{v}\|_F^2 \leq \|a+b\|_F^2 = \|a\|_F^2 + \|b\|_F^2 + 2\langle a, b\rangle \leq 2\big(\|a\|_F^2 + \|b\|_F^2\big).$$

Therefore, using the above inequality, we get

$$\begin{aligned} \|u\|_F^2 + \|v\|_F^2 &= \|u\|_F^2 + \|\tilde{v} - b\|_F^2 \\ &\leq 2\|u\|_F^2 + 2\big(\|\tilde{v}\|_F^2 + \|b\|_F^2\big) \leq 4\|a\|_F^2 + 6\|b\|_F^2. \end{aligned}$$

This completes the proof. □

Comparing the system (3.8) with the system (3.6) and considering $(u, v) = (d_x, d_s)$, $b = \dfrac{P(w^{\frac{1}{2}})}{\sqrt{\mu}}\theta\nu r_q^0$ and $a = v^{-1} - v$ in (3.8), by using Lemma 3.6 and

$$\begin{aligned} \big\|P(w^{\frac{1}{2}})r_q^0\big\|_F^2 &= \big\langle P(w^{\frac{1}{2}})r_q^0, P(w^{\frac{1}{2}})r_q^0\big\rangle = \big\langle P(w)r_q^0, r_q^0\big\rangle \\ &= \big\langle P(w)r_q^0, 2\rho_d e\big\rangle - \big\langle P(w)r_q^0, 2\rho_d e - r_q^0\big\rangle \leq \big\langle P(w)r_q^0, 2\rho_d e\big\rangle \\ &= \big\langle P(w)(2\rho_d e), 2\rho_d e\big\rangle - \big\langle P(w)(2\rho_d e - r_q^0), 2\rho_d e\big\rangle \\ &\leq \big\langle P(w)(2\rho_d e), 2\rho_d e\big\rangle = 4\rho_d^2\big\langle P(w)e, e\big\rangle = 4\rho_d^2 \mathrm{tr}(w^2), \end{aligned}$$

where the inequalities hold since $x^0 = \rho_p e, s^0 = \rho_d e$ and $\|\rho_p Me + q\|_F \leq \rho_d$ imply $2\rho_d e \succeq r_q^0 \succeq 0$, we get

$$\begin{aligned} \omega(v) &\leq 2\|v^{-1} - v\|_F^2 + \frac{3\theta^2\nu^2}{\mu}\left\|P(w^{\frac{1}{2}})r_q^0\right\|_F^2 \\ &\leq 8\delta^2 + \frac{12\theta^2\nu^2\rho_d^2}{\mu}\mathrm{tr}(w^2). \end{aligned}$$

Now, using the inequality $\mathrm{tr}(w^2) \leq \dfrac{\mathrm{tr}(x^2)}{\mu\lambda_{\min}(v)^2}$ (Lemma 4.5 of [5]), Lemma 2.2, $\mathrm{tr}(x) \leq r\rho_p(2+\rho(\delta)^2)$ (Lemma 3.7 of [8]) and $\mu = \nu\rho_p\rho_d$, we obtain:

$$\begin{aligned} \omega(v) \leq 8\delta^2 + \frac{12\theta^2\nu^2\rho_d^2}{\mu} \times \frac{\mathrm{tr}(x^2)}{\mu\lambda_{\min}(v)^2} &\leq 8\delta^2 + \frac{12\theta^2\rho(\delta)^2}{\rho_p^2}\mathrm{tr}(x)^2 \\ &\leq 8\delta^2 + 12\theta^2 r^2\rho(\delta)^2\big(2+\rho(\delta)^2\big)^2. \end{aligned} \tag{3.10}$$

3.3. **Values for θ and τ.** Our aim is to find a positive number τ such that if $\delta := \delta(v) \leq \tau$, then $\delta(v^+) \leq \tau$. By Lemma 3.4, this holds if $\omega(v) < 1$ and

$$\frac{\theta\sqrt{r} + \omega(v)}{2\sqrt{(1-\theta)(1-\omega(v))}} \leq \tau. \tag{3.11}$$

Assuming $\delta(v) \leq \tau$, we therefore need to find τ such that the above inequality holds, with θ as large as possible. We choose

$$\theta = \frac{1}{46r}, \ \tau = \frac{1}{16}. \tag{3.12}$$

Using $\delta \leq \tau$, it follows from (3.10), with the right-hand side of (3.10) being monotonically increasing with respect to δ, that

$$\begin{aligned} omega(v) &\leq 8\tau^2 + 12\theta^2 r^2\rho(\tau)^2\big(2+\rho(\tau)^2\big)^2 \\ &= 8\Big(\frac{1}{16}\Big)^2 + 12\Big(\frac{1}{46r}\Big)^2 r^2\rho\Big(\frac{1}{16}\Big)^2\Big(2+\rho\Big(\frac{1}{16}\Big)^2\Big)^2 \\ &= 0.0943 < 1. \end{aligned}$$

The above inequality means, using Corollary 3.3, that the iterate (x^+, s^+) is strictly feasible. From Lemma 3.4, it follows that

$$\begin{aligned} \delta(v^+) \leq \frac{\theta\sqrt{r} + \omega(v)}{2\sqrt{(1-\theta)(1-\omega(v))}} &\leq \frac{\frac{1}{46\sqrt{r}} + 0.0943}{2\sqrt{(1-\frac{1}{46r})(1-0.0943)}} \\ &= 0.0617 < \frac{1}{16} = \tau. \end{aligned}$$

This implies that (3.11) holds. Therefore, the algorithm is well-defined in the sense that the property $\delta(x, s; \mu) \leq \tau$ is maintained in all iterations.

3.4. **Complexity.** We have found that if at the start of an iteration the iterate satisfies $\delta(x, s; \mu) \leq \tau$ and τ and θ are defined as in (3.12), then after the full-NT step, the iterate is strictly feasible and satisfies $\delta(x^+, s^+; \mu^+) \leq \tau$. This establishes that the algorithm is well-defined.

In each main iteration, both the barrier parameter μ and the norm of the residual vector are reduced by the factor $1 - \theta$. Hence, the total number of main iterations is bounded above by

$$\frac{1}{\theta} \log \frac{\max\{\mathrm{tr}(x^0 \circ s^0), \|r_q^0\|_F\}}{\epsilon}.$$

Thus, we next state the main result of our work.

Theorem 3.7. *If* (1.1) *has a solution* (x^*, s^*) *such that* $\|x^*\|_\infty \leq \rho_p$ *and* $\|s^*\|_\infty \leq \rho_d$*, then after at most*

$$46r \log \frac{\max\{\mathrm{tr}(x^0 \circ s^0), \|r_q^0\|_F\}}{\epsilon}$$

iterations the algorithm finds an ϵ*-solution of* (1.1).

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