GENERALIZED DERIVATIONS ON MODULES

GH. ABBASPOUR*, M. S. MOSLEHIAN AND A. NIKNAM

ABSTRACT. Let $A$ be a Banach algebra and $M$ be a Banach right $A$-module. A linear map $\delta : M \to M$ is called a generalized derivation if there exists a derivation $d : A \to A$ such that

$$\delta(xa) = \delta(x)a + xd(a) \quad (a \in A, x \in M).$$

In this paper we associate a triangular Banach algebra $T$ to a Banach $A$-module $M$ and investigate the relation between generalized derivations on $M$ and derivations on $T$. In particular, we prove that the so-called generalized first cohomology group of $M$ is isomorphic to the first cohomology group of $T$.

1. Introduction

Recently, a number of analysts [1, 6, 17] have studied various generalized notions of derivations in the context of Banach algebras. There are some applications in the other fields of study [11]. Such maps have been extensively studied in pure algebra; cf. [2, 5, 12].

Let, throughout the paper, $A$ denote a Banach algebra (not necessarily unital) and let $M$ be a Banach right $A$-module. A linear mapping $d : A \to A$ is called a derivation if $d(ab) = d(a)b + ad(b)$ ($a, b \in A$). If $a \in A$ and we define $d_a$ by $d_a(x) = ax - xa$ ($x \in A$), then $d_a$ is a derivation and such derivation is called inner. A linear mapping $\delta : M \to M$
is called a generalized derivation if there exists a derivation $d : A \to A$ such that $\delta(xa) = \delta(x)a + xd(a)$ ($x \in M, a \in A$). For convenience, we say that such a generalized derivation $\delta$ is a $d$-derivation. In general, the derivation $d : A \to A$ is not unique and it may happen that $\delta$ (resp. $d$) is bounded but $d$ (resp. $\delta$) is not bounded. For instance, assume that the action of $A$ on $M$ is trivial, i.e. $MA = \{0\}$. Then every linear mapping $\delta : M \to M$ is a $d$-derivation for each derivation $d$ on $A$.

Our notion is a generalization of both concepts of a generalized derivation (cf. [5, 12]) and of a multiplier (cf. [7]) on an algebra (see also [19]). To see this, regard the algebra as a module over itself. The authors in [1] investigated the generalized derivations on Hilbert $C^*$-modules and showed that these maps may appear as the infinitesimal generators of dynamical systems.

**Example 1.1.** Let $M$ be a right Hilbert $C^*$-module over a $C^*$-algebra $A$ of compact operators acting on a Hilbert space (see [15] for more details on Hilbert $C^*$-modules). By Theorem 4 of [3], $M$ has an orthonormal basis so that each element $x$ of $M$ can be expressed as $x = \sum_{\lambda} v_\lambda < v_\lambda, x >$. If $d$ is a derivation on $A$, then the mapping, $\delta : M \to M$ defined by $\delta(x) = \sum_{\lambda} v_\lambda d(< v_\lambda, x >)$ is a $d$-derivation, since

$$
\delta(xa) = \delta \left( \sum_{\lambda} v_\lambda < v_\lambda, xa > \right)
= \sum_{\lambda} v_\lambda d(< v_\lambda, x > a)
= \sum_{\lambda} v_\lambda d(< v_\lambda, x >)a + \sum_{\lambda} v_\lambda < v_\lambda, x > d(a)
= \delta(x)a + xd(a).
$$

The set $\mathcal{B}(M)$ of all bounded module maps on $M$ is a Banach algebra and $M$ is a Banach $\mathcal{B}(M)-A$-bimodule equipped with $Tx = T(x) \quad (x \in M, T \in \mathcal{B}(M))$, since we have $T(xa) = T(xa) = T(x)a = (Tx)a$ and $\|Txa\| \leq \|T\| \|x\| \|a\|$, for all $a \in A, x \in M, T \in \mathcal{B}(M)$.

We call $\delta : M \to M$ a generalized inner derivation if there exist $a \in A$ and $T \in \mathcal{B}(M)$ such that $\delta(x) = Tx - xa = T(x) - xa$. Mathieu in [16] called a map $\delta : A \to A$ a generalized inner derivation if $\delta(x) = bx - xa$ for some $a, b \in A$. If we consider $A$ as a right $A$-module in a natural way, and take $T(x) = bx$, then our definition covers the notion of Mathieu.
In this paper we deal with the derivations on the triangular Banach algebras of the form $T = \begin{pmatrix} \mathcal{B}(M) & M \\ 0 & A \end{pmatrix}$. Such algebras were introduced by Forrest and Marcoux [8] that in turn are motivated by work of Gilfeather and Smith in [10] (these algebras have been also investigated by Y. Zhang who called them module extension Banach algebras [22]). Among some facts on generalized derivations, we investigate the relation between generalized derivations on $M$ and derivations on $T$. In particular, we show that the generalized first cohomology group of $M$ is isomorphic to the first cohomology group of $T$.

2. Main results

If we consider $A$ as an $A$-module in a natural way then we have the following lemma about generalized derivations on $A$.

**Lemma 2.1.** A linear mapping $\delta : A \to A$ is a generalized derivation if and only if there exist a derivation $d : A \to A$ and a module map $\varphi : A \to A$ such that $\delta = d + \varphi$.

**Proof.** Suppose $\delta$ is a generalized derivation on $A$. Then there exists a derivation $d$ on $A$ such that $\delta$ is a $d$-derivation. On putting $\varphi = \delta - d$, we have

$$
\varphi(xa) = \delta(xa) - d(xa) = \delta(x)a + xd(a) - (d(x)a + xd(a))
= (\delta(x) - d(x))a = \varphi(x)a,
$$

for all $a, x \in A$. Thus $\varphi$ is a module map and $\delta = d + \varphi$.

Conversely, let $d$ be a derivation on $A$, $\varphi$ be a module map on $A$ and put $\delta = d + \varphi$. Then clearly $\delta$ is a linear map and

$$
\delta(xa) = d(xa) + \varphi(xa) = d(x)a + xd(a) + \varphi(x)a
= (d(x) + \varphi(x))a + xd(a) = \delta(x)a + xd(a)
$$

for all $a, x \in A$. Therefore $\delta$ is a $d$-derivation. \qed

The next two results concern the boundedness of a generalized derivation.
Theorem 2.2. Let $A$ have a bounded left approximate identity $\{e_n\}_{n \in I}$ and let $\delta$ be a $d$-derivation on $A$. Then $\delta$ is bounded if and only if $d$ is bounded.

Proof. First we show that every module map on $A$ is bounded. Suppose that $\varphi$ is a module map on $A$ and let $\{a_n\}$ be a sequence in $A$ converging to zero in the norm topology. By a consequence of Cohen Factorization Theorem (see Corollary 11.12 of [4]) there exist a sequence $\{b_n\}$ and an element $c$ in $A$ such that $b_n \to 0$ and $a_n = cb_n$, $(n \in \mathbb{N})$. Then $\varphi(a_n) = \varphi(cb_n) = \varphi(c)b_n \to 0$. Thus by the closed graph theorem, $\varphi$ is bounded. Now let $\delta$ be a $d$-derivation. By Lemma 2.1, $\delta = d + \varphi$ for some module map $\varphi$ on $A$. Therefore $\delta$ is bounded if and only if $d$ is bounded. \hfill \Box

Corollary 2.3. Every generalized derivation on a $C^*$-algebra is bounded.

Proof. Every derivation on a $C^*$-algebra is automatically continuous; cf. [13]. \hfill \Box

Let $\varphi : A \to A$ be a homomorphism (algebra morphism). A linear mapping $T : M \to M$ is called a $\varphi$-morphism if $T(\varphi(a)) = T(a)\varphi(a)$ $(a \in A, x \in M)$. If $\varphi$ is an isomorphism and $T$ is a bijective mapping then we say $T$ to be a $\varphi$-isomorphism. An $id_A$-morphism is a module map (module morphism). Here $id_A$ denotes the identity operator on $A$.

Proposition 2.4. Suppose $\delta$ is a bounded $d$-derivation on $M$ and $d$ is bounded. Then $T = \exp(\delta)$ is a bi-continuous $\exp(d)$-isomorphism.

Proof. Using induction one can easily show that

$$\delta^{(n)}(xa) = \sum_{r=0}^{n} \binom{n}{r} \delta^{(n-r)}(x)d^{(r)}(a).$$

For each $a \in A, x \in M$ we have
\[ T(xa) = \exp(\delta)(xa) \]
\[ = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^{(n)}(xa) \]
\[ = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r=0}^{n} \frac{n!}{r!} \delta^{(n-r)}(x)d^{(r)}(a) \]
\[ = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r=0}^{n} \left( \frac{1}{(n-r)!} \delta^{(n-r)}(x) \frac{1}{r!} d^{(r)}(a) \right) \]
\[ = \left( \sum_{n=0}^{\infty} \frac{1}{n!} \delta^{(n)}(x) \right) \left( \sum_{n=0}^{\infty} \frac{1}{n!} d^{(n)}(a) \right) \]
\[ = \exp(\delta)(x) \exp(d)(a). \]

The operators \( \exp(\delta) \) and \( \exp(d) \) are invertible in the Banach algebras of bounded operators on \( M \) and \( A \), respectively. Hence \( T \) is an \( \exp(d) \)-isomorphism.

**Proposition 2.5.** Let \( \delta \) be a bounded generalized derivation on \( M \). Then \( \delta \) is a generalized inner derivation if and only if there exists an inner derivation \( d_a \) on \( A \) such that \( \delta \) is a \( d_a \)-derivation.

**Proof.** Let \( \delta \) be a generalized inner derivation. Then there exist \( a \in A \) and \( T \in B(M) \) such that \( \delta(x) = T(x) - xa \ (x \in M) \). We have
\[ \delta(x)b + xd_a(b) = (T(x) - xa)b + xab - xba = T(x)b - xba = T(xb) - (xb)a = \delta(xb) \quad (b \in A, x \in M). \]
Hence \( \delta \) is a \( d_a \)-derivation.

Conversely, suppose that \( \delta \) is a \( d_a \)-derivation for some \( a \in A \). Define \( T : M \to M \) by \( T(x) = \delta(x) + xa \). Then \( T \) is linear, bounded and
\[ T(xb) = \delta(xb) + xb = (\delta(x)b + xd_a(b)) + xba = \delta(x)b + xab - xba = (\delta(x) + xa)b = T(x)b. \]
It follows that \( T \in B(M) \) and \( \delta(x) = (\delta(x) + xa) - xa = T(x) - xa \). Therefore \( \delta \) is a generalized inner derivation. \( \square \)

The linear spaces of all bounded generalized derivations and generalized inner derivations on \( M \) are denoted by \( GZ^1(M, M) \) and \( GN^1(M, M) \), respectively. We call the quotient space \( GH^1(M, M) = GZ^1(M, M)/GN^1(M, M) \) the generalized first cohomology group of \( M \).
Corollary 2.6. \( GH^1(M, M) = 0 \) whenever \( H^1(A, A) = 0 \).

Proof. Let \( \delta : M \to M \) be a generalized derivation. Then there exists a derivation \( d : A \to A \) such that \( \delta \) is a \( d \)-derivation. Due to \( H^1(A, A) = 0 \), we deduce that \( d \) is inner and, by Proposition 2.5, so is \( \delta \). Hence \( GH^1(M, M) = 0 \).

Using some ideas of [8, 18], we give the following notion.

Definition 2.7. \( \mathcal{T} = \{ \begin{pmatrix} T & x \\ 0 & a \end{pmatrix} : T \in \mathcal{B}(M), x \in M, a \in A \} \) equipped with the usual \( 2 \times 2 \) matrix addition and formal multiplication and with the norm \( \| \begin{pmatrix} T & x \\ 0 & a \end{pmatrix} \| = \| T \| + \| x \| + \| a \| \) is a Banach algebra. We call this algebra the triangular Banach algebra associated to \( M \).

The following two theorems give some interesting relations between generalized derivations on \( M \) and derivations on \( \mathcal{T} \).

Let \( \delta \) be a bounded \( d \)-derivation on \( M \). We define \( \Delta_\delta : \mathcal{B}(M) \to \mathcal{B}(M) \) by \( \Delta_\delta(T) = \delta T - T \delta \). Then \( \Delta_\delta \) is clearly a derivation on \( \mathcal{B}(M) \).

Theorem 2.8. Let \( \delta \) be a bounded \( d \)-derivation on \( M \) and let \( d \) be bounded. Then the map \( D^\delta : \mathcal{T} \to \mathcal{T} \) defined by \( D^\delta \left( \begin{pmatrix} T & x \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} \Delta_\delta(T) & \delta(x) \\ 0 & d(a) \end{pmatrix} \) is a bounded derivation on \( \mathcal{T} \). Also \( \delta \) is a generalized inner derivation if and only if \( D^\delta \) is an inner derivation.

Proof. It is clear that \( D^\delta \) is linear. For any \( T_1, T_2 \in \mathcal{B}(M), x_1, x_2 \in M, a_1, a_2 \in A \) we have

\[
\begin{align*}
D^\delta \left( \begin{pmatrix} T_1 & x_1 \\ 0 & a_1 \end{pmatrix} \right) \left( \begin{pmatrix} T_2 & x_2 \\ 0 & a_2 \end{pmatrix} \right) &= D^\delta \left( \begin{pmatrix} T_1 T_2 & T_1 x_2 + x_1 a_2 \\ 0 & a_1 a_2 \end{pmatrix} \right) \\
&= \left( \Delta_\delta(T_1 T_2) - \delta(T_1, x_2 + x_1 a_2) \right) \\
&\quad - \left( \Delta_\delta(T_1 T_2) - \delta(T_1, x_2 + x_1 a_2) \right) \\
&\quad - \left( \Delta_\delta(T_1, x_2) + \delta(x_1 a_2) + x_1 d(a_2) \right) \\
&\quad - \left( \Delta_\delta(T_1, x_2) + \delta(x_1 a_2) + x_1 d(a_2) \right) \\
&\quad - \left( \Delta_\delta(T_1 T_2) - \delta(T_1, x_2 + x_1 a_2) \right) \\
&\quad - \left( \Delta_\delta(T_1, x_2) + \delta(x_1 a_2) + x_1 d(a_2) \right) \\
&\quad - \left( \Delta_\delta(T_1, x_2) + \delta(x_1 a_2) + x_1 d(a_2) \right) \\
&\quad - \left( \Delta_\delta(T_1, x_2) + \delta(x_1 a_2) + x_1 d(a_2) \right).
\end{align*}
\]
Thus $D^\delta$ is a derivation on $T$. Due to $\| (\begin{array}{cc}
\Delta_\delta(T) & \delta(x) \\
0 & d(a) \end{array}) \| = \| \Delta_\delta(T) \| + \| \delta(x) \| + \| d(a) \| \leq \max \{ \| \Delta_\delta \|, \| \delta \|, \| d \| \} \| (\begin{array}{cc}
T & x \\
0 & a \end{array}) \|$, we infer that $D^\delta$ is bounded. Now suppose that $\delta$ is a generalized inner derivation. Then there exist $a \in A$ and $T \in \mathcal{B}(M)$ such that $\delta(x) = T(x) - xa$ \quad ($x \in M$). For all $S \in \mathcal{B}(M), b \in A$ and $y \in M$ we have

$$
D \begin{pmatrix} T & 0 \\
0 & a \end{pmatrix} \begin{pmatrix} S & y \\
0 & b \end{pmatrix} = \begin{pmatrix} T & 0 \\
0 & a \end{pmatrix} \begin{pmatrix} S & y \\
0 & b \end{pmatrix} - \begin{pmatrix} S & y \\
0 & b \end{pmatrix} \begin{pmatrix} T & 0 \\
0 & a \end{pmatrix}
$$

$$
= \begin{pmatrix} T \, S \, S \, T & T \, y \, y \, a + y \, a \, T \, y \\
0 & a \, y \, - b \, a \, + a \, b \, - b \, a \, b \\
\Delta_\delta(S) & \delta(y) \\
0 & d_a(b) \end{pmatrix}
$$

$$
= D^\delta \begin{pmatrix} S & y \\
0 & b \end{pmatrix}.
$$

Hence $D^\delta = D \begin{pmatrix} T & 0 \\
0 & a \end{pmatrix}$ and so $D^\delta$ is an inner derivation.

Conversely, let $\delta$ be a bounded $d$-derivation such that the associated derivation $D^\delta$ is an inner derivation, say $D^\delta = D \begin{pmatrix} T_0 & x_0 \\
0 & a_0 \end{pmatrix}$. Then for each $T \in \mathcal{B}(M), x \in M, a \in A$ we have

$$
\begin{pmatrix} \Delta_\delta(T) & \delta(x) \\
0 & d(a) \end{pmatrix} = D^\delta \begin{pmatrix} T & x \\
0 & a \end{pmatrix}
$$

$$
= D \begin{pmatrix} T_0 & x_0 \\
0 & a_0 \end{pmatrix} \begin{pmatrix} T & x \\
0 & a \end{pmatrix}
$$

$$
= \begin{pmatrix} T_0 \, T - TT_0 & T_0(x) + x_0 a - T(x) - xa_0 \\
0 & a_0 a - a a_0 \end{pmatrix}
$$

(2.1)

Hence $d = d_{a_0}$ is inner. Putting $a = 0$ and $T = 0$ in (2.1), we conclude that $\delta(x) = T_0(x) - xa_0$ \quad ($x \in M$). Hence $\delta$ is a generalized inner derivation. \hfill \Box

The converse of the above theorem is true in the unital case.
Theorem 2.9. Let $A$ be unital and $\mathcal{T}$ be the triangular Banach algebra associated to a unital Banach right $A$-module $M$. Assume that $D : \mathcal{T} \to \mathcal{T}$ is a bounded derivation. Then there exist $m_0 \in M$, a bounded derivation $d : A \to A$ and a bounded $d$-derivation $\delta : M \to M$ such that

$$D\left(\begin{array}{cc} T & x \\ 0 & a \end{array}\right) = \left(\begin{array}{cc} \Delta_\delta(T) & \delta(x) + m_0 a - T.m_0 \\ 0 & d(a) \end{array}\right).$$

Moreover, $D$ is inner if and only if $\delta$ is a generalized inner derivation.

Proof. We use some ideas of Proposition 2.1 of [8]. By simple computation one can verify that

(i) $D\left(\begin{array}{cc} 0 & 0 \\ 0 & 1_A \end{array}\right) = \left(\begin{array}{cc} 0 & m_0 \\ 0 & 0 \end{array}\right)$ for some $m_0 \in M$;

(ii) $D\left(\begin{array}{cc} 0 & 0 \\ 0 & a \end{array}\right) = \left(\begin{array}{cc} 0 & m_0 a \\ 0 & d(a) \end{array}\right)$ for some bounded derivation $d$ on $A$;

(iii) $D\left(\begin{array}{cc} 0 & x \\ 0 & 0 \end{array}\right) = \left(\begin{array}{cc} 0 & \delta(x) \\ 0 & 0 \end{array}\right)$ for some linear mapping $\delta$ on $M$;

(iv) $D\left(\begin{array}{cc} T & 0 \\ 0 & 0 \end{array}\right) = \left(\begin{array}{cc} \Delta_\delta(T) & -T.m_0 \\ 0 & 0 \end{array}\right)$;

and finally $D\left(\begin{array}{cc} T & x \\ 0 & a \end{array}\right) = \left(\begin{array}{cc} \Delta_\delta(T) & \delta(x) + m_0 a - T.m_0 \\ 0 & d(a) \end{array}\right)$.

We have

$$\begin{array}{l}
\left(\begin{array}{cc} 0 & \delta(xa) \\ 0 & 0 \end{array}\right) = D\left(\begin{array}{cc} 0 & xa \\ 0 & 0 \end{array}\right) = D\left(\begin{array}{cc} 0 & x \\ 0 & 0 \end{array}\right) \left(\begin{array}{cc} 0 & 0 \\ 0 & a \end{array}\right) \\
= \left(\begin{array}{cc} 0 & x \\ 0 & 0 \end{array}\right) D\left(\begin{array}{cc} 0 & 0 \\ 0 & a \end{array}\right) + D\left(\begin{array}{cc} 0 & x \\ 0 & 0 \end{array}\right) \left(\begin{array}{cc} 0 & 0 \\ 0 & a \end{array}\right) \\
= \left(\begin{array}{cc} 0 & x \\ 0 & 0 \end{array}\right) \left(\begin{array}{cc} 0 & m_0 a \\ 0 & d(a) \end{array}\right) + \left(\begin{array}{cc} 0 & \delta(x) \\ 0 & 0 \end{array}\right) \left(\begin{array}{cc} 0 & 0 \\ 0 & a \end{array}\right) \\
= \left(\begin{array}{cc} 0 & \delta(xa) + xd(a) \\ 0 & 0 \end{array}\right).
\end{array}$$

Thus $\delta(xa) = \delta(x) a + xd(a)$ and so $\delta$ is a $d$-derivation. It is clear that $D$ is inner if and only if $d$ is inner and, using Proposition 2.5, the latter holds if and only if $\delta$ is a generalized inner derivation. \hfill $\Box$
Generalized derivations on modules

**Theorem 2.10.** Let $A$ be a unital Banach algebra, $M$ be a unital Banach right $A$-module and $T = \begin{pmatrix} \mathcal{B}(M) & M \\ 0 & A \end{pmatrix}$. Then $H^1(T, T) \cong GH^1(M, M)$

**Proof.** Let $\Psi : GZ^1(M, M) \to H^1(T, T)$ be defined by

$$\Psi(\delta) = [D^\delta],$$

where $[D^\delta]$ represents the equivalence class of $D^\delta$ in $H^1(T, T)$. Clearly $\Psi$ is linear. We shall show that $\Psi$ is surjective. To end this, we assume that $D$ is a bounded derivation on $T$. Let $\delta, d, \Delta_\delta$ and $m_0 \in M$ be as in the Theorem 2.9. Then

$$(D - D^\delta) \begin{pmatrix} T & x \\ 0 & a \end{pmatrix} = \begin{pmatrix} \Delta_\delta(T) & \delta(x) + m_0a - T.m_0 \\ 0 & d(a) \end{pmatrix} - \begin{pmatrix} \Delta_\delta(T) & \delta(x) \\ 0 & d(a) \end{pmatrix} = \begin{pmatrix} 0 & m_0a - T.m_0 \\ 0 & 0 \end{pmatrix} = D \begin{pmatrix} 0 & -m_0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T & x \\ 0 & a \end{pmatrix}.$$

So $[D] = [D^\delta] = \Psi(\delta)$ and thus $\Psi$ is surjective. Therefore $H^1(T, T) \cong GZ^1(M, M)/\text{Ker}(\Psi)$. Note that $\delta \in \text{Ker}(\Psi)$ if and only if $D^\delta$ is inner derivation on $T$. Hence $\text{Ker}(\Psi) = GN^1(M, M)$, by Theorem 2.8. Thus $H^1(T, T) \cong GH^1(M, M)$. \hfill \Box

**Example 2.11.** Suppose that $A$ is unital and $M = A$. Then $B(A) = A$ and so $GH^1(A, A) \cong H^1\left(\begin{pmatrix} A & A \\ 0 & A \end{pmatrix}, \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}\right) = H^1(A, A)$, by Proposition 4.4 of [9]. In particular, every generalized derivation on a unital commutative semisimple Banach algebra [21], a unital simple $C^*$-algebra [20], or a von Neumann algebra [14] is generalized inner.

We have investigated the interrelation between generalized derivations on a Banach algebra and its ordinary derivations. We also studied generalized derivations on a Banach module in virtue of derivations on its associated triangular Banach algebra. Thus, we established a link
between two interesting research areas: Banach algebras and triangular algebras.

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**References**


Gholamreza Abbaspour Tabadkan
Dept. of Pure Math.
School of Mathematical Science
Damghan Univ. of Basic Sciences
36715-364, Damghan, Iran
and
Department of Mathematics
Ferdowsi University
P.O. Box 1159
Mashhad 91775, Iran
e-mail: abbaspour@dmu.ac.ir

Mohammad Sal Moslehian and Assadollah Niknam
Department of Mathematics
Ferdowsi University
P.O. Box 1159
Mashhad 91775, Iran
and
Centre of Excellence in Analysis on Algebraic Structures (CEAAS)
Ferdowsi University, Iran
e-mail: moslehian@ferdowsi.um.ac.ir
e-mail: niknam@math.um.ac.ir