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CONVERGENCE IN A SEQUENTIAL TWO STAGE DECISION MAKING PROCESS

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Abstract. We analyze a sequential decision making process, in which at each step the decision is made in two stages. In the first stage a partially optimal action is chosen, which allows the decision maker to learn how to improve it under the new environment. We show how inertia (cost of changing) may lead the process to converge to a routine where no further changes are made. We illustrate our scheme with some economic models.
Keywords: Sequential decision making, costs to change, convergence.
MSC(2010): 91B06, 47H10

1. **Introduction**

There are many sequential decision making processes in which, at each step, the decision is made in two stages. In the first stage, the decision maker chooses a partially optimal solution, after the implementation of which she learns about the new environment and uses this learning to improve the current solution. After presenting our mathematical model in the next section, we will give several examples in which this situation occurs and fits our model very well. Here, we focus on the interlinked dynamic effects arising in this type of processes, an essential ingredient in our model being the inertia effects (costs to change). We present simple conditions under which the process converges, leading to a routine action choice in the limit. We also discuss some real world situations to which our model applies.

The rest of the paper is organized as follows. Section 2 presents the model and illustrates it by means of some examples. Section 3 presents a convergence theorem. In Section 4, we summarize our conclusions. Our convergence theorem relies upon a new Caristi type lemma.

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2. **The model**

We consider a decision maker who wants to maximize a utility function $U : X \longrightarrow \mathbb{R}$ over a decision space X. We assume that $X \subseteq X_1 \times X_2$, with the sets X_1 and X_2 consisting of partial decisions belonging to two different categories. Each decision $x = (x_1, x_2)$ has two components; the first one, $x_1 \in X_1$, can be implemented at no cost, whereas to implement the second component, $x_2 \in X_2$, one incurs a cost which depends on the amount of change imposed on this variable relative to the preceding decision. We assume that, for every given $x_1 \in X_1$, the problem of maximizing $U(x_1, x_2)$ over the set $F(x_1) := \{x_2 \in X_2 : (x_1, x_2) \in X\}$ is (relatively) easy to solve. Thus, what makes the decision problem difficult is the choice of $x_1 \in X_1$, and we assume that there is a nonnegative cost $C(x_1, x_1')$ of changing from a given $x_1 \in X$ to a new $x_1' \in X$. We assume that the costs are measured in the same units as the utility function.

Notice that the decision maker may be an individual or some relatively complex organization. In the latter case, the tasks of choosing $x_1 \in X_1$ and $x_2 \in X_2$ may be made by different parts within the organization.

We consider the following sequential decision making process. Given an initial $x_1 \in X_1$, the decision maker makes an optimal (relative to x_1) decision by maximizing $U(x_1, x_2)$ over $F(x_1)$. Let $x_2 \in X_2$ be an optimal solution to this maximization problem. The decision maker will search for the optimal $x_1 \in X_1$ relative to x_2 taking into account the costs to change, that is, she will maximize $U(x_1', x_2) - C(x_1, x_1')$ over the set $F^{-1}(x_2) := \{x_1' \in X_1 : (x_1', x_2) \in X\}$. She will then change x_1 accordingly, after which she will proceed to the next iteration, choosing an optimal $x_2 \in X_2$ relative to the new x_1, etc..

We devote the rest of this section to describe a few real world situations which fit into our model.

Consider first a consumer who first chooses a store x_1 from a set X_1 of available stores and then a commodity (say, a computer) x_2 from a set X_2 of commodities of the same type offered by the chosen store. The initial store may be the one closest to her home, and she will choose the optimal computer available at the store. After having decided which computer she wants to buy, she will search for the same computer in other stores, in order to find the store that offers the best conditions (cheaper price, better technical service, etc.). The choice of a new store involves costs of change, including, for instance, the ones incurred by traveling from the current store to the new one. Once an optimal store is chosen, the consumer will search for the optimal computer offered by the store; then she will search for the optimal store where this new computer is available, and so on.

As another example, consider an individual who has a job. Given this job and its associated income, the individual chooses an optimal standard of living.

Once this choice is made, she searches for the most convenient job that is compatible with her chosen way of life (a job closer to her home, a less demanding job, etc.). Changing the job entails some costs, like, for instance, adaptation costs. After changing her job, the individual aims at improving her standard of living relative to the new job, etc..

Another illustration comes from production theory. Let X_1 be a collection of technologies or production sets. We define the set X_2 as the union of all technologies, so that an element $x_2 \in X_2$ is a feasible production plan $(u, v) \in x_1$, for some $x_1 \in X_1$. The decision space is $X := \{(x_1, (u, v)) \in X_1 \times X_2 : (u, v) \in x_1\}$. Given a technology $x_1 \in X_1$, the producer chooses a production plan $(u, v) \in x_1$ so as to maximize its net profit over x_1. After this choice is made, she aims at improving the technology while keeping the chosen production plan feasible. Choosing a new technology is an innovation choice involving some costs of change. But this innovation allows for new production plans, so the producer then chooses the optimal production plan compatible with the new technology. This procedure can repeat indefinitely.

In our last example, the decision maker is a pair consisting of an organization and a set of individuals. The organization and the individuals share a common goal, specified by a utility function. The organization sets an environment x_1 (e.g., a rule), under which the individuals choose an optimal profile of actions x_2. After observing the chosen profile of actions, the organization improves the environment subject to the constraint that the chosen profile of actions be still feasible, taking into account the costs it will incur by changing the environment. Once a new environment is established, the individuals adapt to the new situation by choosing an optimal profile of actions compatible with the new environment. We thus have an iterative process fitting our scheme.

3. **Convergence of the iterative process**

Let as assume that the set X_1 has a metric space structure (X_1, d), the set X_2 is a compact topological space, the decision space X is a subset of $X_1 \times X_2$, the cost to change function $C : X_1 \times X_1 \longrightarrow \mathbb{R}_+$ is continuous and satisfies $C(x_1, x_1) = 0$, for all $x_1 \in X_1$, and the utility function $U : X \longrightarrow \mathbb{R}$ is continous and bounded from above.

The following technical assumptions will also be needed.

(1) There exists a subadditive, non decreasing, continuous function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying $\phi(0) = 0$, $\phi(t) > 0$, for $t > 0$, and $\phi[d(x_1, x_1')] \leq C(x_1, x_1')$, for all $x_1, x_1' \in X_1$. Note that $\phi \circ d$ is a metric inducing a topology equivalent to that induced by d.

(2) The correspondences $X_1 \ni x_1 \rightrightarrows F(x_1) \subseteq X_2$ and $X_2 \ni x_2 \rightrightarrows F^{-1}(x_2) \subseteq X_1$ defined by $F(x_1) := \{x_2 \in X_2 : (x_1, x_2) \in X\}$ and $F^{-1}(x_2) := \{x_1 \in X_1 : (x_1, x_2) \in X\}$, respectively, are compact-valued and continuous. Note that this continuity assumption holds, for instance, if $x_1 \in X_1$ and $x_2 \in X_2$ are independent decisions, that is, if every two such partial decisions are compatible.

Indeed, this amounts to saying that $X = X_1 \times X_2$, so that $F(x_1) = X_2$, for all $x_1 \in X_1$ and $F^{-1}(x_2) = X_1$, for all $x_2 \in X_2$.

(3) For every $x_1 \in X_1$, the function $U(x_1, \cdot)$ has a unique maximizer $Rx_1 := \arg\max_{x_2 \in F(x_1)} U(x_1, x_2)$. This hypothesis holds, for example, if X_2 is a convex subset of a topological vector space, for every $x_1 \in X_1$ the set $F(x_1)$ is convex, and U is strictly quasiconcave in its second argument. By Berge's maximum theorem [1], the mapping $R : X_1 \longrightarrow X_2$ is continuous.

Define $G : X_1 \longrightarrow \mathbb{R}$ by

$$G(x_1) := \max_{x_1' \in F^{-1}(Rx_1)} \{U(x_1', Rx_1) - C(x_1, x_1')\}. \tag{3.1}$$

This function assigns to each partial decision $x_1 \in X_1$ the maximum payoff the decision maker can get by changing it while keeping $x_2 = Rx_1$ unchanged, net of the cost to change.

(4) For every $x_1 \in X_1$, the maximization problem in (3.1) has a unique solution $Tx_1 \in X_1$. This hypothesis holds, for instance, if X_1 is a normed vector space, for every $x_2 \in X_2$ the set $F^{-1}(x_2)$ is convex, U is strictly quasiconcave in its first argument, and C is strictly convex in its second argument. By Berge's maximum theorem, $T : X \longrightarrow X$ is continuous.

The following lemma is similar to the Caristi fixed point theorem [2], but our assumptions are different: Instead of imposing a semicontinuity assumption on the function G, we just assume it to be bounded from above, but unlike the case of Caristi theorem we require the mapping T to be continuous.

Lemma 1. *Let (Y, d) be a complete metric space and $G : Y \longrightarrow \mathbb{R}$ be a bounded from above function. If $T : Y \longrightarrow Y$ is a continuous mapping such that $d(y, Ty) \leq G(Ty) - G(y)$, for each $y \in Y$, then, for any $y_0 \in Y$, the sequence $\{T^n y_0\}$ converges to a fixed point $\overline{y}$ of T.*

Proof. Since $G(T^{n+1}y) \geq G(T^n y) + d(T^n y, T^{n+1}y) \geq G(T^n y)$, the sequence $\{G(T^n y)\}$ is nondecreasing. As it is bounded from above, it is also convergent. For any $m, p \in \mathbb{N}$, one has

$$\begin{aligned} d(T^m y, T^{m+p} y) &\leq \Sigma_{i=0}^{p-1} d(T^{m+i} y, T^{m+i+1} y) \\ &\leq \Sigma_{i=0}^{p-1} \left[G(T^{m+i+1} y) - G(T^{m+i} y)\right] \\ &= G(T^{m+p} y) - G(T^m y). \end{aligned}$$

This proves that $\{T^n y\}$ is a Cauchy sequence. By the completeness of Y, this sequence converges to some point $\overline{y} \in Y$, which, as T is continuous, is a fixed point of T. □

Theorem 2. *For every $x_1^0 \in X_1$, the sequence $\left\{\left(T^n x_1^0, RT^n x_1^0\right)\right\}$ converges to $(\overline{x}_1, R\overline{x}_1)$, for some fixed point $\overline{x}_1 \in X_1$ of T.*

Proof. Since U is bounded from above and C is nonnegative, G is bounded from above, too. For each $x_1 \in X_1$, using that $Tx_1 \in F^{-1}(Rx_1)$ and $RTx_1 \in$

$F(Tx_1)$, that is, $Rx_1 \in F(Tx_1)$ and $Tx_1 \in F^{-1}(RTx_1)$, respectively, and that $C(Tx_1, Tx_1) = 0$, one sees that

$$\begin{aligned} \phi(d(x_1, Tx_1)) &\leq C(x_1, Tx_1) + \max_{x_2 \in F(Tx_1)} U(Tx_1, x_2) - U(Tx_1, Rx_1) \\ &= U(Tx_1, RTx_1) - G(x_1) \\ &\leq \max_{x_1' \in F^{-1}(RTx_1)} \left\{ U(x_1', RTx_1) - C(Tx_1, x_1') \right\} - G(x_1) \\ &= G(Tx_1) - G(x_1). \end{aligned}$$

Hence, by Lemma 1 applied to the metric space $(X_1, \phi \circ d)$, the sequence $\left\{T^n x_1^0\right\}$ converges to some fixed point $\overline{x}_1 \in X_1$ of T. Since R is continuous, it follows that the sequence $\left\{RT^n x_1^0\right\}$converges to $R\overline{x}_1$. Consequently, $\left\{\left(T^n x_1^0, RT^n x_1^0\right)\right\}$converges to $(\overline{x}_1, R\overline{x}_1)$. □

4. Conclusion

We examined a sequential two stages decision making process where the decision maker first makes a costly partial decision and then completes it in a costless optimal way. Our model took into account inertia, that is, costs to change, and was shown to converge to a stable decision under suitable assumptions. The main result essentially showed that high costs to change or, more specifically, costs which increase with the distance between partial decisions were enough to have convergence.

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