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IMAGE PROCESSING BY ALTERNATE DUAL GABOR FRAMES

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ABSTRACT. We present an application of the dual Gabor frames to image processing. Our algorithm is based on finding some dual Gabor frame generators which reconstructs accurately the elements of the underlying Hilbert space. The advantages of these duals constructed by a polynomial of Gabor frame generators are compared with their canonical dual.
Keywords: Gabor frame, dual frame, alternate dual frame, image processing.
MSC(2010): Primary: 42C15; Secondary: 62U10.

1. Introduction and preliminaries

A frame for a separable Hilbert space $\mathcal{H}$ is a sequence of vectors $\{f_i\}_{i=1}^{\infty}$ for which there are constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \qquad (f \in \mathcal{H}). \tag{1.1}$$

If the right-hand side of (1.1) holds, it is said to be a *Bessel* sequence. For a frame $\{f_i\}_{i=1}^{\infty}$ we define the frame operator $S : \mathcal{H} \to \mathcal{H}$ given by $Sf = \sum_{i=1}^{\infty}\langle f, f_i\rangle f_i$. This operator is bounded, invertible and positive. Two Bessel sequences $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ are said to be *dual frames* if

$$f = \sum_{i=1}^{\infty}\langle f, g_i\rangle f_i, \quad (f \in \mathcal{H}).$$

It can be shown that two such Bessel sequences indeed are frames. Every frame has at least one dual, which is called the *canonical dual* and is given by $\{S^{-1}f_i\}_{i=1}^{\infty}$, which is a frame with bounds B^{-1} and A^{-1}. A dual which is not

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the canonical dual is called an *alternate dual*, or simply a dual. Only frames which are not basis (redundant frames) have several duals.

In [1], it is shown how we can construct a sequence of alternate duals from a specific dual. Let $\{f_i\}_{i=1}^{\infty}$ be a frame for $\mathcal{H}$ with the frame operator S and a dual $\{g_i\}_{i=1}^{\infty}$. Put

$$g_i' = S^{-1}f_i - f_i + Sg_i, \tag{1.2}$$

and assume that $f \in \mathcal{H}$. Using the properties of the frame operator,

$$\begin{aligned}\sum_{i=1}^{\infty}\langle f, g_i'\rangle f_i &= \sum_{i=1}^{\infty}\langle f, S^{-1}f_i\rangle f_i - \sum_{i=1}^{\infty}\langle f, f_i\rangle f_i + \sum_{i=1}^{\infty}\langle f, Sg_i\rangle f_i \\ &= SS^{-1}f = f,\end{aligned}$$

so it follows that the sequence $\{g_i'\}_{i=1}^{\infty}$ is also a dual for $\{f_i\}_{i=1}^{\infty}$. The introductory courses on frames can be found in the books [5, 10]. The advantages of frames and their promising features in various application have attracted a lot of interest and effort in recent years. Furthermore, Gabor frames have been widely used in signal and image processing and many other parts of applied mathematics [2, 3, 7, 12–14, 16, 20]. For $f \in L^2(\mathbb{R})$, we define the modulation operator by $E_b f(x) = e^{2\pi ibx} f(x)$ and the translation operator by $T_a f(x) = f(x-a)$ where $a, b \in \mathbb{R}$. A Gabor frame is a frame for $L^2(\mathbb{R})$ of the form $\{E_{mb}T_{na}f\}_{m,n\in\mathbb{Z}}$ with the generator $f \in L^2(\mathbb{R})$ and $a, b > 0$. Various characterizations of Gabor frames have been given by Wexler and Raz [22], Daubechise et al. [9] and Ron and Shen [21]. It is well known that two Gabor frames $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ are called *dual* of each other if

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}h\rangle E_{mb}T_{na}g, \quad (f \in L^2(\mathbb{R})). \tag{1.3}$$

Although a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ when $ab < 1$ has infinitely many duals, the standard choice of h is $S^{-1}g$, where $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the frame operator of $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. There are several duality principles in Gabor frame theory [4, 15, 17]. In particular, an explicit construction of dual Gabor frames can be found in [6,8]. We claim that more precise results can be obtained by using different duals.

We end this section with an explicit expression for the canonical dual generator.

Proposition 1.1. *[6] Let $N \in \mathbb{N}$ and let $g \in L^2(\mathbb{R})$ be a function with support in $[0, N]$. Assume that $b \le \frac{1}{N}$ and that there exist $A, B > 0$ such that*

$$A \le G(x) := \sum_{n\in\mathbb{Z}} |g(x-na)|^2 \le B \quad a.e.\ x.$$

Then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, and the canonical dual generator is given by $S^{-1}g = \frac{b}{G}g$.

2. Tensor product of alternate duals

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces with orthonormal bases $\{e_\alpha\}_{\alpha\in I}$ and $\{u_\beta\}_{\beta\in J}$, respectively. Also let Q be a bounded antilinear map from $\mathcal{K}$ into $\mathcal{H}$, or equivalently, bounded linear operator from $\mathcal{K}^*$ into $\mathcal{H}$. Then $\sum_\beta \|Qu_\beta\|^2$ is independent of the choice of $\{u_\beta\}_{\beta\in J}$ and

$$\sum_\beta \|Qu_\beta\|^2 = \sum_\alpha \|Q^* e_\alpha\|^2. \tag{2.1}$$

Hence, we can define the Hilbert Schmidt norm of Q by $\|Q\|_{HS}^2 = \sum_\beta \|Qu_\beta\|^2$. The set of all antilinear maps $Q : \mathcal{K} \to \mathcal{H}$ such that $\|Q\|_{HS} < \infty$, denoted by $\mathcal{H} \otimes \mathcal{K}$, is a Hilbert space with the inner product

$$\langle Q, P\rangle = \sum_\beta \langle Qu_\beta, Pu_\beta\rangle.$$

If $u \in \mathcal{H}$ and $v \in \mathcal{K}$, the map $w \mapsto \langle v, w\rangle u$ $(w \in \mathcal{K})$ belongs to $\mathcal{H} \otimes \mathcal{K}$; we denote it by $u \otimes v$:

$$(u \otimes v)(w) = \langle v, w\rangle u.$$

It is easy to see that

$$\begin{aligned} \|u \otimes v\| &= \|u\|\|v\|, \\ \langle u \otimes v, u' \otimes v'\rangle &= \langle u, u'\rangle\langle v, v'\rangle \end{aligned}$$

for all $u, u' \in \mathcal{H}$ and $v, v' \in \mathcal{K}$. Moreover, $\{e_\alpha \otimes u_\beta\}_{\alpha,\beta}$ is an orthonormal basis for $\mathcal{H} \otimes \mathcal{K}$. For example, the tensor product $L^1(\mathbb{R}) \otimes L^1(\mathbb{R})$ is isometrically isomorphic to $L^2(\mathbb{R}^2)$ where $(f \otimes g)(x, y) = f(x)g(y)$ for all $f, g \in L^1(\mathbb{R})$ and $x, y \in \mathbb{R}$. For $T \in B(\mathcal{H})$ and $U \in B(\mathcal{K})$ the tensor product of T and U, denoted with $T \otimes U$, is defined by

$$(T \otimes U)Q = TQU^*, \quad (Q \in \mathcal{H} \otimes \mathcal{K}).$$

It is shown that $\|T \otimes U\| = \|T\|\|U\|$ and $(T \otimes U)(u \otimes v) = Tu \otimes Uv$. For more details of these facts see Subsection 7.3 of [11].

We now review some basic facts about frames in the tensor product of Hilbert spaces. As usual in frame theory, we assume that the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ are separable with the orthonormal bases $\{e_i\}_{i=1}^\infty$ and $\{u_i\}_{i=1}^\infty$, respectively.

Lemma 2.1. *Let $\{f_i\}_{i=1}^\infty$ and $\{g_i\}_{i=1}^\infty$ be frames for $\mathcal{H}$ and $\mathcal{K}$, respectively. The sequence $\{f_i \otimes g_j\}_{i=1,j=1}^\infty$ is a frame for $\mathcal{H} \otimes \mathcal{K}$. In particular, the frame bounds of tensor product of two frames is the product of their frames bounds.*

Proof. Let $Q \in \mathcal{H} \otimes \mathcal{K}$. Then we have;

$$\begin{aligned}
\langle Q, f_i \otimes g_j \rangle &= \sum_k \langle Qe_k, (f_i \otimes g_j)e_k \rangle \\
&= \sum_k \langle Qe_k, f_i \rangle \langle e_k, g_j \rangle \\
&= \langle Q \sum_k \langle g_j, e_k \rangle e_k, f_i \rangle = \langle Qg_j, f_i \rangle.
\end{aligned}$$

Now suppose that A_1 and A_2 are the lower frame bounds of $\{f_i\}_{i=1}^{\infty}$ and $\{g_j\}_{j=1}^{\infty}$, respectively. So by (2.1) we obtain

$$\begin{aligned}
\sum_i \sum_j |\langle Q, f_i \otimes g_j \rangle|^2 &= \sum_i \sum_j |\langle Q, f_i \otimes g_j \rangle|^2 \\
&\geq A_1 \sum_j \|Qg_j\|^2 \\
&= A_1 \sum_k \sum_j |\langle g_j, Q^* u_k \rangle|^2 \\
&\geq A_1 A_2 \sum_k \|Q^* u_k\|^2 = A_1 A_2 \|Q\|_{HS}^2.
\end{aligned}$$

A similar argument works for the upper bounds. □

The following theorem summaries basic properties of tensor product frames [18].

Theorem 2.2. *Suppose $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ and $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{K}$ are frames with the frame operator S_1 and S_2, respectively.*

(1) $S = S_1 \otimes S_2$ is the frame operator of $\{f_i \otimes g_j\}_{i=1,j=1}^{\infty}$.
(2) $S^{-1} = S_1^{-1} \otimes S_2^{-1}$. In particular, the canonical dual of $\{f_i \otimes g_j\}_{i=1,j=1}^{\infty}$ is the tensor product of their canonical duals.

Our aim is to prove that by using alternate duals instead of the canonical duals we may obtain more accurate results. Hence, we first show that the above theorem is also true for alternate duals.

Theorem 2.3. *The tensor product of alternate duals of the frames is an alternate dual for their tensor product.*

Proof. Let $\{f_i'\}_{i=1}^{\infty}$ and $\{g_i'\}_{i=1}^{\infty}$ be alternate duals of two frames $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ and $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{K}$, respectively. The elements of $\mathcal{H} \otimes \mathcal{K}$ can be described by a pair of dual frames in $\mathcal{K}$. More precisely, for each $Q \in \mathcal{H} \otimes \mathcal{K}$ and $v \in \mathcal{K}$ we have

$$\begin{aligned} Qv &= Q\left(\sum_i \langle v, g_i\rangle g_i'\right) \\ &= \sum_i \langle g_i, v\rangle Q g_i' = \left(\sum_i Q g_i' \otimes g_i\right) v. \end{aligned}$$

This implies that

$$\begin{aligned} \sum_{i,j=1}^{\infty} \langle Q, f_i' \otimes g_j'\rangle (f_i \otimes g_j) &= \sum_{i,j=1}^{\infty} \langle Q g_j', f_i'\rangle (f_i \otimes g_j) \\ &= \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} \langle Q g_j', f_i'\rangle f_i\right) \otimes g_j \\ &= \sum_{j=1}^{\infty} \left(Q g_j' \otimes g_j\right) = Q. \end{aligned}$$

□

It is straightforward to show that the tensor product of Gabor frames form a Gabor frame of the tensor product Hilbert spaces, but the converse is not true in general [19]. It can be easily seen that if $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$, then $\{E_{m(b,b)}T_{n(a,a)}g \otimes g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R}^2)$.

3. Computational experiments

For frames generated by any compactly supported function g whose integer-translates form a partition of unity, e.g., a B-spline, Christensen and Kim constructed a class of dual frame generators, formed by linear combinations of translates of g [6, 8]:

Theorem 3.1. *Let $N \in \mathbb{N}$ and $b \in (0, \frac{1}{2N-1}]$. Let $g \in L^2(\mathbb{R})$ be a real-valued bounded function with $supp(g) \subseteq [0, N]$, for which*

$$\sum_{n\in\mathbb{Z}} g(x-n) = 1.$$

Then the functions h and k defined by

$$h(x) = bg(x) + 2b\sum_{n=1}^{N-1} g(x+n), \qquad k(x) = \sum_{n=-N+1}^{N-1} a_n g(x+n),$$

where

$$a_0 = b \quad a_n + a_{-n} = 2b, \quad n = 1, 2, ..., N-1,$$

generate two dual frames $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_n k\}_{m,n\in\mathbb{Z}}$ for Gabor frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$.

Now we examine Theorem 3.1 when the generator function g is a B-spline. Recall that the B-splines B_N, $N \in \mathbb{N}$, are given inductively by

$$B_1 = \chi_{[0,1]}, \quad B_{N+1} = B_N * B_1,$$

where $*$ denotes the usual convolution of two functions. The B-spline B_N has its support in the interval $[0, N]$. Furthermore, it is well known that the integer-translates of any B-spline form a partition of unity, see Theorem 6.1.1 of [5]. Thus, by Proposition 1.1, $\{E_{mb}T_nB_N\}_{m,n\in\mathbb{Z}}$ is a Gabor frame with the frame operator

$$Sf = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_nB_N\rangle E_{mb}T_nB_N, \quad (f \in L^2(\mathbb{R})), \tag{3.1}$$

where $N \in \mathbb{N}$ and $b \in (0, \frac{1}{2N-1}]$.

Using Theorem 3.1 on the B-splines and applying (1.2) we can construct two sequences $\{h_l\}_{l=1}^{\infty}$ and $\{k_l\}_{l=1}^{\infty}$ of dual generators

$$h_{l+1} = S^{-1}B_N - B_N + Sh_l, \quad l \in \mathbb{N}, \tag{3.2}$$

$$k_{l+1} = S^{-1}B_N - B_N + Sk_l, \quad l \in \mathbb{N}, \tag{3.3}$$

where $h_1 = h$ and $k_1 = k$ is given by Theorem 3.1. So, $\{E_{mb}T_nh_l\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_nk_l\}_{m,n\in\mathbb{Z}}$ are two alternate duals for $\{E_{mb}T_nB_N\}_{m,n\in\mathbb{Z}}$ for all $l \in \mathbb{N}$ and we may rewrite (1.3) as

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_nh_l\rangle E_{mb}T_nB_N = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_nk_l\rangle E_{mb}T_nB_N. \tag{3.4}$$

For each $f \in L^2(\mathbb{R})$ the finite terms of (3.4) can be considered as an estimation of f. The benefit of using the alternate duals based model (3.4) for approximation of signals has been discussed in [1].

Image processing is a growing field covering a wide range of techniques for the manipulation of digital images. There are a variety of methods available for getting the desired results. Most image-processing techniques involve treating the image as a two-dimensional signal and applying standard signal processing techniques to it. More precisely, the reconstruction formula (3.4) in two-dimension can be read as

$$M = \sum_{m,n\in\mathbb{Z}} \langle M, E_{m(b,b)}T_{(n,n)}g\rangle E_{m(b,b)}T_{(n,n)}B_N \otimes B_N, \tag{3.5}$$

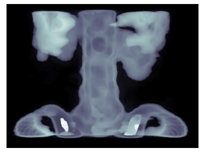
where $N \in \mathbb{N}$, $b \in (0, \frac{1}{2N-1}]$, M is an image matrix and g is one of the following tensor product functions:

$$h_l \otimes h_l, \quad h_l \otimes k_l, \quad k_l \otimes k_l, \quad (l \in \mathbb{N}).$$

Moreover, one may rewrite (3.5) with respect to the canonical as follows

$$M = \sum_{m,n\in\mathbb{Z}} \langle M, E_{m(b,b)}T_{(n,n)}S^{-1}B_N \otimes S^{-1}B_N\rangle E_{m(b,b)}T_{(n,n)}B_N \otimes B_N, \tag{3.6}$$

(A) tire



(B) spine

(C) peppers

FIGURE 1. Original images.

(A) Original

(B) canonical

(C) h_1

(D) h_2

(E) k_1

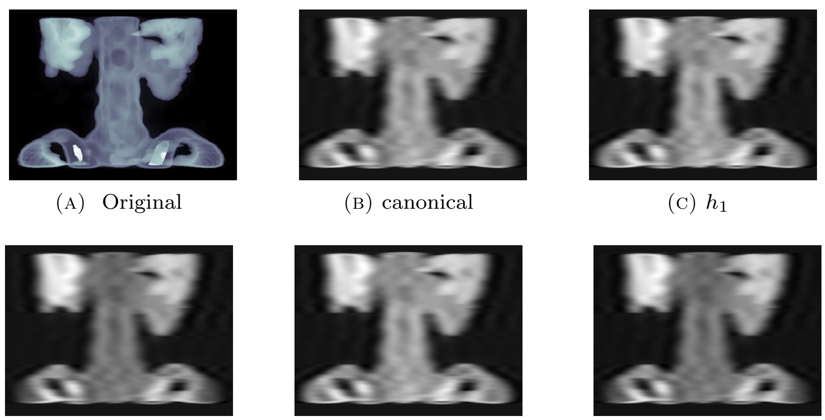
(F) k_2

FIGURE 2. Approximation of Image Tire by using different generators.

where S, given by (3.1), is the Gabor frame operator of $\{E_{mb}T_nB_N\}_{m,n\in\mathbb{Z}}$ and therefore $S^{-1}B_N = bB_N$ by Proposition 1.1. To demonstrate the benefit of dual Gabor frames presented in this paper, we have decomposed some test images shown in Figure 1. A finite terms of (3.5) and (3.6) can be considered as an estimation of the test image M when $m = -3, ..., 3$, $n = -50, ..., 50$, $N = 3$ and $b = 1/(2N-1)$. The estimations of test images by using five different generators are shown in Figures 2-4.

	AMSE		
Dual generator	spine	tire	peppers
canonical	0.1251	**0.0090**	**0.0052**
h_1	0.1250	**0.0090**	0.0054
k_1	0.1253	**0.0090**	0.0053
h_2	0.0947	0.0174	0.0151
k_2	0.0916	0.0171	0.0151
h_3	**0.0751**	0.0768	0.0276

TABLE 1. AMSEs for the approximation of test images obtained by (3.5) and (3.6)



(A) Original (B) canonical (C) h_1

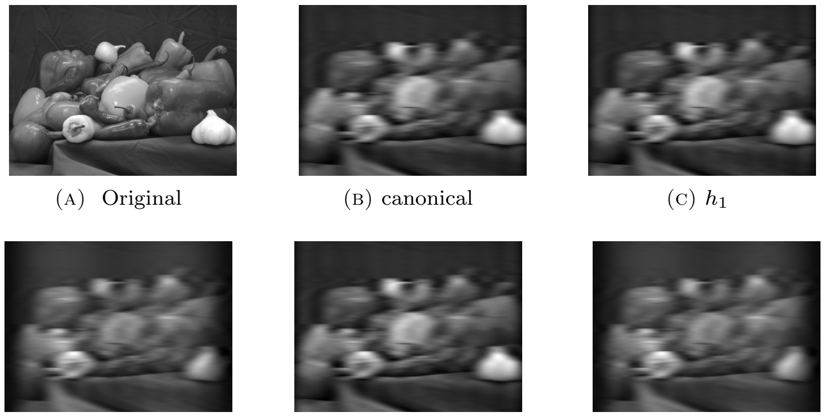
(D) h_2 (E) k_1 (F) k_2

FIGURE 3. Approximation of Image Spine by using different generators.

The performance of each estimator was measured by its average mean-square error (AMSE) defined as the average over simulated replications $\widehat{x}_i$ of

$$n^{-1}\sum_{i=1}^{n}|\widehat{x}_i - x_i|^2.$$

Table 1 presents the average mean-square errors for the obtained estimations of test images with different dual generators. The advantage of choosing a dual Gabor frame generator with the minimal AMSE is highlighted in Table 1.



(A) Original (B) canonical (C) h_1

(D) h_2 (E) k_1 (F) k_2

FIGURE 4. Approximation of Image Peppers by using different generators.

Finally, it is worthwhile to point out that this technique can be applied to any approximation method which uses Gabor frames, see for example face and marker detection algorithms introduced in [12].

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