ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 42 (2016), No. 6, pp. 1315–1330

Title:

A remark on the means of the number of divisors

Author(s):

M. Hassani

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 6, pp. 1315–1330
Online ISSN: 1735-8515

A REMARK ON THE MEANS OF THE NUMBER OF DIVISORS

M. HASSANI

(Communicated by Rahim Zaare-Nahandi)

Dedicated to Professor Jean-Marc Deshouillers, an inspiring mathematician

Abstract. We obtain the asymptotic expansion of the sequence with general term $\frac{A_n}{G_n}$, where A_n and G_n are the arithmetic and geometric means of the numbers $\mathrm{d}(1), \mathrm{d}(2), \dots, \mathrm{d}(n)$, with $\mathrm{d}(n)$ denoting the number of positive divisors of n. Also, we obtain some explicit bounds concerning G_n and $\frac{A_n}{G_n}$.
Keywords: Arithmetic function, arithmetic mean, geometric mean, growth of arithmetic functions.
MSC(2010): Primary: 11A25; Secondary: 11N56, 11N05.

1. Introduction and summary of the results

Assume that $(a_n)_{n\in\mathbb{N}}$ is a real sequence with $a_n > 0$. We will denote the arithmetic and geometric means of the numbers $a_1, a_2, \dots, a_n$, by $A(a_1, \dots, a_n)$ and $G(a_1, \dots, a_n)$, respectively. In this paper, we are motivated by a classical result asserting that

$$\lim_{n\to\infty} \frac{A(1,\dots,n)}{G(1,\dots,n)} = \frac{\mathrm{e}}{2}.$$

More precisely, by using Stirling's approximation for $n!$ one obtains

$$\frac{A(1,\dots,n)}{G(1,\dots,n)} = \frac{\mathrm{e}}{2} + O\Big(\frac{\log n}{n}\Big).$$

We refer the reader to [6] for more details. The ratio $\frac{\mathrm{e}}{2}$ appears surprisingly in studying the ratio of the arithmetic to the geometric means of several number theoretic sequences, including the sequence of prime numbers. More precisely, in [8] we proved that

$$\frac{A(p_1,\dots,p_n)}{G(p_1,\dots,p_n)} = \frac{\mathrm{e}}{2} + O\Big(\frac{1}{\log n}\Big),$$

Article electronically published on December 18, 2016.
Received: 14 April 2015, Accepted: 23 August 2015.

where p_n denotes the nth prime number. As further examples of this phenomenon, in [7] we showed that

$$\frac{A(\varrho_1, \ldots, \varrho_{\phi(n)})}{G(\varrho_1, \ldots, \varrho_{\phi(n)})} = \frac{\mathrm{e}}{2} + O\Big(\frac{\log n \log\log n}{n}\Big),$$

where $\{\varrho_1, \ldots, \varrho_{\phi(n)}\}$ is the least positive reduced set of residues modulo n, and in [5] we proved validity of the expansion

$$\frac{A(\gamma_1, \ldots, \gamma_n)}{G(\gamma_1, \ldots, \gamma_n)} = \frac{\mathrm{e}}{2}\left(1 - \frac{1}{2\log n} - \frac{\log\log n}{2\log^2 n} - \frac{1}{2\log^2 n}\right) + O\Big(\frac{(\log\log n)^2}{\log^3 n}\Big),$$

where $0 < \gamma_1 < \gamma_2 < \gamma_3 < \cdots$ denote the consecutive ordinates of the imaginary parts of non-real zeros of the Riemann zeta-function, which is defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\Re(s) > 1$, and extended by analytic continuation to the complex plane with a simple pole at $s = 1$.

On the other hand, we note that the appearance of the similar limit value $\frac{\mathrm{e}}{2}$ in the above results is not trivial and a global property. As an example, we consider the asymptotic behaviour of the ratio under study for the values of the Euler function. By using the asymptotic expansions for $A(\phi(1), \ldots, \phi(n))$ and $G(\phi(1), \ldots, \phi(n))$ (see [14] for the arithmetic mean, and [10] for the geometric mean), we get

$$\frac{A(\phi(1), \ldots, \phi(n))}{G(\phi(1), \ldots, \phi(n))} = \frac{3\mathrm{e}}{\pi^2} \prod_p \left(1 - \frac{1}{p}\right)^{-\frac{1}{p}} + O\Big(\frac{\log n}{n}\Big),$$

where the product runs over all primes. This gives a limit value different from $\frac{\mathrm{e}}{2}$, for the case of Euler function.

In this note we study the asymptotic behaviour of the ratio $\frac{A_n}{G_n}$ as $n \to \infty$, where for the whole text of the paper we let

$$A_n := A\left(\mathrm{d}(1), \mathrm{d}(2), \ldots, \mathrm{d}(n)\right), \quad \text{and} \quad G_n := G\left(\mathrm{d}(1), \mathrm{d}(2), \ldots, \mathrm{d}(n)\right),$$

and

$$\mathrm{d}(n) = \sum_{\substack{d \mid n \\ d > 0}} 1,$$

denotes the number of positive divisors of n. While the asymptotic expansion of A_n is known in the literature, the function G_n has been less studied. In [3] an asymptotic expansion for $\log G_n$ has been suggested as special case of a general result. In the present paper, to approximate $\log G_n$ we develop an argument based on the average order of the omega function

$$\omega(k) = \sum_{p \mid k} 1,$$

which counts the number of distinct prime divisors of the positive integer k. This connection allows us to improve on the remainder of $\log G_n$, because there are some very good known results on the average value of $\omega(k)$. Let us write

$$\frac{1}{n}\sum_{k\leqslant n}\omega(k)=\log\log n+M+R(n), \tag{1.1}$$

where M is the Meissel–Mertens constant defined by

$$M=\gamma+\sum_{p}\left(\log\left(1-p^{-1}\right)+p^{-1}\right), \tag{1.2}$$

and γ refers to Euler's constant, and the sum runs over all primes. Hardy and Ramanujan [4] proved that $R(n)\ll\frac{1}{\log n}$, and Diaconis [1] improved on this approximation by showing that

$$R(n)=\sum_{j=1}^{m}\frac{a_j}{\log^j n}+O\Big(\frac{1}{\log^{m+1} n}\Big), \tag{1.3}$$

for each fixed $m\geqslant 1$ with the precise value $a_1=\gamma-1$, and ensuring that other coefficients a_j are computable constants. By utilizing the expansion (1.3) we will prove the following results.

Theorem 1.1. *Assume that M is the Meissel–Mertens constant, defined by (1.2), and β is the absolute constant defined by*

$$\beta=\sum_{\substack{p^\alpha\\ \alpha\geqslant 2}}\frac{1}{p^\alpha}\log\Big(1+\frac{1}{\alpha}\Big), \tag{1.4}$$

where the sum runs over all prime powers p^α with $\alpha\geqslant 2$. Then, for any fixed integer $m\geqslant 1$ one has

$$G_n=B\,(\log n)^{\log 2}\left(1+\sum_{k=1}^{m}\frac{b_k}{\log^k n}+O\Big(\frac{1}{\log^{m+1} n}\Big)\right), \tag{1.5}$$

where

$$B=\mathrm{e}^{\beta+M\log 2}, \tag{1.6}$$

and the coefficients b_k are computable constants, with $b_1=(\gamma-1)\log 2\approxeq -0.293$.

Approximation of A_n is related to the so called Dirichlet's divisor problem. By using Dirichlet's hyperbola method (for example see [13]) one obtains

$$A_n=\frac{2}{n}\sum_{k\leqslant\sqrt{n}}\left[\frac{n}{k}\right]-\frac{[\sqrt{n}]^2}{n}, \tag{1.7}$$

and this implies that

$$A_n = \log n + (2\gamma - 1) + O\Big(\frac{1}{\sqrt{n}}\Big). \tag{1.8}$$

Now, by dividing the right hand side of (1.8) by (1.5) we obtain the following asymptotic expansion for the ratio $\frac{A_n}{G_n}$.

Theorem 1.2. *For every fixed integer $m \geqslant 1$ one has*

$$\frac{A_n}{G_n} = B^{-1}(\log n)^{1-\log 2}\left(1 + \sum_{k=1}^{m}\frac{r_k}{\log^k n} + O\Big(\frac{1}{\log^{m+1} n}\Big)\right), \tag{1.9}$$

where B is the constant defined as in (1.6), and the coefficients r_k are computable constants, with $r_1 = 2\gamma - 1 + (1-\gamma)\log 2 \approxeq 0.448$.

Next, we follow the steps in deduction of the above results in numerical details to obtain an explicit bound for G_n and then for $\frac{A_n}{G_n}$. To do this, we require some explicit bounds for $R(n)$, which can be found in [9] asserting that the double side inequality

$$-\frac{3.8854}{\log n} < R(n) < \frac{1}{\log^2 n}, \tag{1.10}$$

is valid for any $n \geqslant 2$. Moreover, we need the following useful result.

Lemma 1.3. *For every real $\alpha > 1$ and every real $z > 1$ we have*

$$\sum_{p>z}\frac{1}{p^{\alpha}} < \frac{1}{(\alpha-1)z^{\alpha-1}\log z} + \frac{3(3\alpha-1)}{4(\alpha-1)z^{\alpha-1}\log^2 z}, \tag{1.11}$$

where the sum runs over primes p larger than z.

By using the above explicit bounds, we obtain the following explicit form of the expansion (1.5).

Lemma 1.4. *For every $n \geqslant 2$ we have*

$$B\,(\log n)^{\log 2}\Big(1 - \frac{2.958}{\log n}\Big) < G_n < B\,(\log n)^{\log 2}\Big(1 + \frac{0.695}{\log^2 n}\Big). \tag{1.12}$$

To get an explicit form of the expansion of (1.9), we also need an explicit bound concerning A_n, namely as follows.

Lemma 1.5. *For any $n \geqslant 1$ one has*

$$\log n + (2\gamma - 1) - \frac{6}{\sqrt{n}} < A_n < \log n + (2\gamma - 1) + \frac{6}{\sqrt{n}}. \tag{1.13}$$

Now, we are able to obtain the following explicit form of Theorem 1.2, providing sharp bounds for the ratio $\frac{A_n}{G_n}$.

Theorem 1.6. *For any $n \geqslant 2$ one has*

$$B^{-1}(\log n)^{1-\log 2}\Big(1+\frac{0.091}{\log n}\Big) < \frac{A_n}{G_n} < B^{-1}(\log n)^{1-\log 2}\Big(1+\frac{4.053}{\log n}\Big). \tag{1.14}$$

While the proof of the above asymptotic results follows some standard number theoretic methods, in order to prove explicit results we need to follow several computational steps. Hence we give proofs for all of them, separately. Before introducing the proofs, we give some remarks on the constants β and B.

Remark 1.7. We observe that

$$\beta = \sum_{k=2}^{\infty} P(k)\log\Big(1+\frac{1}{k}\Big), \tag{1.15}$$

where

$$P(s) = \sum_{p}\frac{1}{p^s},$$

with the sum running over all primes is the prime zeta function defined for complex values of s with $\Re(s) > 1$. The convergence of the Euler product $\zeta(s) = \prod_p (1-p^{-s})^{-1}$ guarantees that $\zeta(s)$ does not vanish for $\Re(s) > 1$. Thus, by taking logarithm from both sides and utilizing the Maclaurin expansion of the logarithm function, we get

$$\log\zeta(s) = \sum_{m=1}^{\infty}\sum_{p}\frac{1}{mp^{ms}},$$

and consequently $P(s) < \log\zeta(s)$ is valid for each real $s > 1$. On the other hand, for every real $s > 1$ we have

$$\zeta(s) = 1 + \sum_{n=2}^{\infty}\frac{1}{n^s} < 1 + \int_1^{\infty}\frac{\mathrm{d}t}{t^s} = 1 + \frac{1}{s-1}. \tag{1.16}$$

Also, for every real $t > 0$ the inequality

$$\log(1+t) < t, \tag{1.17}$$

is valid. Hence, by using (1.16) and (1.17), we obtain $P(s) < \frac{1}{s-1}$ for any real $s > 1$, and also we get $\log(1+\frac{1}{k}) < k^{-1}$ for each $k > 0$. These bounds imply that $\beta < \sum_{k=2}^{\infty}(k(k-1))^{-1} = 1$, and this ensures that β defined as in (1.4) is indeed an absolute constant. Moreover, we observe that the series (1.15) converges rapidly. Also, it is known [2] that

$$M = \gamma + \sum_{k=2}^{\infty}\frac{\mu(k)\log\zeta(k)}{k},$$

and the later sum converges rapidly, too. Hence, by computation one may approximate the values of β and M, and then B, as follows

$$\begin{aligned} \beta &\approxeq 0.26194755498513799388260073634848006426660733053984, \\ M &\approxeq 0.26149721284764278375542683860869585905156664826120, \\ B &\approxeq 1.55768946444498105779234755862859827712273243065993. \end{aligned}$$

All computations performed in this paper have been done using Maple and Mathematica.

2. **Proof of the asymptotic results**

Connecting $\log G_n$ to the mean value of $\omega(n)$. From here on we let

$$\ell(\alpha) = \log\left(1 + \frac{1}{\alpha}\right).$$

We consider the identity $n \log G_n = \sum_{k\leqslant n} \log \mathrm{d}(k)$, and we write

$$\begin{aligned} \sum_{k\leqslant n} \log \mathrm{d}(k) &= \sum_{k\leqslant n} \sum_{p^\alpha \| k} \log(\alpha+1) = \sum_{k\leqslant n} \sum_{\substack{p^\alpha | k \\ \alpha\geqslant 1}} (\log(\alpha+1) - \log\alpha) \\ &= \sum_{\substack{p^\alpha\leqslant n \\ \alpha\geqslant 1}} (\log(\alpha+1) - \log\alpha) \left[\frac{n}{p^\alpha}\right] = \sum_{\substack{p^\alpha\leqslant n \\ \alpha\geqslant 1}} \ell(\alpha) \left[\frac{n}{p^\alpha}\right] \\ &= (\log 2) \sum_{p\leqslant n} \left[\frac{n}{p}\right] + \sum_{\substack{p^\alpha\leqslant n \\ \alpha\geqslant 2}} \ell(\alpha) \left[\frac{n}{p^\alpha}\right]. \end{aligned}$$

The key point to connect $\log G_n$ and the mean value of $\omega(k)$ is the following

$$\sum_{k\leqslant n} \omega(k) = \sum_{k\leqslant n} \sum_{p|k} 1 = \sum_{p\leqslant n} \sum_{\substack{k\leqslant n \\ p|k}} 1 = \sum_{p\leqslant n} \left[\frac{n}{p}\right].$$

Also, we write

$$\sum_{\substack{p^\alpha\leqslant n \\ \alpha\geqslant 2}} \ell(\alpha) \left[\frac{n}{p^\alpha}\right] = \beta n - nR_1(n) - R_2(n),$$

where β is the absolute constant defined as in (1.4), and

$$R_1(n) = \sum_{\substack{p^\alpha > n \\ \alpha\geqslant 2}} \frac{\ell(\alpha)}{p^\alpha}, \qquad \text{and} \qquad R_2(n) = \sum_{\substack{p^\alpha\leqslant n \\ \alpha\geqslant 2}} \ell(\alpha) \left\{\frac{n}{p^\alpha}\right\}. \tag{2.1}$$

Thus, we obtain

$$\log G_n = \frac{\log 2}{n} \sum_{k\leqslant n} \omega(k) + \beta - R_0(n), \tag{2.2}$$

where

$$R_0(n) = R_1(n) + \frac{R_2(n)}{n}.$$

Proof of Theorem 1.1. By using the inequality (1.17) we get $\ell(\alpha) < \alpha^{-1}$ for each $\alpha > 0$. Hence

$$0 \leqslant R_1(n) \ll \sum_{\substack{p^\alpha > n \\ \alpha \geqslant 2}} \frac{1}{\alpha p^\alpha} = \sum_{\alpha=2}^{\infty} \sum_{p^\alpha > n} \frac{1}{\alpha p^\alpha}.$$

For sufficiently large values of n we write

$$\sum_{\alpha=2}^{\infty} \sum_{p^\alpha > n} \frac{1}{\alpha p^\alpha} \leqslant \sum_{2 \leqslant \alpha < \log n} \sum_{p^\alpha > n} \frac{1}{\alpha p^\alpha} + \sum_{\alpha \geqslant \log n} \sum_{p} \frac{1}{\alpha p^\alpha} := T_1(n) + T_2(n),$$

say. We note that $\pi(n) - \pi(n-1) = 1$ or 0, depending if n is prime or not. For any arbitrary sequences a_n and b_n, and for any positive integers M and N, the transformation

$$\sum_{n=M}^{N} a_n(b_{n+1} - b_n) = a_{N+1}b_{N+1} - a_M b_M - \sum_{n=M}^{N} b_{n+1}(a_{n+1} - a_n),$$

is known as summation by parts (see [15], page 2891). We take in this formula $a_n = \frac{1}{n^\alpha}$, $b_n = \pi(n-1)$, $M = [z] + 1$ and also we let $N \to \infty$. Hence, by considering the approximation $\pi(x) \ll \frac{x}{\log x}$ and assuming that $\alpha > 1$ we imply

$$\begin{aligned}
\sum_{p > z} \frac{1}{p^\alpha} &= \sum_{n > z} \frac{\pi(n) - \pi(n-1)}{n^\alpha} = -\frac{\pi(z)}{([z]+1)^\alpha} - \sum_{n > z} \pi(n) \Big(\frac{1}{(n+1)^\alpha} - \frac{1}{n^\alpha} \Big) \\
&< \sum_{n > z} \pi(n) \Big(\frac{1}{n^\alpha} - \frac{1}{(n+1)^\alpha} \Big) \ll \sum_{n > z} \frac{\pi(n)}{n(n+1)^\alpha} \ll \sum_{n > z} \frac{1}{(n+1)^\alpha \log n} \\
&< \frac{1}{\log z} \sum_{n > z} \frac{1}{(n+1)^\alpha} \ll \frac{1}{\log z} \int_z^\infty \frac{\mathrm{d}t}{(t+1)^\alpha} \ll \frac{1}{z^{\alpha-1} \log z}.
\end{aligned}$$

This approximation implies that

$$\begin{aligned}
T_1(n) &= \sum_{2 \leqslant \alpha < \log n} \frac{1}{\alpha} \sum_{p > n^{\frac{1}{\alpha}}} \frac{1}{p^\alpha} \\
&\ll \sum_{2 \leqslant \alpha < \log n} \frac{1}{n^{1-\frac{1}{\alpha}} \log n} < \frac{1}{n^{\frac{1}{2}} \log n} \sum_{2 \leqslant \alpha < \log n} 1 \ll \frac{1}{n^{\frac{1}{2}}}.
\end{aligned}$$

To approximate $T_2(n)$ we write

$$T_2(n) = \sum_{p} \sum_{\alpha \geqslant \log n} \frac{1}{\alpha p^\alpha} < \frac{1}{\log n} \sum_{p} \sum_{\alpha \geqslant \log n} \frac{1}{p^\alpha} \ll \frac{1}{\log n} \sum_{p} \frac{1}{p^{\log n}}.$$

Now, we note that

$$\sum_p \frac{1}{p^{\log n}} = \frac{1}{2^{\log n}} + \sum_{p\geqslant 3} \frac{1}{p^{\log n}} \ll \frac{1}{2^{\log n}} + \int_2^\infty \frac{\mathrm{d}t}{t^{\log n}} \ll \frac{1}{2^{\log n}}.$$

Thus, we obtain

$$T_2(n) \ll \frac{1}{n^{\log 2}\log n}.$$

We combine the above approximations to get

$$R_1(n) \ll T_1(n) + T_2(n) \ll \frac{1}{n^{\frac{1}{2}}} + \frac{1}{n^{\log 2}\log n} \ll \frac{1}{n^{\frac{1}{2}}}. \tag{2.3}$$

To approximate $R_2(n)$ we write

$$0 \leqslant R_2(n) \ll \sum_{\substack{p^\alpha \leqslant n \\ \alpha\geqslant 2}} 1 = \sum_{\substack{p\leqslant n^{\frac{1}{\alpha}} \\ \alpha\geqslant 2}} 1 = \sum_{2\leqslant\alpha\leqslant\frac{\log n}{\log 2}} \pi(n^{\frac{1}{\alpha}})$$

$$\ll \sum_{2\leqslant\alpha\leqslant\frac{\log n}{\log 2}} \frac{n^{\frac{1}{\alpha}}}{\log n^{\frac{1}{\alpha}}} \leqslant \frac{n^{\frac{1}{2}}}{\log n} \sum_{2\leqslant\alpha\leqslant\frac{\log n}{\log 2}} \alpha \ll n^{\frac{1}{2}}\log n.$$

By using (2.3) and the last approximation we get $R_0(n) \ll \frac{\log n}{\sqrt{n}}$, and then by using (2.2) we deduce

$$\log G_n = \frac{\log 2}{n}\sum_{k\leqslant n}\omega(k) + \beta + O\Big(\frac{\log n}{\sqrt{n}}\Big).$$

Now we put (1.1) in the last equality, and also we apply the approximation (1.3) to obtain

$$\log G_n = (\log 2)\log\log n + (\beta + M\log 2) + \sum_{j=1}^m \frac{a_j\log 2}{\log^j n} + O\Big(\frac{1}{\log^{m+1} n}\Big), \tag{2.4}$$

for every fixed $m \geqslant 1$, where the coefficients a_j are the constants as in the expansion (1.3) with the precise value $a_1 = \gamma - 1$. Now we take exponent of both sides of (2.4) to get (1.5). For instance, we note that as $n \to \infty$, one has

$$\exp\left(\sum_{j=1}^m \frac{a_j\log 2}{\log^j n} + O\Big(\frac{1}{\log^{m+1} n}\Big)\right)$$

$$= 1 + \sum_{i=1}^m \frac{1}{i!}\Big(\sum_{j=1}^m \frac{a_j\log 2}{\log^j n}\Big)^i + O\Big(\frac{1}{\log^{m+1} n}\Big)$$

$$= 1 + \sum_{k=1}^m \frac{b_k}{\log^k n} + O\Big(\frac{1}{\log^{m+1} n}\Big),$$

where the coefficients b_k are computable constants in terms of the coefficients a_j, with $b_1 = a_1 \log 2$. This completes the proof. Finally, we let $A_j = a_j \log 2$, and we list some more initial values of the coefficients b_k as follows

$$\begin{aligned}
b_2 &= \frac{1}{2!}A_1^2 + A_2,\\
b_3 &= \frac{1}{3!}A_1^3 + A_1A_2 + A_3,\\
b_4 &= \frac{1}{4!}A_1^4 + \frac{1}{2}A_1^2A_2 + \frac{1}{2}A_2^2 + A_1A_3 + A_4,\\
b_5 &= \frac{1}{5!}A_1^5 + \frac{1}{6}A_1^3A_2 + \frac{1}{2}A_1A_2^2 + \frac{1}{2}A_1^2A_3 + A_2A_3 + A_1A_4 + A_5.
\end{aligned}$$

Proof of Theorem 1.2. If we let

$$\mathcal{E}_m(n) := \sum_{j=1}^{m} \frac{b_j}{\log^j n} + O\Big(\frac{1}{\log^{m+1} n}\Big),$$

where the coefficients b_j are the coefficients as in (1.5), then by applying the expansions (1.8) and (1.5) we get

$$\frac{A_n}{G_n} = B^{-1}(\log n)^{1-\log 2} F_m(n),$$

where

$$F_m(n) = \left(1 + \frac{2\gamma - 1}{\log n} + O\Big(\frac{1}{\sqrt{n}\log n}\Big)\right)(1 + \mathcal{E}_m(n))^{-1}.$$

We have

$$\begin{aligned}
(1 + \mathcal{E}_m(n))^{-1} &= 1 + \sum_{i=1}^{m}(-1)^i\Big(\sum_{j=1}^{m}\frac{b_j}{\log^j n}\Big)^i + O\Big(\frac{1}{\log^{m+1} n}\Big)\\
&= 1 + \sum_{k=1}^{m}\frac{c_k}{\log^k n} + O\Big(\frac{1}{\log^{m+1} n}\Big),
\end{aligned}$$

where the coefficients c_k are computable constants in terms of the coefficients b_j (and consequently in terms of the coefficients a_j), with $c_1 = -b_1$. This gives

$$F_m(n) = 1 + \frac{c_1 + 2\gamma - 1}{\log n} + \sum_{k=2}^{m}\frac{c_k + (2\gamma - 1)c_{k-1}}{\log^k n} + O\Big(\frac{1}{\log^{m+1} n}\Big),$$

and by taking $r_1 = c_1 + 2\gamma - 1 = 2\gamma - 1 + (1-\gamma)\log 2$ and $r_k = c_k + (2\gamma - 1)c_{k-1}$ for $k \geqslant 2$, we obtain (1.9). Finally, we list some more initial values of the coefficients c_k as follows

$$\begin{aligned}
c_2 &= +b_1^2 - b_2,\\
c_3 &= -b_1^3 + 2b_1b_2 - b_3,\\
c_4 &= +b_1^4 - 3b_1^2b_2 + b_2^2 + 2b_1b_3 - b_4,\\
c_5 &= -b_1^5 + 4b_1^3b_2 - 3b_1b_2^2 - 3b_1^2b_3 + 2b_2b_3 + 2b_1b_4 - b_5.
\end{aligned}$$

3. **Proof of the explicit results**

Proof of Lemma 1.3. For every real α we write

$$\sum_{p>z} \frac{1}{p^\alpha} = \lim_{b\to\infty} \sum_{z<p\leqslant b} \frac{1}{p^\alpha}.$$

We set $\varpi(n)$ to be 1 when n is prime and 0 otherwise. We have $\sum_{n\leqslant x} \varpi(n) = \pi(x)$, and by partial summation (see [13], page 3) we obtain

$$\sum_{z<p\leqslant b} \frac{1}{p^\alpha} = \sum_{z<n\leqslant b} \frac{\varpi(n)}{n^\alpha} = \frac{\pi(b)}{b^\alpha} - \frac{\pi(z)}{z^\alpha} + \alpha \int_z^b \frac{\pi(t)}{t^{\alpha+1}} \mathrm{d}t.$$

Thus, for each real $\alpha > 1$

$$\sum_{p>z} \frac{1}{p^\alpha} = \alpha \int_z^\infty \frac{\pi(t)}{t^{\alpha+1}} \mathrm{d}t - \frac{\pi(z)}{z^\alpha}.$$

Theorem 1 of [12] asserts that

$$\pi(x) < \frac{x}{\log x}\Big(1 + \frac{3}{2\log x}\Big), \tag{3.1}$$

for each real $x > 1$. This implies

$$\begin{aligned}
\int_z^\infty \frac{\pi(t)}{t^{\alpha+1}} \mathrm{d}t &< \int_z^\infty \frac{1}{t^\alpha \log t}\Big(1 + \frac{3}{2\log t}\Big) \mathrm{d}t\\
&< \frac{1}{\log z}\Big(1 + \frac{3}{2\log z}\Big) \int_z^\infty \frac{1}{t^\alpha} \mathrm{d}t = \frac{1}{(\alpha-1)z^{\alpha-1}\log z}\Big(1 + \frac{3}{2\log z}\Big).
\end{aligned}$$

Theorem 1 of [12] also asserts validity of $\frac{x}{\log x}(1 + \frac{1}{2\log x}) < \pi(x)$ for each real $x \geqslant 59$. By using this

$$\frac{x}{\log x}\Big(1 - \frac{3}{4\log x}\Big) < \pi(x), \tag{3.2}$$

for each real $x > 1$. Thus, for $z > 1$ we get

$$-\frac{\pi(z)}{z^\alpha} < -\frac{1}{z^{\alpha-1}\log z}\Big(1 - \frac{3}{4\log z}\Big),$$

and consequently

$$\sum_{p>z}\frac{1}{p^{\alpha}}<\frac{\alpha}{(\alpha-1)z^{\alpha-1}\log z}\Big(1+\frac{3}{2\log z}\Big)-\frac{1}{z^{\alpha-1}\log z}\Big(1-\frac{3}{4\log z}\Big).$$

This gives (1.11), and completes the proof.

Explicit approximation of $R_1(n)$. We apply the inequality $\ell(\alpha)<\alpha^{-1}$, which is valid for each $\alpha>0$, to get

$$0\leqslant R_1(n)<\sum_{\substack{p^{\alpha}>n\\ \alpha\geqslant 2}}\frac{1}{\alpha p^{\alpha}}=\sum_{\alpha=2}^{\infty}\sum_{p^{\alpha}>n}\frac{1}{\alpha p^{\alpha}}.$$

We let $n>\mathrm{e}^6$, and we write

$$\sum_{\alpha=2}^{\infty}\sum_{p^{\alpha}>n}\frac{1}{\alpha p^{\alpha}}\leqslant\sum_{2\leqslant\alpha<\log n}\sum_{p^{\alpha}>n}\frac{1}{\alpha p^{\alpha}}+\sum_{\alpha\geqslant\log n}\sum_{p}\frac{1}{\alpha p^{\alpha}}:=\Sigma_1(n)+\Sigma_2(n),$$

say. We utilize the bound (1.11) to get

$$\begin{aligned}\Sigma_1(n)&=\sum_{2\leqslant\alpha<\log n}\frac{1}{\alpha}\sum_{p>n^{\frac{1}{\alpha}}}\frac{1}{p^{\alpha}}\\&<\sum_{2\leqslant\alpha<\log n}\Big(\frac{1}{(\alpha-1)n^{1-\frac{1}{\alpha}}\log n}+\frac{3\alpha(3\alpha-1)}{4(\alpha-1)n^{1-\frac{1}{\alpha}}\log^2 n}\Big)\\&<\frac{1}{n^{\frac{1}{2}}\log n}+\frac{15}{2n^{\frac{1}{2}}\log^2 n}+\frac{1}{n^{\frac{2}{3}}\log n}S_1(n)+\frac{3}{4n^{\frac{2}{3}}\log^2 n}S_2(n),\end{aligned}$$

where

$$S_1(n)=\sum_{3\leqslant\alpha<\log n}\frac{1}{\alpha-1},\qquad\text{and}\qquad S_2(n)=\sum_{3\leqslant\alpha<\log n}\frac{\alpha(3\alpha-1)}{\alpha-1}.$$

Since $\sum_{1\leqslant\alpha\leqslant y}\frac{1}{\alpha}<1+\log y$, we get $S_1(n)<\log\log n$. Also, for every real $y\geqslant 6$ we have $\sum_{3\leqslant\alpha<y}\frac{\alpha(3\alpha-1)}{\alpha-1}<\frac{3}{2}y^2+4y$. Since we have assumed that $n>\mathrm{e}^6$, it follows that $S_2(n)<\frac{3}{2}\log^2 n+4\log n$. Thus, for $n>\mathrm{e}^6$ we get

$$\Sigma_1(n)<\frac{1}{n^{\frac{1}{2}}\log n}+E_1(n),$$

where

$$E_1(n)=\frac{15}{2n^{\frac{1}{2}}\log^2 n}+\frac{\log\log n}{n^{\frac{2}{3}}\log n}+\frac{3}{4n^{\frac{2}{3}}\log^2 n}\Big(\frac{3}{2}\log^2 n+4\log n\Big).$$

To approximate $\Sigma_2(n)$ we write

$$\Sigma_2(n)=\sum_{p}\sum_{\alpha\geqslant\log n}\frac{1}{\alpha p^{\alpha}}<\frac{1}{\log n}\sum_{p}\sum_{\alpha\geqslant\log n}\frac{1}{p^{\alpha}}.$$

By using geometric sum, we get

$$\sum_{\alpha\geqslant\log n}\frac{1}{p^{\alpha}}=\sum_{\alpha=[\log n]+1}^{\infty}\frac{1}{p^{\alpha}}=\frac{1}{(p-1)p^{[\log n]}}\leqslant\frac{2}{p^{[\log n]+1}}<\frac{2}{p^{\log n}}.$$

Hence, we obtain

$$\Sigma_2(n)<\frac{2}{\log n}\sum_{p}\frac{1}{p^{\log n}}.$$

Now, we note that

$$\sum_{p}\frac{1}{p^{\log n}}=\frac{1}{2^{\log n}}+\sum_{p\geqslant 3}\frac{1}{p^{\log n}}<\frac{1}{2^{\log n}}+\int_2^{\infty}\frac{\mathrm{d}t}{t^{\log n}}=\frac{1}{2^{\log n}}\Big(1+\frac{2}{\log n-1}\Big).$$

Thus, for $n\geqslant \mathrm{e}$ we obtain $\Sigma_2(n)<E_2(n)$, with

$$E_2(n)=\frac{2}{n^{\log 2}\log n}\Big(1+\frac{2}{\log n-1}\Big).$$

We combine the above bounds to get

$$R_1(n)<\sum_{\alpha=2}^{\infty}\sum_{p^{\alpha}>n}\frac{1}{\alpha p^{\alpha}}\leqslant\Sigma_1(n)+\Sigma_2(n)<\frac{1}{n^{\frac{1}{2}}\log n}+E_1(n)+E_2(n), \tag{3.3}$$

for $n>\mathrm{e}^6$. Meanwhile, if we let $f_1(n)=(E_1(n)+E_2(n))\,n^{\frac{1}{2}}\log n$ then we have $f_1(n)=o(1)$ as $n\to\infty$. Also, $f_1(n)$ is strictly decreasing for $n>\mathrm{e}$. By computation we observe that $f_1(n)<4$, and consequently $R_1(n)<\frac{5}{\sqrt{n}\log n}$, is valid for $n\geqslant 37683$.

Explicit approximation of $R_2(n)$. We have

$$0\leqslant R_2(n)<\sum_{\substack{p^{\alpha}\leqslant n\\ \alpha\geqslant 2}}\ell(2)=\ell(2)\sum_{\substack{p\leqslant n^{\frac{1}{\alpha}}\\ \alpha\geqslant 2}}1=\ell(2)\sum_{2\leqslant\alpha\leqslant\frac{\log n}{\log 2}}\pi(n^{\frac{1}{\alpha}}).$$

By using (3.1), for $x>1$ we get

$$\pi(x)\leqslant\frac{x}{\log x}\Big(1+\frac{3}{2\log 2}\Big). \tag{3.4}$$

We let

$$c_2=\Big(1+\frac{3}{2\log 2}\Big)\ell(2). \tag{3.5}$$

By using (3.4) we obtain

$$0\leqslant R_2(n)<c_2\sum_{2\leqslant\alpha\leqslant\frac{\log n}{\log 2}}\frac{n^{\frac{1}{\alpha}}}{\log n^{\frac{1}{\alpha}}}=\frac{c_2}{\log n}\sum_{2\leqslant\alpha\leqslant\frac{\log n}{\log 2}}\alpha n^{\frac{1}{\alpha}}\leqslant\frac{c_2 n^{\frac{1}{2}}}{\log n}\sum_{2\leqslant\alpha\leqslant\frac{\log n}{\log 2}}\alpha.$$

Now, we apply the inequality $\sum_{2\leqslant\alpha\leqslant y}\alpha < \frac{y(y+1)}{2}$ to get

$$0 \leqslant R_2(n) < \frac{c_2}{2\log 2}n^{\frac{1}{2}}\log(2n) := E_3(n), \tag{3.6}$$

say, for any $n \geqslant 1$. Meanwhile, we note that the function $f_2(n) = \frac{c_2}{2\log 2}\frac{\log(2n)}{\log n}$ is strictly decreasing for $n > 1$, and hence $\lim_{n\to\infty} f_2(n) = \frac{c_2}{2\log 2} < f_2(n) \leqslant f_2(v)$ for each $n \geqslant v$. Thus, for any $n \geqslant 5439$ we obtain $0 \leqslant R_2(n) < n^{\frac{1}{2}}\log n$.

Explicit approximation of $R_0(n)$. By using (3.3) and (3.6) we obtain $0 \leqslant R_0(n) < E_0(n)$ for $n > \mathrm{e}^6$, where

$$E_0(n) = \frac{1}{n^{\frac{1}{2}}\log n} + E_1(n) + E_2(n) + \frac{E_3(n)}{n}.$$

We observe that the function $f_3(n) = \frac{E_0(n)\sqrt{n}}{\log n}$ is strictly decreasing for $n > \mathrm{e}$, and hence $f_3(n) < 1$ for $n \geqslant 394657 := n_0$, say. Thus $E_0(n) < \frac{\log n}{\sqrt{n}}$, and consequently

$$0 \leqslant R_0(n) < \frac{\log n}{\sqrt{n}}, \tag{3.7}$$

hold for every $n \geqslant n_0$.

Completing the proof of Lemma 1.4. We recall the value $n_0 = 394657$. By using (2.2) and (1.1) we write

$$\log G_n = (\log 2)\log\log n + (\beta + M\log 2) + E(n),$$

where

$$E(n) = R(n)\log 2 - R_0(n).$$

By applying the bounds (1.10) and (3.7) we get

$$-\Big(\frac{3.8854\log 2}{\log n} + \frac{\log n}{\sqrt{n}}\Big) < E(n) < \frac{\log 2}{\log^2 n},$$

for any $n \geqslant n_0$. We note that the inequality

$$\frac{3.8854\log 2}{\log n} + \frac{\log n}{\sqrt{n}} \leqslant \frac{\eta}{\log n}, \tag{3.8}$$

is equivalent by $\eta \geqslant 3.8854\log 2 + \frac{\log^2 n}{\sqrt{n}}$, and since the ratio $\frac{\log^2 n}{\sqrt{n}}$ is strictly decreasing for $n \geqslant 55$, hence we take $\eta = 3.8854\log 2 + \frac{\log^2 n_0}{\sqrt{n_0}} \approxeq 2.95746$, for which (3.8) holds for every $n \geqslant n_0$. Thus, for any $n \geqslant 394657$ we obtain

$$-\frac{2.958}{\log n} < E(n) < \frac{\log 2}{\log^2 n}.$$

Now we note that the function $f_4(n) = (\mathrm{e}^{\frac{\log 2}{\log^2 n}} - 1)\log^2 n$ is strictly decreasing for $n \geqslant \mathrm{e}$. Hence $f_4(n) \leqslant f_4(n_0) < 0.695$, and consequently $\mathrm{e}^{E(n)} < 1 + \frac{0.695}{\log^2 n}$, for each $n \geqslant n_0$. Also, the function $f_5(n) = (1 - \mathrm{e}^{-\frac{2.958}{\log n}}) \log n$ is strictly increasing for $n \geqslant \mathrm{e}$, and $\lim_{n\to\infty} f_5(n) = 2.958$. This implies the validity of $f_5(n) \leqslant 2.958$ for each $n \geqslant \mathrm{e}$. Hence $1 - \frac{2.958}{\log n} < \mathrm{e}^{E(n)}$ holds for every $n \geqslant n_0$. This completes the proof of both sides of (1.12) for every $n \geqslant n_0$. Now, we define

$$J_l(n) = \Big(\frac{G_n}{B\,(\log n)^{\log 2}} - 1\Big) \log n, \quad \text{and} \quad J_u(n) = \Big(\frac{G_n}{B\,(\log n)^{\log 2}} - 1\Big) \log^2 n.$$

By computation, we have

$$\min_{2\leqslant n\leqslant n_0} J_l(n) = J_l(47) \approxeq -0.4321 > -2.958,$$

which confirms validity of the left hand side of (1.12) for each $2 \leqslant n \leqslant n_0$. Also, we observe that $J_u(n) < 0$ for each $3 \leqslant n \leqslant n_0$, and $J_u(2) \approxeq 0.0819 < 0.695$. This confirms validity of the left hand side of (1.12) for each $2 \leqslant n \leqslant n_0$.

Proof of Lemma 1.5. We start from (1.7). We assume that $n > 1$, and we let $N = [\sqrt{n}]$. Hence

$$A_n = \frac{2}{n} \sum_{k=1}^{N} \left[\frac{n}{k}\right] - \frac{N^2}{n}.$$

We let $H_N = \sum_{k=1}^{N} \frac{1}{k}$, and we apply the double side inequality $y - 1 < [y] \leqslant y$ to write

$$2H_N - \frac{N^2 + 2N}{n} < A_n \leqslant 2H_N - \frac{N^2}{n}.$$

The inequality $N > \sqrt{n} - 1$ gives $N^2 > n - 2\sqrt{n}$, and so $-\frac{N^2}{n} < -1 + \frac{2}{\sqrt{n}}$. Also, from $N \leqslant \sqrt{n}$ we get $-\frac{N^2+2N}{n} > -1 - \frac{2}{\sqrt{n}}$. Thus

$$2H_N - 1 - \frac{2}{\sqrt{n}} < A_n < 2H_N - 1 + \frac{2}{\sqrt{n}}.$$

We use the Euler–Maclaurin summation formula (see for example [11], page 27) with $m = 1$ to get

$$|H_N - (\log N + \gamma)| \leqslant \frac{3N + 1}{6N^2},$$

for each $N \geqslant 1$. Thus, for each $n > 1$ we obtain

$$\log n + (2\gamma - 1) + f_6(n) < A_n < \log n + (2\gamma - 1) + f_7(n),$$

where $f_6(n) = 2\log(1 - \frac{1}{\sqrt{n}}) - \frac{3\sqrt{n}+1}{3(\sqrt{n}-1)^2} - \frac{2}{\sqrt{n}}$ and $f_7(n) = \frac{3\sqrt{n}+1}{3(\sqrt{n}-1)^2} + \frac{2}{\sqrt{n}}$. The function $g_6(n) = \sqrt{n} f_6(n)$ is strictly increasing for $n > 1$, and $g_6(22) \approxeq -5.9795 > -6$. Thus, $f_6(n) > -\frac{6}{\sqrt{n}}$ is valid for $n \geqslant 22$. Also, the function $g_7(n) = \sqrt{n} f_7(n)$ is strictly decreasing for $n > 1$, and $g_7(5) \approxeq 5.7604 < 6$. This implies that $f_7(n) < \frac{6}{\sqrt{n}}$ for any $n \geqslant 5$. Thus, we obtain validity of (1.13) for $n \geqslant 22$. Computations verify its validity for $1 \leqslant n \leqslant 21$, as well.

Proof of Theorem 1.6. We recall the value $n_0 = 394657$. By using the explicit bounds (1.12) and (1.13), we get

$$B^{-1}(\log n)^{1-\log 2} f_8(n) < \frac{A_n}{G_n} < B^{-1}(\log n)^{1-\log 2} f_9(n),$$

for any $n \geqslant n_0$ with

$$f_8(n) = \frac{\log n + (2\gamma - 1) - \frac{6}{\sqrt{n}}}{\log n + \frac{0.695}{\log n}}, \qquad \text{and} \qquad f_9(n) = \frac{\log n + (2\gamma - 1) + \frac{6}{\sqrt{n}}}{\log n - 2.958}.$$

The function $g_8(n) = (f_8(n) - 1)\log n$ is strictly increasing for $n \geqslant 4$, and hence $g_8(n) \geqslant g_8(n_0) \approxeq 0.0906 > 0.091$. This proves the left hand inequality of (1.14). By following a similar argument, we observe that the function $g_9(n) = (f_9(n) - 1)\log n$ is strictly decreasing for $n \geqslant 20$, and hence $g_9(n) \leqslant g_9(n_0) \approxeq 4.05218 < 4.053$. This completes the proof of both inequalities in (1.14) for every $n \geqslant n_0$. We define

$$K(n) = \Big(\frac{BA_n}{G_n(\log n)^{1-\log 2}} - 1\Big)\log n.$$

While Theorem 1.2 implies that

$$\lim_{n\to\infty} K(n) = 2\gamma - 1 + (1-\gamma)\log 2 \approxeq 0.448,$$

by computation we get

$$\min_{2\leqslant n\leqslant n_0} K(n) = K(389759) \approxeq 0.4880 > 0.091,$$

and

$$\max_{2\leqslant n\leqslant n_0} K(n) = K(12) \approxeq 0.7590 < 4.053,$$

confirming the validity of the left hand side and the right hand side of (1.14) for $2 \leqslant n \leqslant n_0$, respectively. This completes the proof.

Acknowledgements

I express my gratitude to Prof. Florian Luca for many helpful suggestions during the preparation of the paper. Also, I thank the anonymous referee(s) and to Prof. Terence Mills for careful reading of the manuscript and giving many valuable suggestions and corrections, which improved the presentation of the paper.

References

[1] P. Diaconis, Asymptotic expansions for the mean and variance of the number of prime factors of a number n, Technical Report No. 96, Department of Statistics, Stanford University, December 14, 1976.

[2] S. R. Finch, Mathematical Constants, Encyclopedia Math. Appl., 94, Cambridge Univ. Press, 2003.

[3] J. -M. De Koninck and F. Luca, Analytic Number Theory: Exploring the Anatomy of Integers, Grad. Stud. Math. 134, Amer. Math. Soc., Providence, 2012.
[4] G. Hardy and S. Ramanujan, The normal number of prime factors of a number n, *Q. J. Math.* **48** (1917) 76–92.
[5] M. Hassani, A note on the distribution of the non-trivial zeros of the Riemann zeta function, submitted.
[6] M. Hassani, On the arithmetic-geometric means of positive integers and the number e, *Appl. Math. E-Notes* **14** (2014) 250–255.
[7] M. Hassani, Restricted factorial and a remark on the reduced residue classes, *Appl. Math. E-Notes* **16** (2016) 244–250.
[8] M. Hassani, On the ratio of the arithmetic and geometric means of the prime numbers and the number e, *Int. J. Number Theory* **9** (2014), no. 6, 1593–1603.
[9] M. Hassani, Remarks on the number of prime divisors of integers, *Math. Inequal. Appl.* **16** (2013), no. 3, 843–849.
[10] M. Hassani, Uniform distribution modulo one of some sequences concerning the Euler function, *Rev. Un. Mat. Argentina* **54** (2013), no. 1, 55–68.
[11] A. M. Odlyzko, Asymptotic enumeration methods, in: R. L. Graham, M. Groetschel, and L. Lovasz (eds.), Handbook of Combinatorics, 2, pp. 1063–1229, Elsevier, 1995.
[12] J. B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois J. Math.* **6** (1962) 64–94.
[13] G. Tenenbaum, Introduction to Analytic and Probabilistic Number Theory, Cambridge Stud. Adv. Math. 46, Cambridge Univ. Press, 1995.
[14] A. Walfisz, Weylsche Exponentialsummen in der Neueren Zahlentheorie, Mathematische Forschungsberichte, XV. VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.
[15] E. W. Weisstein, CRC Concise Encyclopedia of Mathematics, Chapman & Hall/CRC, 2nd edition, Boca Raton, 2003.

(Mehdi Hassani) Department of Mathematics, University of Zanjan, University Blvd., 45371-38791, Zanjan, Iran.
E-mail address: mehdi.hassani@znu.ac.ir