Title:
A remark on the means of the number of divisors

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A REMARK ON THE MEANS OF THE NUMBER OF DIVISORS

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(Communicated by Rahim Zaare-Nahandi)

Dedicated to Professor Jean-Marc Deshouillers, an inspiring mathematician

Abstract. We obtain the asymptotic expansion of the sequence with general term \( \frac{A_n}{G_n} \), where \( A_n \) and \( G_n \) are the arithmetic and geometric means of the numbers \( d(1), d(2), \ldots, d(n) \), with \( d(n) \) denoting the number of positive divisors of \( n \). Also, we obtain some explicit bounds concerning \( G_n \) and \( A_n/G_n \).

Keywords: Arithmetic function, arithmetic mean, geometric mean, growth of arithmetic functions.


1. Introduction and summary of the results

Assume that \((a_n)_{n\in\mathbb{N}}\) is a real sequence with \( a_n > 0 \). We will denote the arithmetic and geometric means of the numbers \( a_1, a_2, \ldots, a_n \), by \( A(a_1, \ldots, a_n) \) and \( G(a_1, \ldots, a_n) \), respectively. In this paper, we are motivated by a classical result asserting that

\[
\lim_{n \to \infty} \frac{A(1, \ldots, n)}{G(1, \ldots, n)} = e \frac{e}{2}.
\]

More precisely, by using Stirling’s approximation for \( n! \) one obtains

\[
\frac{A(1, \ldots, n)}{G(1, \ldots, n)} = e \frac{e}{2} + O \left( \frac{\log n}{n} \right).
\]

We refer the reader to [6] for more details. The ratio \( e \frac{e}{2} \) appears surprisingly in studying the ratio of the arithmetic to the geometric means of several number theoretic sequences, including the sequence of prime numbers. More precisely, in [8] we proved that

\[
\frac{A(p_1, \ldots, p_n)}{G(p_1, \ldots, p_n)} = e \frac{e}{2} + O \left( \frac{1}{\log n} \right),
\]

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where \( p_n \) denotes the \( n \)th prime number. As further examples of this phenomenon, in [7] we showed that
\[
\frac{A(q_1, \ldots, q_{\phi(n)})}{G(q_1, \ldots, q_{\phi(n)})} = \frac{e}{2} + O\left(\frac{\log n \log \log n}{n}\right),
\]
where \( \{q_1, \ldots, q_{\phi(n)}\} \) is the least positive reduced set of residues modulo \( n \), and in [5] we proved validity of the expansion
\[
\frac{A(\gamma_1, \ldots, \gamma_n)}{G(\gamma_1, \ldots, \gamma_n)} = \frac{e}{2}\left(1 - \frac{1}{2\log n} - \frac{\log \log n}{2\log^2 n} - \frac{1}{2\log^2 n}\right) + O\left(\frac{(\log \log n)^2}{\log^3 n}\right),
\]
where \( 0 < \gamma_1 < \gamma_2 < \gamma_3 < \cdots \) denote the consecutive ordinates of the imaginary parts of non-real zeros of the Riemann zeta-function, which is defined by \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \) for \( \Re(s) > 1 \), and extended by analytic continuation to the complex plane with a simple pole at \( s = 1 \).

On the other hand, we note that the appearance of the similar limit value \( \frac{e}{2} \) in the above results is not trivial and a global property. As an example, we consider the asymptotic behaviour of the ratio under study for the values of the Euler function. By using the asymptotic expansions for \( A(\phi(1), \ldots, \phi(n)) \) and \( G(\phi(1), \ldots, \phi(n)) \) (see [14] for the arithmetic mean, and [10] for the geometric mean), we get
\[
\frac{A(\phi(1), \ldots, \phi(n))}{G(\phi(1), \ldots, \phi(n))} = \frac{3e}{\pi^2} \prod_p \left(1 - \frac{1}{p}\right)^{-\frac{1}{2}} + O\left(\frac{\log n}{n}\right),
\]
where the product runs over all primes. This gives a limit value different from \( \frac{e}{2} \), for the case of Euler function.

In this note we study the asymptotic behaviour of the ratio \( \frac{A_n}{G_n} \) as \( n \to \infty \), where for the whole text of the paper we let
\[
A_n := A(d(1), d(2), \ldots, d(n)), \quad \text{and} \quad G_n := G(d(1), d(2), \ldots, d(n)),
\]
and
\[
d(n) = \sum_{\substack{d|n \\text{d}} \ \text{d} > 0} 1,
\]
denotes the number of positive divisors of \( n \). While the asymptotic expansion of \( A_n \) is known in the literature, the function \( G_n \) has been less studied. In [3] an asymptotic expansion for \( \log G_n \) has been suggested as special case of a general result. In the present paper, to approximate \( \log G_n \) we develop an argument based on the average order of the omega function
\[
\omega(k) = \sum_{p|k} 1,
\]
which counts the number of distinct prime divisors of the positive integer \( k \). This connection allows us to improve on the remainder of \( \log G_n \), because there are some very good known results on the average value of \( \omega(k) \). Let us write

\[
\frac{1}{n} \sum_{k \leq n} \omega(k) = \log \log n + M + R(n),
\]

where \( M \) is the Meissel–Mertens constant defined by

\[
M = \gamma + \sum_p \left( \log \left( 1 - \frac{1}{p} \right) + \frac{1}{p} \right),
\]

and \( \gamma \) refers to Euler’s constant, and the sum runs over all primes. Hardy and Ramanujan [4] proved that \( R(n) \ll \frac{1}{\log n} \), and Diaconis [1] improved on this approximation by showing that

\[
R(n) = \sum_{j=1}^m a_j \log j n + O\left( \frac{1}{\log^{m+1} n} \right),
\]

for each fixed \( m \geq 1 \) with the precise value \( a_1 = \gamma - 1 \), and ensuring that other coefficients \( a_j \) are computable constants. By utilizing the expansion (1.3) we will prove the following results.

**Theorem 1.1.** Assume that \( M \) is the Meissel–Mertens constant, defined by (1.2), and \( \beta \) is the absolute constant defined by

\[
\beta = \sum_{\alpha \geq 2} \frac{1}{p^\alpha} \log \left( 1 + \frac{1}{\alpha} \right),
\]

where the sum runs over all prime powers \( p^\alpha \) with \( \alpha \geq 2 \). Then, for any fixed integer \( m \geq 1 \) one has

\[
G_n = B \left( \log n \right)^{\log 2} \left( 1 + \sum_{k=1}^m b_k \log k n + O\left( \frac{1}{\log^{m+1} n} \right) \right),
\]

where

\[
B = e^{\beta + M \log 2},
\]

and the coefficients \( b_k \) are computable constants, with \( b_1 = (\gamma - 1) \log 2 \approx -0.293 \).

Approximation of \( A_n \) is related to the so called Dirichlet’s divisor problem. By using Dirichlet’s hyperbola method (for example see [13]) one obtains

\[
A_n = \frac{2}{n} \sum_{k \leq \sqrt{n}} \left[ \frac{n}{k} \right] - \frac{[\sqrt{n}]^2}{n},
\]
and this implies that

\begin{equation}
A_n = \log n + (2\gamma - 1) + O\left(\frac{1}{\sqrt{n}}\right).
\end{equation}

Now, by dividing the right-hand side of (1.5) by (1.8) we obtain the following asymptotic expansion for the ratio $\frac{A_n}{G_n}$.

**Theorem 1.2.** For every fixed integer $m \geq 1$ one has

\begin{equation}
\frac{A_n}{G_n} = B^{-1}(\log n)^{1-\log 2} \left(1 + \sum_{k=1}^{m} \frac{r_k}{\log^k n} + O\left(\frac{1}{\log^{m+1} n}\right)\right),
\end{equation}

where $B$ is the constant defined as in (1.6), and the coefficients $r_k$ are computable constants, with $r_1 = 2\gamma - 1 + (1 - \gamma)\log 2 \approx 0.448$.

Next, we follow the steps in deduction of the above results in numerical details to obtain an explicit bound for $G_n$ and then for $\frac{A_n}{G_n}$. To do this, we require some explicit bounds for $R(n)$, which can be found in [9] asserting that the double side inequality

\begin{equation}
-3.8854 \frac{\log n}{\log n} < R(n) < \frac{1}{\log^2 n},
\end{equation}

is valid for any $n \geq 2$. Moreover, we need the following useful result.

**Lemma 1.3.** For every real $\alpha > 1$ and every real $z > 1$ we have

\begin{equation}
\sum_{p > z} \frac{1}{p^\alpha} < \frac{1}{(\alpha - 1)z^{\alpha-1} \log z} + \frac{3(3\alpha - 1)}{4(\alpha - 1)z^{\alpha-1} \log^2 z},
\end{equation}

where the sum runs over primes $p$ larger than $z$.

By using the above explicit bounds, we obtain the following explicit form of the expansion (1.5).

**Lemma 1.4.** For every $n \geq 2$ we have

\begin{equation}
B(\log n)^{\log 2} \left(1 - \frac{2.958}{\log n}\right) < G_n < B(\log n)^{\log 2} \left(1 + \frac{0.695}{\log^2 n}\right).
\end{equation}

To get an explicit form of the expansion of (1.9), we also need an explicit bound concerning $A_n$, namely as follows.

**Lemma 1.5.** For any $n \geq 1$ one has

\begin{equation}
\log n + (2\gamma - 1) - \frac{6}{\sqrt{n}} < A_n < \log n + (2\gamma - 1) + \frac{6}{\sqrt{n}}.
\end{equation}

Now, we are able to obtain the following explicit form of Theorem 1.2, providing sharp bounds for the ratio $\frac{A_n}{G_n}$. 
Theorem 1.6. For any $n \geq 2$ one has

\begin{equation}
(1.14) \quad B^{-1}(\log n)^{1-\log 2} \left(1 + \frac{0.091}{\log n}\right) < \frac{A_n}{G_n} < B^{-1}(\log n)^{1-\log 2} \left(1 + \frac{4.053}{\log n}\right).
\end{equation}

While the proof of the above asymptotic results follows some standard number theoretic methods, in order to prove explicit results we need to follow several computational steps. Hence we give proofs for all of them, separately. Before introducing the proofs, we give some remarks on the constants $\beta$ and $B$.

Remark 1.7. We observe that

\begin{equation}
(1.15) \quad \beta = \sum_{k=2}^{\infty} P(k) \log \left(1 + \frac{1}{k}\right),
\end{equation}

where

\[ P(s) = \sum_p \frac{1}{p^s}, \]

with the sum running over all primes is the prime zeta function defined for complex values of $s$ with $\Re(s) > 1$. The convergence of the Euler product $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$ guarantees that $\zeta(s)$ does not vanish for $\Re(s) > 1$. Thus, by taking logarithm from both sides and utilizing the Maclaurin expansion of the logarithm function, we get

\[ \log \zeta(s) = \sum_{m=1}^{\infty} \sum_p \frac{1}{mp^{ms}}, \]

and consequently $P(s) < \log \zeta(s)$ is valid for each real $s > 1$. On the other hand, for every real $s > 1$ we have

\begin{equation}
(1.16) \quad \zeta(s) = 1 + \sum_{n=2}^{\infty} \frac{1}{n^s} < 1 + \int_{1}^{\infty} \frac{dt}{t^s} = 1 + \frac{1}{s-1}.
\end{equation}

Also, for every real $t > 0$ the inequality

\begin{equation}
(1.17) \quad \log(1 + t) < t,
\end{equation}

is valid. Hence, by using (1.16) and (1.17), we obtain $P(s) < \frac{1}{s-1}$ for any real $s > 1$, and also we get $\log(1 + \frac{1}{k}) < k^{-1}$ for each $k > 0$. These bounds imply that $\beta < \sum_{k=2}^{\infty} (k(k-1))^{-1} = 1$, and this ensures that $\beta$ defined as in (1.4) is indeed an absolute constant. Moreover, we observe that the series (1.15) converges rapidly. Also, it is known [2] that

\[ M = \gamma + \sum_{k=2}^{\infty} \frac{\mu(k) \log \zeta(k)}{k}, \]
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and the later sum converges rapidly, too. Hence, by computation one may approximate the values of $\beta$ and $M$, and then $B$, as follows

$$
\beta \approx 0.26194755498513799388260073634848006426660733053984,
M \approx 0.26149721284764278375542683860869585905156664826120,
B \approx 1.55768946444498105779234755862859827712273243065993.
$$

All computations performed in this paper have been done using Maple and Mathematica.

2. Proof of the asymptotic results

**Connecting $\log G_n$ to the mean value of $\omega(n)$**. From here on we let

$$
\ell(\alpha) = \log \left(1 + \frac{1}{\alpha}\right).
$$

We consider the identity $n \log G_n = \sum_{k \leq n} \log d(k)$, and we write

$$
\sum_{k \leq n} \log d(k) = \sum_{k \leq n} \sum_{p || k} \log(\alpha + 1) = \sum_{k \leq n} \sum_{p^\alpha || k, \alpha > 1} (\log(\alpha + 1) - \log \alpha)
$$

$$
= \sum_{p^\alpha \leq n, \alpha > 1} (\log(\alpha + 1) - \log \alpha) \left[ \frac{n}{p^\alpha} \right] = \sum_{p^\alpha \leq n, \alpha > 1} \ell(\alpha) \left[ \frac{n}{p^\alpha} \right]
$$

$$
= (\log 2) \sum_{p \leq n} \left[ \frac{n}{p} \right] + \sum_{p^\alpha \leq n, \alpha > 1} \ell(\alpha) \left[ \frac{n}{p^\alpha} \right].
$$

The key point to connect $\log G_n$ and the mean value of $\omega(k)$ is the following

$$
\sum_{k \leq n} \omega(k) = \sum_{k \leq n} \sum_{p || k} 1 = \sum_{p \leq n} \sum_{k \leq n} 1 = \sum_{p \leq n} \left[ \frac{n}{p} \right].
$$

Also, we write

$$
\sum_{p^\alpha \leq n, \alpha > 2} \ell(\alpha) \left[ \frac{n}{p^\alpha} \right] = \beta n - nR_1(n) - R_2(n),
$$

where $\beta$ is the absolute constant defined as in (1.4), and

$$
(2.1) \quad R_1(n) = \sum_{p^\alpha > n, \alpha > 2} \ell(\alpha) \left\{ \frac{n}{p^\alpha} \right\}, \quad \text{and} \quad R_2(n) = \sum_{p^\alpha \leq n, \alpha > 2} \ell(\alpha) \left\{ \frac{n}{p^\alpha} \right\}.
$$

Thus, we obtain

$$
\log G_n = \frac{\log 2}{n} \sum_{k \leq n} \omega(k) + \beta - R_0(n),
$$

(2.2)
where
\[ R_0(n) = R_1(n) + \frac{R_2(n)}{n}. \]

**Proof of Theorem 1.1.** By using the inequality (1.17) we get \( \ell(\alpha) < \alpha^{-1} \) for each \( \alpha > 0 \). Hence
\[ 0 < R_1(n) \ll \sum_{\alpha > 2} \frac{1}{\alpha^\alpha} = \sum_{\alpha=2}^{\infty} \sum_{p^\alpha > n} \frac{1}{\alpha^\alpha}. \]

For sufficiently large values of \( n \) we write
\[ \sum_{\alpha=2}^{\infty} \sum_{p^\alpha > n} \frac{1}{\alpha^\alpha} \ll \sum_{\alpha=2}^{\infty} \sum_{p^\alpha > n} \frac{1}{\alpha^\alpha} + \sum_{\alpha > \log n} \sum_p \frac{1}{\alpha^\alpha} := T_1(n) + T_2(n), \]
say. We note that \( \pi(n) - \pi(n-1) = 1 \) or 0, depending if \( n \) is prime or not. For any arbitrary sequences \( a_n \) and \( b_n \), and for any positive integers \( M \) and \( N \), the transformation
\[ \sum_{n=M}^{N} a_n(b_{n+1} - b_n) = a_{N+1}b_{N+1} - a_Mb_M - \sum_{n=M}^{N} b_{n+1}(a_{n+1} - a_n), \]
is known as summation by parts (see [15], page 2891). We take in this formula \( a_n = \frac{1}{n^\alpha}, \ b_n = \pi(n-1), \ M = [z] + 1 \) and also we let \( N \to \infty \). Hence, by considering the approximation \( \pi(x) \ll \frac{x}{\log x} \) and assuming that \( \alpha > 1 \) we imply
\[ \sum_{p > z} \frac{1}{p^\alpha} = \sum_{n > z} \frac{\pi(n) - \pi(n-1)}{n^\alpha} = - \frac{\pi(z)}{(z+1)^\alpha} - \sum_{n > z} \pi(n) \left( \frac{1}{(n+1)^\alpha} - \frac{1}{n^\alpha} \right) \ll \sum_{n > z} \frac{\pi(n)}{n(n+1)^\alpha} \ll \sum_{n > z} \frac{1}{(n+1)^\alpha \log n} \approx \frac{1}{\log z} \sum_{n > z} \frac{1}{(n+1)^\alpha} \ll \frac{1}{\log z} \int_z^{\infty} \frac{dt}{(t+1)^\alpha} \ll \frac{1}{z^{\alpha-1} \log z}. \]

This approximation implies that
\[ T_1(n) = \sum_{\alpha < \log n} \frac{1}{\alpha} \sum_{p > \alpha n^\frac{1}{\alpha}} \frac{1}{p^\alpha} \ll \sum_{\alpha < \log n} \frac{1}{n^{1-\frac{1}{\alpha}}} \cdot \frac{1}{\log n} \sum_{2 < \alpha < \log n} \frac{1}{n^2} \ll \frac{1}{n^{1+\frac{1}{2}}}. \]

To approximate \( T_2(n) \) we write
\[ T_2(n) = \sum_p \sum_{\alpha > \log n} \frac{1}{\alpha^\alpha} \ll \frac{1}{\log n} \sum_p \sum_{\alpha > \log n} \frac{1}{p^\alpha} \ll \frac{1}{\log n} \sum_p \frac{1}{p^\alpha} \ll \frac{1}{\log n} \sum_p \frac{1}{p^{\alpha \log n}}. \]
Now, we note that
\[ \sum_p \frac{1}{p \log n} = \frac{1}{2 \log n} + \sum_{p > 3} \frac{1}{p \log n} \ll \frac{1}{2 \log n} + \int_2^\infty \frac{dt}{t \log n} \ll \frac{1}{2 \log n}. \]

Thus, we obtain
\[ T_2(n) \ll \frac{1}{n \log^2 \log n}. \]

We combine the above approximations to get
\[ R_1(n) \ll T_1(n) + T_2(n) \ll \frac{1}{n^{1/2}} + \frac{1}{n \log^2 \log n} \ll \frac{1}{n^{1/2}}. \]

To approximate \( R_2(n) \) we write
\[ 0 \leq R_2(n) \ll \sum_{p^\alpha \leq n, \alpha \geq 2} 1 = \sum_{p^{\alpha \log n/\log 2} \leq n, \alpha \geq 2} \pi(n^{1/\alpha}) \]
\[ \ll \sum_{2 \leq \alpha \leq \log_2 n} \frac{n^{1/\alpha}}{\log n} \leq \frac{n^{1/2}}{\log n} \sum_{2 \leq \alpha \leq \log_2 n} \alpha \ll n^{1/2} \log n. \]

By using (2.3) and the last approximation we get \( R_0(n) \ll \frac{\log n}{\sqrt{n}} \), and then by using (2.2) we deduce
\[ \log G_n = \frac{\log 2}{n} \sum_{k \leq n} \omega(k) + \beta + O\left(\frac{\log n}{\sqrt{n}}\right). \]

Now we put (1.1) in the last equality, and also we apply the approximation (1.3) to obtain
\[ \log G_n = (\log 2) \log \log n + (\beta + M \log 2) + \sum_{j=1}^{m} \frac{a_j}{\log^j n} + O\left(\frac{1}{\log^{m+1} n}\right), \]

for every fixed \( m \geq 1 \), where the coefficients \( a_j \) are the constants as in the expansion (1.3) with the precise value \( a_1 = \gamma - 1 \). Now we take exponent of both sides of (2.4) to get (1.5). For instance, we note that as \( n \to \infty \), one has
\[ \exp\left(\sum_{j=1}^{m} \frac{a_j \log 2}{\log^j n} + O\left(\frac{1}{\log^{m+1} n}\right)\right) \]
\[ = 1 + \sum_{i=1}^{m} \frac{1}{i!} \left(\sum_{j=1}^{m} \frac{a_j \log 2}{\log^j n}\right)^i + O\left(\frac{1}{\log^{m+1} n}\right) \]
\[ = 1 + \sum_{k=1}^{m} \frac{b_k}{\log^k n} + O\left(\frac{1}{\log^{m+1} n}\right), \]
where the coefficients $b_k$ are computable constants in terms of the coefficients $a_j$, with $b_1 = a_1 \log 2$. This completes the proof. Finally, we let $A_j = a_j \log 2$, and we list some more initial values of the coefficients $b_k$ as follows

\begin{align*}
b_2 &= \frac{1}{2!} A_1^2 + A_2, \\
b_3 &= \frac{1}{3!} A_1^3 + A_1 A_2 + A_3, \\
b_4 &= \frac{1}{4!} A_1^4 + \frac{1}{2} A_1^2 A_2 + \frac{1}{2} A_2^2 + A_1 A_3 + A_4, \\
b_5 &= \frac{1}{5!} A_1^5 + \frac{1}{6} A_1^2 A_2 + \frac{1}{2} A_1 A_3^2 + \frac{1}{2} A_2^2 A_3 + A_2 A_3 + A_1 A_4 + A_5.
\end{align*}

**Proof of Theorem 1.2.** If we let

\[ E_m(n) := \sum_{j=1}^{m} \frac{b_j}{\log^j n} + O\left(\frac{1}{\log^{m+1} n}\right), \]

where the coefficients $b_j$ are the coefficients as in (1.5), then by applying the expansions (1.8) and (1.5) we get

\[ \frac{A_n}{G_n} = B^{-1}(\log n)^{1-\log 2} F_m(n), \]

where

\[ F_m(n) = \left(1 + \frac{2\gamma - 1}{\log n} + O\left(\frac{1}{\sqrt{n \log n}}\right)\right) (1 + E_m(n))^{-1}. \]

We have

\[
(1 + E_m(n))^{-1} = 1 + \sum_{i=1}^{m} (-1)^i \left( \sum_{j=1}^{m} \frac{b_j}{\log^j n} \right)^i + O\left(\frac{1}{\log^{m+1} n}\right)
\]

\[ = 1 + \sum_{k=1}^{m} \frac{c_k}{\log^k n} + O\left(\frac{1}{\log^{m+1} n}\right), \]

where the coefficients $c_k$ are computable constants in terms of the coefficients $b_j$ (and consequently in terms of the coefficients $a_j$), with $c_1 = -b_1$. This gives

\[ F_m(n) = 1 + \frac{c_1 + 2\gamma - 1}{\log n} + \sum_{k=2}^{m} \frac{c_k + (2\gamma - 1)c_{k-1}}{\log^k n} + O\left(\frac{1}{\log^{m+1} n}\right), \]

and by taking $r_1 = c_1 + 2\gamma - 1 = 2\gamma - 1 + (1-\gamma) \log 2$ and $r_k = c_k + (2\gamma - 1)c_{k-1}$ for $k \geq 2$, we obtain (1.9). Finally, we list some more initial values of the
coefficients $c_k$ as follows

$$
\begin{align*}
  c_2 &= +b_1^2 - b_2, \\
  c_3 &= -b_1^3 + 2b_1b_2 - b_3, \\
  c_4 &= +b_1^4 - 3b_1^2b_2 + b_2^2 + 2b_1b_3 - b_4, \\
  c_5 &= -b_1^5 + 4b_1^3b_2 - 3b_1b_2^2 - 3b_1^2b_3 + 2b_2b_3 + 2b_1b_4 - b_5.
\end{align*}
$$

3. Proof of the explicit results

**Proof of Lemma 1.3.** For every real $\alpha$ we write

$$
\sum_{p > z} \frac{1}{p^\alpha} = \lim_{b \to \infty} \sum_{z < p \leq b} \frac{1}{p^\alpha}.
$$

We set $\varpi(n)$ to be 1 when $n$ is prime and 0 otherwise. We have $\sum_{n \leq x} \varpi(n) = \pi(x)$, and by partial summation (see [13], page 3) we obtain

$$
\sum_{z < p \leq b} \frac{1}{p^\alpha} = \sum_{z < n \leq b} \frac{\varpi(n)}{n^\alpha} = \frac{\pi(b)}{b^\alpha} - \frac{\pi(z)}{z^\alpha} + \alpha \int_z^b \frac{\pi(t)}{t^{\alpha+1}} dt.
$$

Thus, for each real $\alpha > 1$

$$
\sum_{p > z} \frac{1}{p^\alpha} = \alpha \int_z^\infty \frac{\pi(t)}{t^{\alpha+1}} dt - \frac{\pi(z)}{z^\alpha}.
$$

Theorem 1 of [12] asserts that

$$
\pi(x) < \frac{x}{\log x} \left(1 + \frac{3}{2 \log x}\right),
$$

for each real $x > 1$. This implies

$$
\int_z^\infty \frac{\pi(t)}{t^{\alpha+1}} dt < \int_z^\infty \frac{1}{t^{\alpha} \log t} \left(1 + \frac{3}{2 \log t}\right) dt
$$

$$
< \frac{1}{\log z} \left(1 + \frac{3}{2 \log z}\right) \int_z^\infty \frac{1}{t^\alpha} dt = \frac{1}{(\alpha - 1)z^{\alpha-1} \log z} \left(1 + \frac{3}{2 \log z}\right).
$$

Theorem 1 of [12] also asserts validity of

$$
\frac{x}{\log x} \left(1 + \frac{1}{2 \log x}\right) < \pi(x)
$$

for each real $x \geq 59$. By using this

$$
\frac{x}{\log x} \left(1 - \frac{3}{4 \log x}\right) < \pi(x),
$$

for each real $x > 1$. Thus, for $z > 1$ we get

$$
-\frac{\pi(z)}{z^\alpha} < -\frac{1}{z^{\alpha-1} \log z} \left(1 - \frac{3}{4 \log z}\right),
$$

On the means of the number of divisors 1324
and consequently
\[ \sum_{p > z} \frac{1}{p^\alpha} < \frac{\alpha}{(\alpha - 1)z^{\alpha - 1} \log z} \left(1 + \frac{3}{2 \log z}\right) - \frac{1}{z^{\alpha - 1} \log z} \left(1 - \frac{3}{4 \log z}\right). \]

This gives (1.11), and completes the proof.

**Explicit approximation of** $R_1(n)$. We apply the inequality $\ell(\alpha) < \alpha^{-1}$, which is valid for each $\alpha > 0$, to get
\[ 0 \leq R_1(n) = \sum_{p > n} \frac{1}{\alpha p^{\alpha}} = \sum_{\alpha=2}^{\infty} \sum_{p > n} \frac{1}{\alpha p^{\alpha}}. \]

We let $n > e^6$, and we write
\[ \sum_{\alpha=2}^{\infty} \sum_{p > n} \frac{1}{\alpha p^{\alpha}} \leq \sum_{\alpha < \log n} \sum_{p > n} \frac{1}{\alpha p^{\alpha}} + \sum_{\alpha \geq \log n} \sum_{p} \frac{1}{\alpha p^{\alpha}} := \Sigma_1(n) + \Sigma_2(n), \]
say. We utilize the bound (1.11) to get
\[ \Sigma_1(n) = \sum_{2 \leq \alpha < \log n} \frac{1}{\alpha} \sum_{p > n^\frac{1}{\alpha}} \frac{1}{p^{\alpha}} \leq \sum_{2 \leq \alpha < \log n} \left(\frac{1}{(\alpha - 1)n^{1 - \frac{1}{\alpha}} \log n} + \frac{3\alpha(3\alpha - 1)}{4(\alpha - 1)n^{1 - \frac{1}{\alpha}} \log^2 n}\right) \]
\[ \leq \frac{1}{n^2 \log n} + \frac{15}{2n^2 \log^3 n} + \frac{1}{n^2 \log n} S_1(n) + \frac{3}{4n^2 \log^2 n} S_2(n), \]
where
\[ S_1(n) = \sum_{3 \leq \alpha < \log n} \frac{1}{\alpha - 1}, \quad \text{and} \quad S_2(n) = \sum_{3 \leq \alpha < \log n} \frac{\alpha(3\alpha - 1)}{\alpha - 1}. \]

Since $\sum_{1 \leq \alpha \leq y} \frac{1}{\alpha} < 1 + \log y$, we get $S_1(n) < \log \log n$. Also, for every real $y \geq 6$ we have $\sum_{3 \leq \alpha < y} \frac{\alpha(3\alpha - 1)}{\alpha - 1} < \frac{3}{2} y^2 + 4y$. Since we have assumed that $n > e^6$, it follows that $S_2(n) < \frac{3}{2} \log^2 n + 4 \log n$. Thus, for $n > e^6$ we get
\[ \Sigma_1(n) < \frac{1}{n^2 \log n} + E_1(n), \]
where
\[ E_1(n) = \frac{15}{2n^2 \log^2 n} + \frac{\log \log n}{n^2 \log n} + \frac{3}{4n^2 \log^2 n} \left(\frac{3}{2} \log^2 n + 4 \log n\right). \]

To approximate $\Sigma_2(n)$ we write
\[ \Sigma_2(n) = \sum_{p} \sum_{\alpha \geq \log n} \frac{1}{\alpha p^{\alpha}} < \frac{1}{\log n} \sum_{p} \sum_{\alpha \geq \log n} \frac{1}{p^{\alpha}}. \]
By using geometric sum, we get
\[
\sum_{\alpha \geq \log n} \frac{1}{p^\alpha} = \sum_{\alpha = \lceil \log n \rceil + 1}^{\infty} \frac{1}{p^\alpha} = \frac{1}{(p - 1)p^{\lceil \log n \rceil}} \leq \frac{2}{p^{\lceil \log n \rceil + 1}} < \frac{2}{p^{\log n}}.
\]
Hence, we obtain
\[
\sum_{p} \frac{1}{p^{\log n}} < \frac{2}{\log n} \sum_{p} \frac{1}{p^{\log n}}.
\]
Now, we note that
\[
\sum_{p} \frac{1}{p^{\log n}} = \frac{1}{2^{\log n}} + \sum_{p \geq 3} \frac{1}{p^{\log n}} < \frac{1}{2^{\log n}} + \int_{2}^{\infty} \frac{dt}{t^{\log n}} = \frac{1}{2^{\log n}} \left(1 + \frac{2}{\log n - 1}\right).
\]
Thus, for \(n \geq e\) we obtain \(\Sigma_2(n) < E_2(n)\), with
\[
E_2(n) = \frac{2}{n^{\log 2} \log n} \left(1 + \frac{2}{\log n - 1}\right).
\]
We combine the above bounds to get
\[
R_1(n) < \sum_{\alpha = 2}^{\infty} \sum_{\alpha \geq \log n} \frac{1}{\alpha p^\alpha} \leq \Sigma_1(n) + \Sigma_2(n) < \frac{1}{n^{\log n}} + E_1(n) + E_2(n),
\]
for \(n > e^6\). Meanwhile, if we let \(f_1(n) = (E_1(n) + E_2(n)) n^{\frac{1}{2}} \log n\) then we have \(f_1(n) = o(1)\) as \(n \to \infty\). Also, \(f_1(n)\) is strictly decreasing for \(n > e\). By computation we observe that \(f_1(n) < 4\), and consequently \(R_1(n) < \frac{5}{\sqrt{\log n}}\), is valid for \(n \geq 37683\).

**Explicit approximation of \(R_2(n)\).** We have
\[
0 \leq R_2(n) < \sum_{\alpha \geq 2} \sum_{\alpha \leq n} \frac{1}{\alpha p^\alpha} = \ell(2) \sum_{\alpha \leq n} 1 = \ell(2) \sum_{2 \leq \alpha \leq \frac{\log n}{\log 2}} \pi(n^{\frac{1}{\alpha}}).
\]
By using (3.1), for \(x > 1\) we get
\[
\pi(x) \leq \frac{x}{\log x} \left(1 + \frac{3}{2 \log 2}\right).
\]
We let
\[
c_2 = \left(1 + \frac{3}{2 \log 2}\right) \ell(2).
\]
By using (3.4) we obtain
\[
0 \leq R_2(n) < c_2 \sum_{2 \leq \alpha \leq \frac{\log n}{\log 2}} \frac{n^{\frac{1}{\alpha}}}{\log n^{\frac{1}{\alpha}}} = \frac{c_2}{\log n} \sum_{2 \leq \alpha \leq \frac{\log n}{\log 2}} \alpha n^{\frac{1}{\alpha}} \leq \frac{c_2 n^{\frac{1}{3}}}{\log n} \sum_{2 \leq \alpha \leq \frac{\log n}{\log 2}} \alpha.
\]
Now, we apply the inequality $\sum_{2 \leq \alpha \leq y} \alpha < \frac{y(y+1)}{2}$ to get

\[(3.6) \quad 0 \leq R_2(n) < \frac{c_2}{2 \log 2} n^{1/2} \log(2n) := E_3(n),\]

say, for any $n \geq 1$. Meanwhile, we note that the function $f_2(n) = \frac{c_2}{2 \log 2} \frac{\log(2n)}{\log n}$ is strictly decreasing for $n > 1$, and hence $\lim_{n \to \infty} f_2(n) = \frac{c_2}{2 \log 2} < f_2(n) \leq f_2(\nu)$ for each $n \geq \nu$. Thus, for any $n \geq 5439$ we obtain $0 \leq R_2(n) < n^{1/2} \log n$.

**Explicit approximation of $R_0(n)$**. By using (3.3) and (3.6) we obtain $0 \leq R_0(n) < E_0(n)$ for $n > e^6$, where

\[E_0(n) = \frac{1}{n^{1/2} \log n} + E_1(n) + E_2(n) + \frac{E_3(n)}{n},\]

We observe that the function $f_3(n) = \frac{E_0(n) \sqrt{n}}{\log n}$ is strictly decreasing for $n > e$, and hence $f_3(n) < 1$ for $n \geq 394657 := n_0$, say. Thus $E_0(n) < \frac{\log n}{\sqrt{n}}$, and consequently

\[(3.7) \quad 0 \leq R_0(n) < \frac{\log n}{\sqrt{n}},\]

hold for every $n \geq n_0$.

**Completing the proof of Lemma 1.4**. We recall the value $n_0 = 394657$. By using (2.2) and (1.1) we write

\[
\log G_n = (\log 2) \log \log n + (\beta + M \log 2) + E(n),
\]

where

\[
E(n) = R(n) \log 2 - R_0(n).
\]

By applying the bounds (1.10) and (3.7) we get

\[-\left(\frac{3.8854 \log 2}{\log n} + \frac{\log n}{\sqrt{n}}\right) < E(n) < \frac{\log 2}{\log^2 n},\]

for any $n \geq n_0$. We note that the inequality

\[(3.8) \quad \frac{3.8854 \log 2}{\log n} + \frac{\log n}{\sqrt{n}} \leq \frac{\eta}{\log n},\]

is equivalent by $\eta \geq 3.8854 \log 2 + \frac{\log^2 n}{\sqrt{n}}$, and since the ratio $\frac{\log^2 n}{\sqrt{n}}$ is strictly decreasing for $n \geq 55$, hence we take $\eta = 3.8854 \log 2 + \frac{\log^2 n_0}{\sqrt{n_0}} \simeq 2.95746$, for which (3.8) holds for every $n \geq n_0$. Thus, for any $n \geq 394657$ we obtain

\[-\frac{2.958}{\log n} < E(n) < \frac{\log 2}{\log^2 n}.\]

Now we note that the function $f_4(n) = (e^{\log^2 n} - 1) \log^2 n$ is strictly decreasing for $n \geq e$. Hence $f_4(n) \leq f_4(n_0) < 0.695$, and consequently $e^{E(n)} < 1 + \frac{0.695}{\log^2 n}$,
for each \( n \geq n_0 \). Also, the function \( f_3(n) = (1 - e^{-\frac{2.958}{\log n}}) \log n \) is strictly increasing for \( n \geq e \), and \( \lim_{n \to \infty} f_3(n) = 2.958 \). This implies the validity of \( f_3(n) \leq 2.958 \) for each \( n \geq e \). Hence \( 1 - \frac{2.958}{\log n} < e^{E(n)} \) holds for every \( n \geq n_0 \).

This completes the proof of both sides of (1.12) for every \( n \geq n_0 \). Now, we define

\[
J_l(n) = \left( \frac{G_n}{B(\log n)^{\log 2}} - 1 \right) \log n, \quad \text{and} \quad J_u(n) = \left( \frac{G_n}{B(\log n)^{\log 2}} - 1 \right) \log^2 n.
\]

By computation, we have

\[
\min_{2 \leq n \leq n_0} J_l(n) = J_l(47) \approx -0.4321 > -2.958,
\]

which confirms validity of the left hand side of (1.12) for each \( 2 \leq n \leq n_0 \). Also, we observe that \( J_u(n) < 0 \) for each \( 3 \leq n \leq n_0 \), and \( J_u(2) \approx 0.0819 < 0.695 \).

This confirms validity of the left hand side of (1.12) for each \( 2 \leq n \leq n_0 \).

**Proof of Lemma 1.5.** We start from (1.7). We assume that \( n > 1 \), and we let \( N = \lceil \sqrt{n} \rceil \). Hence

\[
A_n = \frac{2}{n} \sum_{k=1}^{N} \left\lfloor \frac{n}{k} \right\rfloor - \frac{N^2}{n}.
\]

We let \( H_N = \sum_{k=1}^{N} \frac{1}{k} \), and we apply the double side inequality \( y - 1 < \lceil y \rceil \leq y \) to write

\[
2H_N - \frac{N^2 + 2N}{n} < A_n \leq 2H_N - \frac{N^2}{n}.
\]

The inequality \( N > \sqrt{n} - 1 \) gives \( N^2 > n - 2\sqrt{n} \), and so \( -\frac{N^2}{n} < -1 + \frac{2}{\sqrt{n}} \).

Also, from \( N \leq \sqrt{n} \) we get \( -\frac{N^2 + 2N}{n} > -1 - \frac{2}{\sqrt{n}} \). Thus

\[
2H_N - 1 - \frac{2}{\sqrt{n}} < A_n < 2H_N - 1 + \frac{2}{\sqrt{n}}.
\]

We use the Euler–Maclaurin summation formula (see for example [11], page 27) with \( m = 1 \) to get

\[
|H_N - (\log N + \gamma)| \leq \frac{3N + 1}{6N^2},
\]

for each \( N \geq 1 \). Thus, for each \( n > 1 \) we obtain

\[
\log n + (2\gamma - 1) + f_6(n) < A_n < \log n + (2\gamma - 1) + f_7(n),
\]

where \( f_6(n) = 2\log(1 - \frac{1}{\sqrt{n}}) - \frac{3\sqrt{n} + 1}{3(\sqrt{n} - 1)^2} - \frac{2}{\sqrt{n}} \) and \( f_7(n) = \frac{3\sqrt{n} + 1}{3(\sqrt{n} - 1)^2} + \frac{2}{\sqrt{n}} \).

The function \( g_6(n) = \sqrt{n}f_6(n) \) is strictly increasing for \( n > 1 \), and \( g_6(22) \approx -5.9795 > -6 \). Thus, \( f_6(n) > -\frac{6}{\sqrt{n}} \) is valid for \( n \geq 22 \). Also, the function \( g_7(n) = \sqrt{n}f_7(n) \) is strictly decreasing for \( n > 1 \), and \( g_7(5) \approx 5.7604 < 6 \). This implies that \( f_7(n) < \frac{6}{\sqrt{n}} \) for any \( n \geq 5 \). Thus, we obtain validity of (1.13) for \( n \geq 22 \). Computations verify its validity for \( 1 \leq n \leq 21 \), as well.
Proof of Theorem 1.6. We recall the value $n_0 = 394657$. By using the explicit bounds (1.12) and (1.13), we get

$$B^{-1}(\log n)^{1-\log^2} f_8(n) < \frac{A_n}{G_n} < B^{-1}(\log n)^{1-\log^2} f_9(n),$$

for any $n \geq n_0$ with

$$f_8(n) = \frac{\log n + (2\gamma - 1) - \frac{6}{\sqrt{n}}}{\log n + 0.695 \log n}, \quad \text{and} \quad f_9(n) = \frac{\log n + (2\gamma - 1) + \frac{6}{\sqrt{n}}}{\log n - 2.958}.$$

The function $g_8(n) = (f_8(n) - 1) \log n$ is strictly increasing for $n \geq 4$, and hence $g_8(n) \geq g_8(n_0) \approx 0.0906 > 0.091$. This proves the left hand inequality of (1.14). By following a similar argument, we observe that the function $g_9(n) = (f_9(n) - 1) \log n$ is strictly decreasing for $n \geq 20$, and hence $g_9(n) \leq g_9(n_0) \approx 4.05218 < 4.053$. This completes the proof of both inequalities in (1.14) for every $n \geq n_0$. We define

$$K(n) = \left( \frac{BA_n}{G_n(\log n)^{1-\log^2}} - 1 \right) \log n.$$

While Theorem 1.2 implies that

$$\lim_{n \to \infty} K(n) = 2\gamma - 1 + (1 - \gamma) \log 2 \approx 0.448,$$

by computation we get

$$\min_{2 \leq n \leq n_0} K(n) = K(389759) \approx 0.4880 > 0.091,$$

and

$$\max_{2 \leq n \leq n_0} K(n) = K(12) \approx 0.7590 < 4.053,$$

confirming the validity of the left hand side and the right hand side of (1.14) for $2 \leq n \leq n_0$, respectively. This completes the proof.

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References


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