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# CO-CENTRALIZING GENERALIZED DERIVATIONS ACTING ON MULTILINEAR POLYNOMIALS IN PRIME RINGS

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Abstract. Let $R$ be a noncommutative prime ring of characteristic different from 2, $U$ the Utumi quotient ring of $R$, $C$ $(= Z(U))$ the extended centroid of $R$. Let $0 \neq a \in R$ and $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$ which is noncentral valued on $R$. Suppose that $G$ and $H$ are two nonzero generalized derivations of $R$ such that $a(H(f(x))f(x) - f(x)G(f(x))) \in C$ for all $x = (x_1, \ldots, x_n) \in R^n$. In this paper, we prove that one of the following holds:

(1) $f(x_1, \ldots, x_n)^2$ is central valued on $R$ and there exist $b, p, q \in U$ such that $H(x) = px + xb$ for all $x \in R$, $G(x) = bx + xq$ for all $x \in R$ with $a(p - q) \in C$;
(2) there exist $p, q \in U$ such that $H(x) = px + xq$ for all $x \in R$, $G(x) = qx$ for all $x \in R$ with $ap = 0$;
(3) $f(x_1, \ldots, x_n)^2$ is central valued on $R$ and there exist $q \in U$, $\lambda \in C$ and an outer derivation $g$ of $U$ such that $H(x) = xq + \lambda x - g(x)$ for all $x \in R$, $G(x) = qx + g(x)$ for all $x \in R$, with $a \in C$;
(4) $R$ satisfies $s_4$ and there exist $b, p \in U$ such that $H(x) = px + xb$ for all $x \in R$, $G(x) = bx + xp$ for all $x \in R$.

**Keywords:** Prime ring, generalized derivation, extended centroid.
**MSC(2010):** Primary: 16W25; Secondary: 16N60, 16R50.

## 1. Introduction

Throughout this paper, $R$ always denotes a prime ring with center $Z(R)$, $C$ be the extended centroid of $R$ and $U$ be the Utumi quotient ring of $R$. For $x, y \in R$, the Lie commutator of $x, y$ is denoted by $[x, y]$ and defined by $[x, y] = xy - yx$. By a derivation $d$ of $R$ we mean an additive mapping $d : R \to R$ satisfying $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A derivation $d$ is called inner if $d(x) = [q, x]$ for all $x \in R$ for some $q \in U$. A derivation which is not inner is called an outer derivation. An additive subgroup $L$ of $R$ is said to be Lie

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ideal of $R$, if $[L,R] \subseteq L$. The standard polynomial identity $s_4$ in four variables is defined as $s_4(x_1,x_2,x_3,x_4) = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}$ where $(-1)^\sigma$ is $+1$ or $-1$ according to $\sigma$ being an even or odd permutation in symmetric group $S_4$.

A well known result proved by Posner [25], states that if for a derivation $d$ the commutators $[d(x),x] \in Z(R)$ for all $x \in R$ holds, then either $d=0$ or $R$ is commutative. Then many related generalizations of Posner's result have been obtained by a number of authors in literature. Brešar proved in [4] that if $d(x)x - xg(x) \in Z(R)$ for all $x \in R$, where $d$ and $g$ are derivations of $R$, then either $d=g=0$ or $R$ is commutative.

In [24], Niu and Wu studied the left annihilator of the set $\{d(u)u - u\delta(u) | u \in L\}$, where $d$ and $\delta$ are two derivations of $R$ and $L$ is a noncentral Lie ideal of $R$. They proved that if the annihilator is not zero, then $R$ satisfies $s_4$ and $d = -\delta$ are inner derivations of $R$.

In [5], Carini et al. studied the result of Niu and Wu [24] by replacing derivations with generalized derivations. An additive function $F : R \to R$ is called a generalized derivation of $R$, if there exists a derivation $d$ of $R$ such that $F(xy) = F(x)y + xd(y)$ for all $x,y \in R$. In [5], the authors proved that if $H$ and $G$ are two nonzero generalized derivations of a prime ring $R$ with char $(R) \neq 2$ and $L$ a noncentral Lie ideal of $R$ such that $a(H(u)u - uG(u)) = 0$ for all $u \in L$ and for some $0 \neq a \in R$, then one of the following holds: (1) there exist $b', c' \in U$ such that $G(x) = c'x$ and $H(x) = b'x + xc'$ with $ab' = 0$; (2) $R$ satisfies $s_4$ and there exist $b', c', q' \in U$ such that $G(x) = c'x + xq'$ and $H(x) = b'x + xc'$ with $a(b' - q') = 0$.

Lee and Shiue [19] extended the Bršar's result [4] taking $x$ from the set $A = \{f(x_1,\ldots,x_n) : x_1,\ldots,x_n \in I\}$, where $f(x_1,\ldots,x_n)$ is a noncentral polynomial over $C$ and $I$ a nonzero ideal of prime ring $R$. Lee and Shiue [19] proved that if for two derivations $d$ and $\delta$ of $R$, $d(x)x - x\delta(x) \in C$ holds for all $x \in A$ then either $d = \delta = 0$ or $d = -\delta$ and $f(x_1,\ldots,x_n)^2$ is central valued on $RC$ unless $char(R) = 2$ and $dim_C RC = 4$.

Recently, Argaç and De Filippis [1] studied the previous result [19] replacing derivations by generalized derivations and without considering central values. They proved the following:

*Let $K$ be a commutative ring with unity, $R$ be a noncommutative prime $K$-algebra with center $Z(R)$, $U$ be the Utumi quotient ring of $R$, $C = Z(U)$ the extended centroid of $R$, $I$ a nonzero ideal of $R$. Suppose that $f(x_1,\ldots,x_n)$ is a noncentral multilinear polynomial over $K$, $G$ and $H$ are two nonzero generalized derivations of $R$ such that $G(f(x))f(x) - f(x)H(f(x)) = 0$ for all $x = (x_1,\ldots,x_n) \in I^n$. Then one of the following holds:*

*(1) $f(x_1,\ldots,x_n)^2$ is central valued on $R$ and there exist $b, a \in U$ such that $G(x) = ax + xb$ for all $x \in R$, $H(x) = bx + xa$ for all $x \in R$;*

*(2) there exists $a \in U$ such that $G(x) = xa$ for all $x \in R$, $H(x) = ax$ for all $x \in R$;*

*(3) $char(R) = 2$ and $R$ satisfies $s_4$.*

In [11] De Filippis et al. studied the situation with left annihilator condition. They proved the following:

*Let $K$ be a commutative ring with unity, $R$ be a noncommutative prime $K$-algebra of characteristic different from 2, $U$ be the Utumi quotient ring of $R$, $C = Z(U)$ the extended centroid of $R$. Suppose that $f(x_1, \ldots, x_n)$ is a noncentral multilinear polynomial over $K$, $G$ and $H$ are two nonzero generalized derivations of $R$ and there exists $0 \neq a \in R$ such that $a(G(f(x))f(x) - f(x)H(f(x))) = 0$ for all $x = (x_1, \ldots, x_n) \in R^n$. Then one of the following holds:*

*(1) $f(x_1, \ldots, x_n)^2$ is central valued on $R$ and there exist $b', c', q' \in U$ such that $G(x) = b'x + xc'$ for all $x \in R$, $H(x) = c'x + xq'$ for all $x \in R$ with $a(b' - q') = 0$;*

*(2) there exist $b', c' \in U$ such that $G(x) = b'x + xc'$ for all $x \in R$, $H(x) = c'x$ for all $x \in R$ with $ab' = 0$.*

Recently, in [10] De Filippis and Dhara investigated the situation $G(f(x))f(x) - f(x)H(f(x)) \in C$ for all $x = (x_1, \ldots, x_n) \in I^n$ in prime ring and then determined the structure of the maps, where $G, H$ two generalized derivations of $R$ and $I$ a nonzero right ideal of $R$.

So it is natural to consider the situation $a(G(f(x))f(x) - f(x)H(f(x))) \in C$ for all $x = (x_1, \ldots, x_n) \in R^n$, where $G, H$ are two generalized derivations of $R$, $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$ and $0 \neq a \in R$. In the present paper, our main object is to investigate this situation.

## 2. **Main results**

First we fix the following remarks:

*Remark* 2.1. Let $R$ be a prime ring and $L$ a noncentral Lie ideal of $R$. If char $(R) \neq 2$, by [3, Lemma 1] there exists a nonzero ideal $I$ of $R$ such that $0 \neq [I, R] \subseteq L$. Moreover, if $R$ is a simple ring, then $[R, R] \subseteq L$.

*Remark* 2.2. It is well known that each generalized derivation of a prime ring $R$ can be uniquely extended to a generalized derivation of $U$, with the form $ax + \delta(x)$ for all $x \in U$, where $a \in U$ and $\delta$ is a derivation of $U$. We refer to [12, 18, 19].

In order to prove the main theorem, we need the following lemmas.

**Lemma 2.3.** *[10, Lemma 6] Let $R$ be a noncommutative prime ring with Utumi quotient ring $U$ and extended centroid $C$, and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over $C$, which is not central valued on $R$. Suppose that* *there exist* $a, w, b \in U$ *such that* $af(r)^2 + f(r)wf(r) + f(r)^2b \in C$ *for all* $r = (r_1, \ldots, r_n) \in R^n$. *Then one of the following holds:*

(1) $a, b, w \in C$ *with* $a + w + b = 0$;
(2) $w, a + b \in C$ *and* $f(x_1, \ldots, x_n)^2$ *is central valued on* $R$;
(3) *char* $(R) = 2$ *and* $R$ *satisfies* $s_4$;
(4) $R$ *satisfies* $s_4$, $a - b \in C$ *and* $w \in C$.

**Lemma 2.4.** *Let* $R = M_k(C)$, *where* $k \geq 2$ *be ring of all* $k \times k$ *matrices over the field* $C$ *of characteristic different from* 2. *Let* $a$ *be an invertible matrix in* $R$ *and* $f(x_1, \ldots, x_n)$ *a multilinear polynomial over* $C$ *which is not central valued on* $R$. *Suppose that* $b, c, p, q \in R$ *such that* $a(bf(x)^2 + f(x)(c-p)f(x) - f(x)^2q) \in C \cdot I_k$ *for all* $x = (x_1, \ldots, x_n) \in R^n$. *Then* $c - p \in C \cdot I_k$ *and one of the following holds:*

(1) $f(x_1, \ldots, x_n)^2$ *is central valued on* $R$ *and* $a(b + c - p - q) \in C \cdot I_k$;
(2) $b, q \in C \cdot I_k$ *and* $b + c - p - q = 0$;
(3) $k = 2$.

*Proof.* Since $f(x_1, \ldots, x_n)$ is not central valued on $R$, by [20] (see also [23]), there exist $u_1, \ldots, u_n \in M_k(C)$ and $\gamma \in C - \{0\}$ such that $f(u_1, \ldots, u_n) = \gamma e_{kl}$, with $k \neq l$. Moreover, since the set $\{f(r_1, \ldots, r_n) : r_1, \ldots, r_n \in M_m(C)\}$ is invariant under the action of all $C$-automorphisms of $M_k(C)$, then for any $i \neq j$ there exist $r_1, \ldots, r_n \in M_k(C)$ such that $f(r_1, \ldots, r_n) = \gamma e_{ij}$.

Thus, by our hypothesis we have, $a\gamma e_{ij}(c-p)\gamma e_{ij} \in C \cdot I_k$, that is, $a(c-p)_{ji}e_{ij} \in C \cdot I_k$. Since the rank of $a(c-p)_{ji}e_{ij}$ is at most one, $a(c-p)_{ji}e_{ij} = 0$. Again since $a$ is invertible, we have $(c-p)_{ji} = 0$ for any $i \neq j$. Thus, $(c-p)$ is a diagonal matrix. By using a standard argument, it follows that $c - p \in C \cdot I_k$.

In the sequel one may assume $k \geq 3$, if not the proof is finished. Then our hypothesis reduces to

$$a((b + c - p)f(x)^2 - f(x)^2q) \in C \cdot I_k \tag{2.1}$$

for all $x = (x_1, \ldots, x_n) \in R^n$. If $f(x_1, \ldots, x_n)^2$ is central valued on $R$, then by (2.1) and since $f(x_1, \ldots, x_n)$ cannot be an identity for $R$, the conclusion $a(b + c - p - q) \in C \cdot I_k$ follows easily.

Now we assume that $f(x_1, \ldots, x_n)^2$ is not central valued on $R$. Since the relation (2.1) holds for any element of the additive subgroup generated by the polynomial $f(x_1, \ldots, x_n)^2$, then by [9], it follows that $a((b+c-p)x - xq) \in C \cdot I_k$ for any $x \in L$, a noncentral Lie ideal of $R$. Since $a$ is a nonzero invertible element and $R = M_k(C)$, then $(b + c - p)x - xq$ is a zero or invertible matrix, for any $x \in L$. As a consequence of Theorem 3 and Theorem 1.2 in [22], and since $k \geq 3$, one has $q = b + c - p \in C \cdot I_k$, that is $b \in C \cdot I_k$ and $b + c - p - q = 0$, as required. □

**Lemma 2.5.** *Let $R$ be a noncommutative prime ring of characteristic different from 2, $U$ the Utumi quotient ring of $R$, $C$ the extended centroid of $R$. Let $0 \neq a \in R$ and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over $C$ which is not central valued on $R$. Suppose that $b, c, p, q \in R$ such that $a(bf(x)^2 + f(x)(c-p)f(x) - f(x)^2q) \in C$ for all $x = (x_1, \ldots, x_n) \in R^n$. Then $c - p \in C$ and one of the following holds:*

*(1) $f(x_1, \ldots, x_n)^2$ is central valued on $R$ and $a(b + c - p - q) \in C$;*
*(2) $q \in C$ and $a(b + c - p - q) = 0$;*
*(3) $R$ satisfies $s_4$.*

*Proof.* Let us assume that $a(bf(x)^2 + f(x)(c-p)f(x) - f(x)^2q) = 0$ for all $x = (x_1, \ldots, x_n) \in R^n$. Then by [11, Proposition 2.4], $c - p \in C$ and one of the following hold:

(1) $f(x_1, \ldots, x_n)^2$ is central valued on $R$ and $a(b + c - p - q) = 0$, which is our conclusion (1);

(2) $q \in C$ and $a(b + c - p - q) = 0$, which is our conclusion (2).

Next we assume that there exists $r = (r_1, \ldots, r_n) \in R^n$ such that $a(bf(r)^2 + f(r)(c-p)f(r) - f(r)^2q) \neq 0$. Since $a(bf(x)^2 + f(x)(c-p)f(x) - f(x)^2q) \in C$ is a nonzero central generalized identity for $R$, by [6, Theorem 1] $R$ is a PI- ring and hence $RC = U$ is a nontrivial GPI-ring simple with 1. By Lemma 2 in [14] and Theorem 2.3.29 in [26], there exists a field $E$ such that $U \subseteq M_k(E)$ and $U$ and $M_k(E)$ satisfy the same generalized identities. Thus $a(bf(x)^2 + f(x)(c-p)f(x) - f(x)^2q) \in Z(M_k(E))$ for all $x = (x_1, \ldots, x_n) \in (M_k(E))^n$. Since for some $r = (r_1, \ldots, r_n) \in U^n \subseteq (M_k(E))^n$, $a(bf(r)^2 + f(r)(c-p)f(r) - f(r)^2q) \neq 0$, $a$ must be invertible. Then by Lemma 2.4, $c - p \in Z(M_k(E))$ and one of the following holds:

(1) $f(x_1, \ldots, x_n)^2$ is central valued on $R$ and $a(b + c - p - q) \in C \cdot I_k$;
(2) $b, q \in C \cdot I_k$ and $b + c - p - q = 0$;
(3) $R \subseteq U \subseteq M_2(E)$, that is, $R$ satisfies $s_4$.

Thus conclusions (1) to (3) are obtained.

□

**Theorem 2.6.** *Let $R$ be a noncommutative prime ring of characteristic different from 2, $U$ the Utumi quotient ring of $R$, $C$ $(= Z(U))$ the extended centroid of $R$. Let $0 \neq a \in R$ and $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$ which is noncentral valued on $R$. Suppose that $G$ and $H$ are two nonzero generalized derivations of $R$ such that $a(H(f(x))f(x) - f(x)G(f(x))) \in C$ for all $x = (x_1, \ldots, x_n) \in R^n$. Then one of the following holds:*

*(1) $f(x_1, \ldots, x_n)^2$ is central valued on $R$ and there exist $b, p, q \in U$ such that $H(x) = px + xb$ for all $x \in R$, $G(x) = bx + xq$ for all $x \in R$ with $a(p - q) \in C$;*
*(2) there exist $p, q \in U$ such that $H(x) = px + xq$ for all $x \in R$, $G(x) = qx$ for all $x \in R$ with $ap = 0$;*

(3) *$f(x_1,\dots,x_n)^2$ is central valued on $R$ and there exist $q\in U$, $\lambda\in C$ and an outer derivation $g$ of $U$ such that $H(x)=xq+\lambda x-g(x)$ for all $x\in R$, $G(x)=qx+g(x)$ for all $x\in R$, with $a\in C$;*

(4) *$R$ satisfies $s_4$ and there exist $b,p\in U$ such that $H(x)=px+xb$ for all $x\in R$, $G(x)=bx+xp$ for all $x\in R$.*

*Proof.* Since every generalized derivation on $R$ can uniquely be defined on the whole $U$ with the form $ax+d(x)$ for some $a\in U$ and a derivation $d$ on $U$ ([18, Theorem 3]), there exist $p,q\in U$ and derivations $h,g$ on $U$ such that $H(x)=px+h(x)$ and $G(x)=qx+g(x)$ for all $x\in U$. By [8] and [21], both $R$ and $U$ satisfy the same generalized polynomial identities and same differential identities. So we have

$$a\Big((pf(x)+h(f(x)))f(x)-f(x)(qf(x)+g(f(x)))\Big)\in C \tag{2.2}$$

for all $x=(x_1,\dots,x_n)\in U^n$.

Now we divide the proof into three cases:

**Case I:** Assume that both $h$ and $g$ are inner derivations, say $h(x)=[b,x]$ and $g(x)=[c,x]$ for all $x\in U$ and for some $b,c\in U$. Then (2.2) becomes

$$a\Big((p+b)f(x)^2-f(x)(b+q+c)f(x)+f(x)^2c\Big)\in C$$

for all $x=(x_1,\dots,x_n)\in U^n$. Then by Lemma 2.5, $b+q+c\in C$ and one of the following holds:

1) $f(x_1,\dots,x_n)^2$ is central valued on $R$ and $a(p+b-b-q-c+c)=a(p-q)\in C$. This implies $H(x)=px+[b,x]=(b+p)x-xb$ and $G(x)=qx+[c,x]=(b+q+c)x-bx-xc=-bx+x(b+q+c-c)=-bx+x(b+q)$ for all $x\in U$ and so for all $x\in R$. This is our conclusion (1).

2) $c\in C$ and $a(p+b-b-q-c+c)=a(p-q)=0$. Since $b+q+c\in C$, $b+q\in C$ and so $[b,x]=-[q,x]$ for all $x\in U$. Thus $H(x)=px+[b,x]=px-[q,x]=(p-q)x+xq$ and $G(x)=qx+[c,x]=qx$ for all $x\in U$ and so for all $x\in R$, with $a(p-q)=0$, which is our conclusion (2).

3) $R$ satisfies $s_4$. Since $b+q+c\in C$, in this case $H(x)=px+[b,x]=(b+p)x-xb$ and $G(x)=qx+[c,x]=(b+q+c)x-bx-xc=-bx+x(b+q+c-c)=-bx+x(b+q)$ for all $x\in U$ and so for all $x\in R$. This gives conclusion (4).

**Case II:** Assume now that both $h$ and $g$ are not inner derivations of $U$. Let $g$ and $h$ be $C$-dependent modulo inner derivations of $U$. Then there exist $\beta,\gamma\in C$, not all zero and $c'\in U$ such that $\beta h+\gamma g=ad_{c'}$.

First assume that $\beta=0$. Then $\gamma\neq 0$ and $g(x)=[t,x]$ for all $x\in U$, where $t=\gamma^{-1}c'$. Thus (2.2) becomes

$$a\Big((pf(x)+h(f(x)))f(x)-f(x)(qf(x)+[t,f(x)])\Big)\in C$$

that is
(2.3)

$$a\Big((pf(x_1,\ldots,x_n)+f^h(x_1,\ldots,x_n)+\sum_{i=1}^{n}f(x_1,\ldots,h(x_i),\ldots,x_n))f(x_1,\ldots,x_n)$$
$$-f(x_1,\ldots,x_n)(qf(x_1,\ldots,x_n)+[t,f(x_1,\ldots,x_n)])\Big)\in C$$

for all $x=(x_1,\ldots,x_n)\in U^n$. By Kharchenko's theorem [13], (2.3) implies that $U$ satisfies

$$a\Big((pf(x_1,\ldots,x_n)+f^h(x_1,\ldots,x_n)+\sum_{i=1}^{n}f(x_1,\ldots,y_i,\ldots,x_n))f(x_1,\ldots,x_n)$$
$$-f(x_1,\ldots,x_n)(qf(x_1,\ldots,x_n)+[t,f(x_1,\ldots,x_n)])\Big)\in C.$$

In particular, $U$ satisfies

$$a(\sum_{i=1}^{n}f(x_1,\ldots,y_i,\ldots,x_n))f(x_1,\ldots,x_n)\in C. \tag{2.4}$$

Putting $y_i=[b,x_i]$ for all $i=1,\ldots,n$, where $b\in U-C$, we get that $U$ satisfies $a[b,f(x_1,\ldots,x_n)]f(x_1,\ldots,x_n)\in C$. This implies

$$a(bf(x_1,\ldots,x_n)^2-f(x_1,\ldots,x_n)bf(x_1,\ldots,x_n))\in C$$

for all $x_1,\ldots,x_n\in U$. Then by Lemma 2.5, we get $b\in C$, which is a contradiction.

Next assume that $\beta\neq 0$. Then $h=\alpha g+ad_c$, where $\alpha=-\beta^{-1}\gamma$ and $c=\beta^{-1}c'$. In this case $g$ cannot be inner, otherwise $g$ and $h$ both will be inner. Now (2.2) becomes

$$a\Big\{\Big(pf(x)+\alpha g(f(x))+[c,f(x)]\Big)f(x)-f(x)\Big(qf(x)+g(f(x))\Big)\Big\}\in C \tag{2.5}$$

for all $x=(x_1,\ldots,x_n)\in U^n$. Thus, $U$ satisfies

$$a\Big\{\Big(pf(x_1,\ldots,x_n)+\alpha f^g(x_1,\ldots,x_n)$$
$$+\alpha\textstyle\sum_{i=1}^{n}f(x_1,\ldots,g(x_i),\ldots,x_n)+[c,f(x_1,\ldots,x_n)]\Big)f(x_1,\ldots,x_n)$$
$$-f(x_1,\ldots,x_n)\Big(qf(x_1,\ldots,x_n)+f^g(x_1,\ldots,x_n)$$
$$+\textstyle\sum_{i=1}^{n}f(x_1,\ldots,g(x_i),\ldots,x_n))\Big)\Big\}\in C.$$

Then by Kharchenko's theorem [13], we have that $U$ satisfies

$$a\Bigg\{\Bigg(pf(x_1,\dots,x_n)+\alpha f^g(x_1,\dots,x_n)$$
$$+\alpha\sum_{i=1}^n f(x_1,\dots,y_i,\dots,x_n)+[c,f(x_1,\dots,x_n)]\Bigg)f(x_1,\dots,x_n)$$
$$-f(x_1,\dots,x_n)\Bigg(qf(x_1,\dots,x_n)+f^g(x_1,\dots,x_n)$$
$$+\sum_{i=1}^n f(x_1,\dots,y_i,\dots,x_n))\Bigg)\Bigg\}\in C.$$

In particular, $U$ satisfies the blended component,

$$a\Bigg(\alpha\sum_{i=1}^n f(x_1,\dots,y_i,\dots,x_n)f(x_1,\dots,x_n)$$
$$-f(x_1,\dots,x_n)\sum_{i=1}^n f(x_1,\dots,y_i,\dots,x_n)\Bigg)\in C. \tag{2.6}$$

Putting $y_i=[b,x_i]$ for all $i=1,\dots,n$, where $b\in U-C$, we get

$$a\Bigg(\alpha[b,f(x_1,\dots,x_n)]f(x_1,\dots,x_n)-f(x_1,\dots,x_n)[b,f(x_1,\dots,x_n)]\Bigg)\in C,$$

that is

$$a(\alpha bf(x)^2+f(x)(-\alpha b-b)f(x)+f(x)^2b)\in C$$

for all $x=(x_1,\dots,x_n)\in U^n$. Then by Lemma 2.5, we have $(\alpha+1)b\in C$. Now $\alpha\neq -1$ implies $b\in C$, which is a contradiction. Therefore $\alpha=-1$ and so $h=-g+ad_c$. Hence (2.6) becomes

$$a\Bigg(-\sum_{i=1}^n f(x_1,\dots,y_i,\dots,x_n)f(x_1,\dots,x_n)$$
$$-f(x_1,\dots,x_n)\sum_{i=1}^n f(x_1,\dots,y_i,\dots,x_n)\Bigg)\in C. \tag{2.7}$$

Putting $y_1=x_1$ and $y_2=\dots=y_n=0$, we get $a(-f(x_1,\dots,x_n)^2-f(x_1,\dots,x_n)^2)=-2af(x_1,\dots,x_n)^2\in C$ for all $x_1,\dots,x_n\in U$. Since char $(R)\neq 2$, $U$ satisfies $af(x_1,\dots,x_n)^2\in C$. By Lemma 2.3 we have $a\in C$. Moreover, since $a\neq 0$, $af(x_1,\dots,x_n)^2\in C$ yields $f(x_1,\dots,x_n)^2\in C$ for all $x_1,\dots,x_n\in U$. Therefore, (2.5) becomes

$$\Bigg(pf(x)-g(f(x))+[c,f(x)]\Bigg)f(x)-f(x)\Bigg(qf(x)+g(f(x))\Bigg)\in C \tag{2.8}$$

for all $x = (x_1, \dots, x_n) \in U^n$. Since $f(x_1, \dots, x_n)^2 \in C$ implies $g(f(x_1, \dots, x_n)^2) \in C$, from (2.8) we have

$$(pf(x) + [c, f(x)])f(x) - f(x)qf(x) \in C$$

that is,

$$(p + c)f(x_1, \dots, x_n)^2 - f(x_1, \dots, x_n)(c + q)f(x_1, \dots, x_n) \in C \tag{2.9}$$

for all $x = (x_1, \dots, x_n) \in U^n$. Again by Lemma 2.3, $c + q \in C$. Thus (2.9) becomes $(p - q)f(x_1, \dots, x_n)^2 \in C$ for all $x_1, \dots, x_n \in U$. Since $f(x_1, \dots, x_n)^2 \in C$, it yields either $f(x_1, \dots, x_n)^2 = 0$ for all $x_1, \dots, x_n \in U$ or $p - q \in C$. But $f(x_1, \dots, x_n)^2 = 0$ implies $f(x_1, \dots, x_n) = 0$ for all $x_1, \dots, x_n \in U$ (see [7]), a contradiction. Hence $p - q = \lambda \in C$. Therefore, for all $x \in U$, $H(x) = px + h(x) = px - g(x) + [c, x] = (q + \lambda + c)x - xc - g(x) = x(q + c + \lambda) - xc - g(x) = x(q + \lambda) - g(x)$ and $G(x) = qx + g(x)$ for all $x \in U$ and so $x \in R$ with $f(x_1, \dots, x_n)^2$ is central valued on $R$ and $a \in C$. This is our conclusion (3).

**Case III:** Here we assume that $h$ and $g$ are $C$-independent modulo inner derivations of $U$. Note that in this case both $h$ and $g$ are outer derivations of $U$. Then our hypothesis (2.2) becomes

$$\begin{aligned} a\Bigg\{\bigg(&pf(x_1, \dots, x_n) + f^h(x_1, \dots, x_n) \\ &+ \sum_{i=1}^{n} f(x_1, \dots, h(x_i), \dots, x_n)\bigg)f(x_1, \dots, x_n) \\ &- f(x_1, \dots, x_n)\bigg(qf(x_1, \dots, x_n) + f^g(x_1, \dots, x_n) \\ &+ \sum_{i=1}^{n} f(x_1, \dots, g(x_i), \dots, x_n))\bigg)\Bigg\} \in C \end{aligned} \tag{2.10}$$

for all $x_1, \dots, x_n \in U$. Since both $h$ and $g$ are outer derivations, by Kharchenko's theorem [13] we get from (2.10) that $U$ satisfies

$$\begin{aligned} a\Bigg\{\bigg(&pf(x_1, \dots, x_n) + f^h(x_1, \dots, x_n) \\ &+ \sum_{i=1}^{n} f(x_1, \dots, y_i, \dots, x_n)\bigg)f(x_1, \dots, x_n) \\ &- f(x_1, \dots, x_n)\bigg(qf(x_1, \dots, x_n) + f^g(x_1, \dots, x_n) \\ &+ \sum_{i=1}^{n} f(x_1, \dots, z_i, \dots, x_n))\bigg)\Bigg\} \in C. \end{aligned}$$

In particular, $U$ satisfies

$$a\sum_{i=1}^{n} f(x_1,\ldots,y_i,\ldots,x_n)f(x_1,\ldots,x_n)\in C,$$

which is the same as (2.4). Then by same argument as before, it leads a contradiction. □

In the Theorem 2.6, assuming $H=G$, we have the following corollary.

**Corollary 2.7.** *Let $R$ be a noncommutative prime ring of characteristic different from 2, $U$ the Utumi quotient ring of $R$, $C$ $(=Z(U))$ the extended centroid of $R$. Let $0\neq a\in R$ and $f(x_1,\ldots,x_n)$ a multilinear polynomial over $C$ which is noncentral valued on $R$. Suppose that $H$ is a nonzero generalized derivation of $R$ such that $a[H(f(x)),f(x)]\in C$ for all $x=(x_1,\ldots,x_n)\in R^n$. Then one of the following holds:*

- *(1) $f(x_1,\ldots,x_n)^2$ is central valued on $R$ and there exist $p\in U$ and $\lambda\in C$ such that $H(x)=px+xp+\lambda x$ for all $x\in R$;*
- *(2) there exists $\lambda\in C$ such that $H(x)=\lambda x$ for all $x\in R$;*
- *(3) $R$ satisfies $s_4$ and there exist $p\in U$ and $\lambda\in C$ such that $H(x)=px+xp+\lambda x$ for all $x\in R$.*

*Proof.* In Theorem 2.6, assuming $H=G$ we have the following conclusions:

- (1) $f(x_1,\ldots,x_n)^2$ is central valued on $R$ and there exist $b,p,q\in U$ such that $H(x)=px+xb=bx+xq$ for all $x\in R$ with $a(p-q)\in C$. This gives $(p-b)x+x(b-q)=0$, implying $b-q\in C$ and $p-b+b-q=0$, that is $p=q$. Let $b-q=\lambda\in C$. Then $b=q+\lambda=p+\lambda$. Thus $H(x)=px+x(p+\lambda)$ for all $x\in R$, this is our conclusion (1).
- (2) There exist $p,q\in U$ such that $H(x)=px+xq=qx$ for all $x\in R$, with $ap=0$. This gives $(p-q)x+xq=0$, implying $q\in C$ and $p-q+q=0$, that is $p=0$. Thus conclusion (2) is obtained.
- (3) $f(x_1,\ldots,x_n)^2$ is central valued on $R$ and there exist $q\in U$, $\lambda\in C$ and an outer derivation $g$ of $U$ such that $H(x)=xq+\lambda x-g(x)=qx+g(x)$ for all $x\in R$, with $a\in C$. In this case, we have $2g(x)=-[q,x]+\lambda x$, a contradiction, since $g$ is an outer derivation.
- (4) $R$ satisfies $s_4$ and there exist $b,p\in U$ such that $H(x)=px+xb=bx+xp$ for all $x\in R$. In this case $(p-b)x+x(b-p)=0$ for all $x\in R$, implying $b-p\in C$. Let $b-p=\lambda\in C$. Thus $H(x)=px+x(p+\lambda)$ for all $x\in R$. This is our conclusion (3).

□

In Theorem 2.6, assuming $H=d$ and $G=\delta$ two derivations of $R$, we have the following:

**Corollary 2.8.** *Let $R$ be a noncommutative prime ring of characteristic different from 2, $U$ the Utumi quotient ring of $R$, $C$ $(= Z(U))$ the extended centroid of $R$. Let $0 \neq a \in R$ and $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$ which is noncentral valued on $R$. Suppose that $d$ and $\delta$ are two nonzero derivations of $R$ such that $a(d(f(x))f(x) - f(x)\delta(f(x))) \in C$ for all $x = (x_1, \ldots, x_n) \in R^n$. Then one of the following holds:*

(1) *$f(x_1, \ldots, x_n)^2$ is central valued on $R$ and there exist $p \in U$ such that $d(x) = [p, x]$ for all $x \in R$, $\delta(x) = -[p, x]$ for all $x \in R$;*
(2) *$f(x_1, \ldots, x_n)^2$ is central valued on $R$ and $d$, $\delta$ two outer derivations of $R$ such that $d(x) = -\delta(x)$ for all $x \in R$, with $a \in C$;*
(3) *$R$ satisfies $s_4$ and there exist $p \in U$ such that $d(x) = [p, x]$ for all $x \in R$, $\delta(x) = -[p, x]$ for all $x \in R$.*

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