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ON NATURAL HOMOMORPHISMS OF LOCAL COHOMOLOGY MODULES

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Abstract. Let M be a non-zero finitely generated module over a commutative Noetherian local ring $(R, \mathfrak{m})$ with $\dim_R(M) = t$. Let I be an ideal of R with $\operatorname{grade}(I, M) = c$. In this article we will investigate several natural homomorphisms of local cohomology modules. The main purpose of this article is to investigate when the natural homomorphisms $\gamma : \operatorname{Tor}_c^R(k, H_I^c(M)) \to k \otimes_R M$ and $\eta : \operatorname{Ext}_R^d(k, H_I^c(M)) \to \operatorname{Ext}_R^t(k, M)$ are non-zero where $d := t-c$. In fact for a Cohen-Macaulay module M we will show that the homomorphism η is injective (resp. surjective) if and only if the homomorphism $H_{\mathfrak{m}}^d(H_I^c(M)) \to H_{\mathfrak{m}}^t(M)$ is injective (resp. surjective) under the additional assumption of vanishing of Ext modules. The similar results are obtained for the homomorphism γ. Moreover we will construct the natural homomorphism $\operatorname{Tor}_c^R(k, H_I^c(M)) \to \operatorname{Tor}_c^R(k, H_J^c(M))$ for the ideals $J \subseteq I$ with $c = \operatorname{grade}(I, M) = \operatorname{grade}(J, M)$. There are several sufficient conditions on I and J to provide this homomorphism is an isomorphism.

Keywords: Local cohomology, Ext and Tor modules, natural homomorphisms.
MSC(2010): Primary: 13D45.

1. **Introduction**

Let I denote an ideal of a Noetherian local ring $(R, \mathfrak{m})$. We denote $H_I^i(R), i \in \mathbb{Z}$, the local cohomology modules of R with respect to I. For the definition of local cohomology we refer to [6] and [2]. During the last few years many authors have investigated the natural homomorphism $\mu : R \to \operatorname{End}_R(H_I^c(R))$. Firstly, in 2008, M. Hellus and J. Stückrad (see [9, Theorem 2.2]) have shown that this natural homomorphism is an isomorphism for a complete local ring and a cohomologically complete intersection ideal I. After that several authors generalized this idea to an arbitrary module.

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Most recently the extension of this natural homomorphism is proved by the author and Z. Zahid to a finitely generated R-module and the canonical module (see [13, Theorem 1.1] and [17, Theorem 1.1]). Moreover, for a complete local Gorenstein ring, in [16, Theorems 1.1 and 4.4] the author and P. Schenzel have discussed the injectivity and surjectivity of the homomorphism μ in terms of numerical invariants $\tau_{d,d}$, where $\tau_{i,j}$ is defined to be the socle dimension of $H^i_{\mathfrak{m}}(H^{n-j}_I(R))$. M. Hellus and P. Schenzel, in [10, Conjecture 2.7], conjectured that the natural homomorphism $\operatorname{Ext}^d_R(k, H^c_I(R)) \to k$ is non-zero. As a first step it is proved (see [22, Theorem 6.2]) to be true for a regular local ring containing a field. In [16, Theorem 4.4] there is a relation between this conjecture and the above homomorphism μ. One of our main results is a contribution to check the validity of this conjecture for Cohen-Macaulay modules. Moreover we are succeeded to characterize several interpretations such that the natural homomorphism $\operatorname{Tor}^R_c(k, H^c_I(M)) \to k \otimes_R M$ is non-zero. In particular we prove the following result:

Theorem 1.1. *Let M be a non-zero finitely generated R-module of $\dim_R(M) = t$. Suppose that $\operatorname{grade}(I, M) = c$ for an ideal I. Then*

(a) *The following conditions are equivalent:*
  (1) *The natural map $\operatorname{Tor}^R_c(k, H^c_I(M)) \to k \otimes_R M$ is surjective and $\operatorname{Tor}^R_i(k, H^c_I(M)) = 0$ for all $i < c$.*
  (2) *The natural map $\operatorname{Hom}_R(M, k) \to H^c_{\mathfrak{m}}(D(H^c_I(M)))$ is injective and $H^i_{\mathfrak{m}}(D(H^c_I(M))) = 0$ for all $i < c$.*
  (3) *The natural map $D(M) \to H^c_{\mathfrak{m}}(D(H^c_I(M)))$ is injective and $H^i_{\mathfrak{m}}(D(H^c_I(M))) = 0$ for all $i < c$.*

(b) *If M is Cohen-Macaulay and $d := t - c$. Then the following conditions are equivalent:*
  (1) *The natural map $H^d_{\mathfrak{m}}(H^c_I(M)) \to H^t_{\mathfrak{m}}(M)$ is injective and $H^i_{\mathfrak{m}}(H^c_I(M)) = 0$ for all $i < d$.*
  (2) *The natural map $\operatorname{Ext}^d_R(k, H^c_I(M)) \to \operatorname{Ext}^t_R(k, M)$ is injective and $\operatorname{Ext}^i_R(k, H^c_I(M)) = 0$ for all $i < d$.*

Here $D(\cdot)$ denotes the Matlis dual functor. The existence of these natural maps is shown in Propositions 3.1 and 3.3. With the additional assumption of $\operatorname{inj}\dim_R(H^c_I(M)) \leq d$ we succeed to prove that the natural homomorphism $H^d_{\mathfrak{m}}(H^c_I(M)) \to H^t_{\mathfrak{m}}(M)$ is surjective resp. injective (see Theorem 4.2). This actually generalizes the result proven in case of a complete local Gorenstein ring in [16, Theorem 4.4]. The similar results are obtained for the homomorphism $E \to H^c_{\mathfrak{m}}(D(H^c_I(R)))$ (see Theorem 5.5) where $E = E_R(k)$ is the injective hull of k.

Note that, in case of $M = R$ a local Gorenstein ring, the equivalent conditions in Theorem 1.1 are helpful to decide whether an ideal of R is of cohomologically complete intersections (see [14, Thenroem 1.1] and [15, Thenroem 1.1]).

Another part of our investigation is to derive the natural homomorphism

$$\operatorname{Tor}_c^R(k, H_I^c(M)) \to \operatorname{Tor}_c^R(k, H_J^c(M))$$

for two ideals $J \subseteq I$ of R such that $\operatorname{grade}(I, M) = \operatorname{grade}(J, M) = c$ (see Proposition 5.1). Note that there is a necessary condition such that this natural homomorphism is an isomorphism. As a consequence of this we are able to prove the following result:

Theorem 1.2. *Let M be a non-zero finitely generated R-module such that $c = \operatorname{grade}(I, M) = \operatorname{grade}(J, M)$ where $J \subseteq I \subseteq R$ are two ideals. Suppose that $\operatorname{Rad} IR_{\mathfrak{p}} = \operatorname{Rad} JR_{\mathfrak{p}}$ for all $\mathfrak{p} \in V(J) \cap \operatorname{Supp}_R(M)$ such that $\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq c$. Assume in addition that $H_J^i(M) = 0$ for all $i \neq c$. Then the natural homomorphism*

$$\operatorname{Tor}_c^R(k, H_I^c(M)) \to k \otimes_R M$$

is an isomorphism.

Note that the last Theorem 1.2 is actually the dual statement of one of the main results of [20, Theorem 1.2] and [17, Theorem 1.2].

2. Preliminaries

Throughout this paper we will denote $(R, \mathfrak{m})$ by a commutative Noetherian local ring with $k = R/\mathfrak{m}$ its residue field. Let $D(\cdot) := \operatorname{Hom}_R(\cdot, E)$ denote the Matlis dual functor, where $E = E_R(k)$ is the injective hull of k. For the basic results of homological algebra of complexes we refer to [23] and [7]. Moreover in the following we fix some notation which will be used throughout the paper.

Notation 2.1. (1) The symbol " $\cong$ " indicates the isomorphisms between modules.

(2) Let $X \to Y$ be a morphism of complexes of R-modules. Then it is called quasi-isomorphism, i.e. homologically isomorphism, if it induces the isomorphism between the homologies of X and Y. In this case we will write it as $X \xrightarrow{\sim} Y$.

First of all note that the following results are proved in [1]. We add it here for sake of completeness.

Theorem 2.2. *Let $X \xrightarrow{\sim} Y$ be a quasi-isomorphism of R-modules. Then the following statements are true:*

1. *If $F_R^{\cdot}$ is a bounded above complex of flat R-modules. Then it induces a quasi-isomorphism*

$$F_R^{\cdot} \otimes_R X \xrightarrow{\sim} F_R^{\cdot} \otimes_R Y.$$

(2) *If $E^{\cdot}_R$ a bounded below complex of injective R-modules. Then it induces a quasi-isomorphism*

$$\operatorname{Hom}_R(Y, E^{\cdot}_R) \xrightarrow{\sim} \operatorname{Hom}_R(X, E^{\cdot}_R).$$

(3) *If $P^{\cdot}_R$ a bounded above complex of projective R-modules. Then it induces a quasi-isomorphism*

$$\operatorname{Hom}_R(P^{\cdot}_R, X) \xrightarrow{\sim} \operatorname{Hom}_R(P^{\cdot}_R, Y).$$

Lemma 2.3. *Let $(R, \mathfrak{m})$ be a ring, $I \subseteq R$ an ideal, and $M \neq 0$ a maximal Cohen-Macaulay R-module. Suppose that $H^i_I(M) = 0$ for all $i \neq c = \operatorname{grade}(I, M)$ and $\mathfrak{p} \in \operatorname{Supp}_R(M) \cap V(I)$. Then the following hold:*

(1) *$H^i_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$ for all $i \neq c$.*

(2) *The natural homomorphism*

$$H^i_{\mathfrak{p}R_{\mathfrak{p}}}(H^c_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}})) \to H^{\dim(M_{\mathfrak{p}})}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$$

is an isomorphism for $i = \dim(M_{\mathfrak{p}}) - c$ and $H^i_{\mathfrak{p}R_{\mathfrak{p}}}(H^c_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}})) = 0$ for all $i \neq \dim(M_{\mathfrak{p}}) - c$.

Proof. We refer to [14, Lemma 4.4]. □

Note that the following version of Local Duality was already derived in [14, Lemmma 3.1]. For a generalization to arbitrary cohomologically complete intersection ideals we refer to [8, Theorem 6.4.1], [13, Lemma 2.4] or [11, Theorem 3.1]. We include it here for our convenience.

Lemma 2.4. *Let R be a Cohen-Macaulay ring with $\dim(R) = n$. Then for any R-module M and for all $i \in \mathbb{Z}$ we have:*

(1) *$\operatorname{Tor}^R_{n-i}(M, H^n_{\mathfrak{m}}(R)) \cong H^i_{\mathfrak{m}}(M)$.*

(2) *$D(H^i_{\mathfrak{m}}(M)) \cong \operatorname{Ext}^{n-i}_R(M, D(H^n_{\mathfrak{m}}(R)))$.*

In the next we need a result which is actually proved in [10, Proposition 1.4] (see also [24, Lemma 2.5] for a slight generalization). Moreover the elementary proof of the following Lemma is given in [14, Proposition 2.6].

Lemma 2.5. *Suppose that X is an arbitrary R-module. Then $H^i_{\mathfrak{m}}(X) = 0$ for all $i < s$ if and only if $\operatorname{Ext}^i_R(k, X) = 0$ for all $i < s$ where $s \in \mathbb{N}$. Moreover if one of these equivalent conditions holds, then there is an isomorphism*

$$\operatorname{Hom}_R(k, H^s_{\mathfrak{m}}(X)) \cong \operatorname{Ext}^s_R(k, X).$$

Proof. See [10, Proposition 1.4]. □

At the end of this section let us recall the definition of the truncation complex. Note that the idea of the truncation complex was firstly given by P. Schenzel (see [19, Definition 4.1]). Let M be a finitely generated R-module with $\dim_R(M) = t$. Assume that $E^{\cdot}_R(M)$ denotes the minimal injective resolution of M. Then it is easy to see that $\Gamma_I(E_R(R/\mathfrak{p})) = E_R(R/\mathfrak{p})$ for all $\mathfrak{p} \in V(I)$ and zero otherwise. Here $\Gamma_I(-)$ denotes the section functor with support in I.

Definition 2.6. Let $C^{\cdot}_R(I)$ be the cokernel of the embedding of the complexes of R-modules $H^c_I(M)[-c] \to \Gamma_I(E^{\cdot}_R(M))$. This said to be the truncation complex of R with respect to I.

Note that there ia a short exact sequence of complexes of R-modules

$$0 \to H^c_I(M)[-c] \to \Gamma_I(E^{\cdot}_R(M)) \to C^{\cdot}_M(I) \to 0.$$

Then the long exact sequence of cohomologies of this sequence induces that $H^i(C^{\cdot}_M(I)) = 0$ for all $i \leq c$ or $i > t$ and $H^i(C^{\cdot}_M(I)) \cong H^i_I(M)$ for all $c < i \leq t$.

3. Some natural homomorphisms of local cohomology modules

The truncation complex is useful to construct the several natural homomorphisms in the following Proposition. Note that one can also derive these natural homomorphisms as edge homomorphisms of certain spectral sequences. But here for our convenience we will give the elementary proof. Because of we need these construction in the sequel of this paper. Note that the existence of the natural homomorphism φ_1 of the next Proposition is already proved in [11, Lemma 4.2].

Proposition 3.1. *Let $0 \neq M$ be a finitely generated R-module of $\dim(R) = n$ and $\dim_R(M) = t$. Let I be an ideal with $\operatorname{grade}(I, M) = c$ and $d := t - c$. Then*

(a) *There are the natural homomorphisms:*
  (1) $\varphi_1 : H^d_{\mathfrak{m}}(H^c_I(M)) \to H^t_{\mathfrak{m}}(M)$,
  (2) $\varphi_2 : \operatorname{Ext}^d_R(k, H^c_I(M)) \to \operatorname{Ext}^t_R(k, M)$,
  (3) $\varphi_3 : \operatorname{Tor}^R_c(k, H^c_I(M)) \to k \otimes_R M$.

(b) *Suppose that R is Cohen-Macaulay. Then there are the natural homomorphisms:*
  (1) $\varphi_4 : \operatorname{Tor}^R_{n-d}(H^n_{\mathfrak{m}}(R), H^c_I(M)) \to H^t_{\mathfrak{m}}(M)$,
  (2) $\varphi_5 : D(H^t_{\mathfrak{m}}(M)) \to \operatorname{Ext}^{n-d}_R(H^c_I(M), D(H^n_{\mathfrak{m}}(R)))$.

Proof. Firstly we prove statement in (a). To do this let $\underline{x} = x_1, \ldots, x_s \in \mathfrak{m}$ with $\operatorname{Rad}\mathfrak{m} = \operatorname{Rad}(\underline{x})R$. We consider the Čech complex $\check{C}_{\underline{x}}$ with respect to $\underline{x}$. Now apply $\cdot \otimes_R \check{C}_{\underline{x}}$ to the short exact sequence of the truncation complex. Then the resulting sequence of complexes remains exact because $\check{C}_{\underline{x}}$ is a bounded complex of flat R-modules. That is there is the following short exact sequence of complexes

(1) $$0 \to (\check{C}_{\underline{x}} \otimes_R H^c_I(M))[-c] \to \check{C}_{\underline{x}} \otimes_R \Gamma_I(E^{\cdot}_R(M)) \to \check{C}_{\underline{x}} \otimes_R C^{\cdot}_M(I) \to 0.$$

But the complex in the middle is quasi-isomorphic to the following complex

$$\Gamma_{\mathfrak{m}}(\Gamma_I(E^{\cdot}_R(M))) \cong \Gamma_{\mathfrak{m}}(E^{\cdot}_R(M))$$

(see [18, Theorem 3.2]). Recall that $\Gamma_I(E^{\cdot}_R(M))$ is a complex of injective R-modules and $\Gamma_{\mathfrak{m}}(\Gamma_I(\cdot)) = \Gamma_{\mathfrak{m}}(\cdot)$. Then the long exact sequence of cohomologies of the last sequence, concentrated in homological degree t, induces the following natural map

$$H^d_{\mathfrak{m}}(H^c_I(M)) \to H^t_{\mathfrak{m}}(M)$$

where $d := t - c$. This gives the existence of the natural homomorphism of (1).

Now let $F^R_{\cdot}(k)$ be a free resolution of k. Apply the functor $\operatorname{Hom}_R(F^R_{\cdot}(k), .)$ to the short exact sequence of the truncation complex. It induces the following short exact sequence of complexes of R-modules

$$(2) \qquad 0 \to \operatorname{Hom}_R(F^R_{\cdot}(k), H^c_I(M))[-c] \to \operatorname{Hom}_R(F^R_{\cdot}(k), \Gamma_I(E^{\cdot}_R(M))) \to \operatorname{Hom}_R(F^R_{\cdot}(k), C^{\cdot}_M(I)) \to 0.$$

Moreover Let us assume that $X := \operatorname{Hom}_R(F^{\cdot}_R(k), E^{\cdot}_R(M))$. Then X is a complex of injective R-modules and $\operatorname{Hom}_R(F^R_{\cdot}(k), \Gamma_I(E^{\cdot}_R(M))) \xrightarrow{\sim} \Gamma_I(X)$. By [18, Theorem 1.1] the last complex is quasi-isomorphic to $\check{C}_{\underline{y}} \otimes_R X$ where $\underline{y} = y_1, \ldots, y_r \in I$ such that $\operatorname{Rad} I = \operatorname{Rad}(\underline{y})R$.

Note that $\check{C}_{\underline{y}}$ (resp. $E^{\cdot}_R(M)$) is a right bounded (resp. left bounded) complex of flat (resp. injective) R-modules. So Theorem 2.2 induces the following quasi-isomorphism

$$\check{C}_{\underline{y}} \otimes_R \operatorname{Hom}_R(k, E^{\cdot}_R(M)) \xrightarrow{\sim} \check{C}_{\underline{y}} \otimes_R X.$$

But the complex on the left side is quasi-isomorphic to $\operatorname{Hom}_R(k, E^{\cdot}_R(M))$. This is true because of each R-module of the complex $\operatorname{Hom}_R(k, E^{\cdot}_R(M))$ has support in $V(\mathfrak{m})$. Therefore our middle complex of the exact sequence 2 is quasi-isomorphic to $\operatorname{Hom}_R(k, E^{\cdot}_R(M))$. Then the long exact sequence of cohomologies of 2 , in degree t, induces the required homomorphism of (2).

Now by tensoring with $F^R_{\cdot}(k)$ to the truncation complex induces the following exact sequence of complexes of R-modules

(3)

$$0 \to (F^R_{\cdot}(k) \otimes_R H^c_I(M))[-c] \to F^R_{\cdot}(k) \otimes_R \Gamma_I(E^{\cdot}_R(M)) \to F^R_{\cdot}(k) \otimes_R C^{\cdot}_M(I) \to 0.$$

We are interested to calculate the homologies of the complex in the middle. For this let us denote $Y := F^R_{\cdot}(k) \otimes_R \check{C}_{\underline{y}} \otimes_R E^{\cdot}_R(M)$. Recall that $\check{C}_{\underline{y}}$ denote the Čech complex with respect to $\underline{y}$. Because of Theorem 2.2(1) and [18, Theorem 3.2] we get the quasi-isomorphism of complexes

$$F^R_{\cdot}(k) \otimes_R \Gamma_I(E^{\cdot}_R(M)) \xrightarrow{\sim} Y.$$

Again by Theorem 2.2(1) we have a quasi-isomorphism $\check{C}_{\underline{y}} \otimes_R F^{\cdot}_R(k) \xrightarrow{\sim} \check{C}_{\underline{y}} \otimes_R k$ since $\check{C}_{\underline{y}}$ is a right bounded complex of flat R-modules. Then the last complex is isomorphic to k since $\operatorname{Supp}_R(k) \subseteq V(\mathfrak{m})$. Moreover there is a quasi-isomorphism

$$F^R_{\cdot}(k) \otimes_R \check{C}_{\underline{y}} \otimes_R M \xrightarrow{\sim} Y$$

(see Theorem 2.2(1)). Let $L_\cdot^R$ denote the free resolution of M. Then it follows that the morphism of complexes $F_\cdot^R(k) \otimes_R \check{C}_{\underline{y}} \otimes_R L_\cdot^R \to F_\cdot^R(k) \otimes_R \check{C}_{\underline{y}} \otimes_R M$ induces an isomorphism in homology (see Theorem 2.2(1)). So we have $H^i(Y) \cong H^i(F_\cdot^R(k) \otimes_R \check{C}_{\underline{y}} \otimes_R L_\cdot^R)$ for all $i \in \mathbb{Z}$. Since both of the complexes $\check{C}_{\underline{y}}$ and $L_\cdot^R$ are right bounded complex of flat R-modules. By Theorem 2.2(1) we get the following quasi-isomorphism

$$L_\cdot^R \otimes_R \check{C}_{\underline{y}} \otimes_R F_\cdot^R(k) \xrightarrow{\sim} L_\cdot^R \otimes_R \check{C}_{\underline{y}} \otimes_R k.$$

Since $\operatorname{Supp}_R(k) \subseteq V(\mathfrak{m})$ it follows the last complex is quasi-isomorphic to $L_\cdot^R \otimes_R k$. Hence $H^i(Y) \cong H^i(k \otimes_R L_\cdot^R)$ for all $i \in \mathbb{Z}$. Then the homology in degree 0 induces the homomorphism in (3). This finishes the proof of the statement (a).

Finally to prove the assertion of (b) we assume that R is Cohen-Macaulay. Apply the Local Duality Lemma 2.4 (for $M = H_I^c(M)$) we have the isomorphism

$$H_{\mathfrak{m}}^d(H_I^c(M)) \cong \operatorname{Tor}_{n-d}^R(H_I^c(M), H_{\mathfrak{m}}^n(R)).$$

Therefore by (a) this gives the homomorphism in (1). By Hom-Tensor Duality the Matlis dual of this last isomorphism induces the following isomorphism

$$\operatorname{Ext}_R^{n-d}(H_I^c(M), D(H_{\mathfrak{m}}^n(R))) \cong D(H_{\mathfrak{m}}^d(H_I^c(M))).$$

Then the homomorphism in (2) can be easily derived from (a). This completes the proof of the Proposition. □

Note that for any R-module X there are the natural homomorphisms

$$\operatorname{Ext}_R^i(k, X) \to H_{\mathfrak{m}}^i(X)$$

for all $i \in \mathbb{N}$. In [12, Section 4] M. Hochster has studied about these natural maps in case of canonical modules. In view of the above homomorphism we are interested to relate φ_3 with the homomorphisms of the next Proposition.

Remark 3.2. Suppose that the assumptions of last Proposition 3.1 are true. If $F_\cdot(R/\mathfrak{m}^\alpha)$ denote a minimal free resolution of $R/\mathfrak{m}^\alpha$ for any fixed $\alpha \in \mathbb{N}$. Then by similar arguments as we use in the proof of Proposition 3.1 (2) and (3) we can obtain the following natural homomorphisms

$$\operatorname{Tor}_c^R(R/\mathfrak{m}^\alpha, H_I^c(M)) \to R/\mathfrak{m}^\alpha \otimes_R M$$

$$\operatorname{Ext}_R^d(R/\mathfrak{m}^\alpha, H_I^c(M)) \to \operatorname{Ext}_R^t(R/\mathfrak{m}^\alpha, M)$$

for all $\alpha \in \mathbb{N}$.

Proposition 3.3. *With the previous notation there are the following natural homomorphisms:*

$$\varphi_6 : \operatorname{Hom}_R(M, k) \to H_{\mathfrak{m}}^c(D(H_I^c(M))), \text{ and}$$

$$\varphi_7 : D(M) \to H_{\mathfrak{m}}^c(D(H_I^c(M))).$$

Proof. Since the Matlis dual of φ_3 induces the following natural homomorphism

$$\operatorname{Hom}_R(M,k) \to \operatorname{Ext}^c_R(k, D(H^c_I(M))),$$

because of $D(k \otimes_R M) \cong \operatorname{Hom}_R(M,k)$. Now take composition of this with the homomorphism $\operatorname{Ext}^c_R(k, D(H^c_I(M))) \to H^c_{\mathfrak{m}}(D(H^c_I(M)))$. Then we get the existence of first natural homomorphism as follows:

$$\varphi_6 : \operatorname{Hom}_R(M,k) \to H^c_{\mathfrak{m}}(D(H^c_I(M))).$$

Now we show the existence of φ_7. To do this let $H = H^c_I(M)$. Then by Remark 3.2 we have the following natural homomorphisms

$$\operatorname{Tor}^R_c(R/\mathfrak{m}^\alpha, H) \to R/\mathfrak{m}^\alpha \otimes_R M$$

for all $\alpha \in \mathbb{N}$. Then the Matlis dual of this induces the following homomorphism

$$D(R/\mathfrak{m}^\alpha \otimes_R M) \to \operatorname{Ext}^c_R(R/\mathfrak{m}^\alpha, D(H))$$

for each $\alpha \in \mathbb{N}$. By passing to the direct limit of this gives rise to the homomorphism

$$\varinjlim D(R/\mathfrak{m}^\alpha \otimes_R M) \to H^c_{\mathfrak{m}}(D(H)).$$

By Hom-Tensor Duality we have $\varinjlim D(R/\mathfrak{m}^\alpha \otimes_R M) \cong \varinjlim \operatorname{Hom}_R(R/\mathfrak{m}^\alpha, D(M))$. Moreover support of the module $D(M)$ is contained in $\{\mathfrak{m}\}$. So the last module is isomorphic to $D(M)$. Hence it proves the existence of φ_7. □

In the next we are interested to characterize the injectivity and surjectivity of all the homomorphisms $\varphi_i, i = 1, \ldots, 7$. In this direction the first result is the following:

Lemma 3.4. *Let R be a Cohen-Macaulay ring. With the above notation the following are true:*

(1) *φ_1 is non-zero if and only if φ_4 is non-zero if and only if φ_5 is non-zero.*
(2) *The following conditions are equivalent:*
   (i) *φ_1 is injective (resp. surjective).*
   (ii) *φ_4 is injective (resp. surjective).*
   (iii) *φ_5 is surjective (resp. injective).*

Proof. Since R is Cohen-Macaulay so by the Local Duality (see Lemma 2.4) we have the isomorphism

$$H^d_{\mathfrak{m}}(H^c_I(M)) \cong \operatorname{Tor}^R_{n-d}(H^n_{\mathfrak{m}}(R), H^c_I(M)).$$

So φ_1 is non-zero if and only if φ_4 is non-zero. By Hom-Tensor Duality it implies that φ_4 is non-zero if and only if φ_5 is non-zero. Therefore the statement in (1) is shown to be true. Note that the equivalence of the conditions in the statement (2) can be easily proved by the same arguments. □

In the particular case of a complete Gorenstein ring it was proved in [16, Theorem 3.2] that $\varphi_i, i = 1, 4, 5$, are all non-zero.

The next result tells us that all the homomorphisms of Proposition 3.1 are isomorphisms provided that M is cohomologically complete intersection with respect to I. That is $H_I^i(M) = 0$ for all $i \neq c = \operatorname{grade}(I, M)$. Note that our next result is the generalization of [14, Corollary 4.3].

Corollary 3.5. *Let I be an ideal of a local ring $(R, \mathfrak{m})$. Suppose that M is a non-zero finitely generated R-module with $H_I^i(M) = 0$ for all $i \neq c =$ $\operatorname{grade}(I, M)$. If we set $d := t - c$ then the following hold:*

1. *φ_1 is an isomorphism.*
2. *φ_2 is an isomorphism.*
3. *φ_3 is an isomorphism.*

Proof. Suppose that $E_R^{\cdot}(M)$ is a minimal injective resolution of M. Then $\Gamma_I(E_R^{\cdot}(M))$ is an injective resolution of $H_I^c(M)[-c]$ (because of $H_I^i(M) = 0$ for all $i \neq c$). Assume that $\underline{x} = x_1, \ldots, x_r \in \mathfrak{m}$ such that $\operatorname{Rad} \mathfrak{m} = \operatorname{Rad}(\underline{x})R$. Then, for the Čech complex $\check{C}_{\underline{x}}$ with respect to $\underline{x}$, we have the following quasi isomorphism

$$(\check{C}_{\underline{x}} \otimes_R H_I^c(M))[-c] \xrightarrow{\sim} \check{C}_{\underline{x}} \otimes_R \Gamma_I(E_R^{\cdot}(M)),$$

(see Theorem 2.2(1)). Recall that $\check{C}_{\underline{x}}$ is a right bounded complex of flat R-modules. But $\Gamma_I(E_R^{\cdot}(M))$ is a complex of injective R-modules and $\Gamma_{\mathfrak{m}}(\Gamma_I(\cdot)) = \Gamma_{\mathfrak{m}}(\cdot)$. So the complex $\check{C}_{\underline{x}} \otimes_R \Gamma_I(E_R^{\cdot}(M)$ is quasi-isomorphic to the following complex

$$\Gamma_{\mathfrak{m}}(\Gamma_I(E_R^{\cdot}(M))) \cong \Gamma_{\mathfrak{m}}(E_R^{\cdot}(M))$$

(see [18, Theorem 3.2]). Then it implies that $H^i(\check{C}_{\underline{x}} \otimes_R \Gamma_I(E_R^{\cdot}(M))) \cong H_{\mathfrak{m}}^i(M)$ for all $i \in \mathbb{Z}$. This proves that φ_1 is an isomorphism in view of the above quasi-isomorphism.

Now suppose that $F_R^{\cdot}(k)$ is a minimal free resolution of k. Since $F_R^{\cdot}(k)$ is a right bounded complex of free R-modules. By Theorem 2.2(1) we have the following quasi isomorphism

$$(H_I^c(M) \otimes_R F_R^{\cdot}(k))[-c] \xrightarrow{\sim} \Gamma_I(E_R^{\cdot}(M)) \otimes_R F_R^{\cdot}(k).$$

By the proof of Proposition 3.1 $(a)(3)$ it follows that $H^i(\Gamma_I(E_R^{\cdot}(M)) \otimes_R F_R^{\cdot}(k)) \cong \operatorname{Tor}_i^R(k, M)$ for all $i \in \mathbb{Z}$. This proves that φ_3 is an isomorphism..

Since $\Gamma_I(E_R^{\cdot}(M))$ is an injective resolution of $H_I^c(M)[-c]$ and $F_R^{\cdot}(k)$ is a right bounded complex of free R-modules. It induces the following quasi isomorphism

$$\operatorname{Hom}_R(F_{\cdot}^R(k), H_I^c(M))[-c] \xrightarrow{\sim} \operatorname{Hom}_R(F_{\cdot}^R(k), \Gamma_I(E_R^{\cdot}(M)))$$

(see Theorem 2.2(1)). Again by the proof of Proposition 3.1 $(a)(2)$ it follows that the latter complex is quasi isomorphic to $\operatorname{Hom}_R(k, E_R^{\cdot}(M))$. Hence we get φ_2 is an isomorphism. $\square$

Corollary 3.6. *With the above notation suppose in addition that R is Cohen-Macaulay. Then the following hold:*

(1) *φ_4 is an isomorphism.*
(2) *φ_5 is an isomorphism.*

Proof. See Lemma 3.4(2) and Corollary 3.5(1). ☐

4. The homomorphisms φ_1 and φ_2

In this section we investigate the natural homomorphisms φ_1 and φ_2 of Proposition 3.1. Here we will relate several interpretations of φ_1 and φ_2. In particular our first main result is the following:

Theorem 4.1. *Let M be a non-zero Cohen-Macaulay R-module of $\dim_R(M) = t$. Suppose that $\operatorname{grade}(I, M) = c$ for an ideal I and $d := t-c$. Then the following conditions are equivalent:*

(1) *φ_1 is injective and $H^i_{\mathfrak{m}}(H^c_I(M)) = 0$ for all $i < d$.*
(2) *φ_2 is injective and $\operatorname{Ext}^i_R(k, H^c_I(M)) = 0$ for all $i < d$.*

Proof. Note that the equivalence of both of the vanishing statements in (1) and (2) follows by Lemma 2.5 (for $X = H^c_I(M)$). Then by Proposition 3.1 it induces the commutative diagram

$$\begin{array}{ccc} \operatorname{Ext}^d_R(k, H^c_I(M)) & \to & H^d_{\mathfrak{m}}(H^c_I(M)) \\ \downarrow \varphi_2 & & \downarrow \varphi_1 \\ \operatorname{Ext}^t_R(k, M) & \to & H^t_{\mathfrak{m}}(M) \end{array}$$

This gives rise to the following commutative diagram

$$\begin{array}{ccc} \operatorname{Ext}^d_R(k, H^c_I(M)) & \to & \operatorname{Hom}_R(k, H^d_{\mathfrak{m}}(H^c_I(M))) \\ \downarrow \varphi_2 & & \downarrow \\ \operatorname{Ext}^t_R(k, M) & \to & \operatorname{Hom}_R(k, H^t_{\mathfrak{m}}(M)) \end{array}$$

Since M is Cohen-Macaulay so $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq t$. Then, by the equivalence of the vanishing statements, both of the horizontal homomorphisms of the last diagram are isomorphisms (see Lemma 2.5).

We only need to prove the equivalence of the injectivity. For this let us assume that φ_1 is injective. It implies that $\operatorname{Hom}_R(k, H^d_{\mathfrak{m}}(H^c_I(M))) \to \operatorname{Hom}_R(k, H^t_{\mathfrak{m}}(M))$ is injective. Hence, by the above commutative diagram, φ_2 is injective.

Conversely, suppose that φ_2 is injective. Then the cohomology sequence of the exact sequence 1 provides the following exact sequence of R-modules

$$0 \to H^{t-1}_{\mathfrak{m}}(C^{\cdot}_M(I)) \to H^d_{\mathfrak{m}}(H^c_I(M)) \to H^t_{\mathfrak{m}}(M) \tag{4}$$

To this end recall that $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq t$ (since M is Cohen-Macaulay). We will prove that $H^{t-1}_{\mathfrak{m}}(C^{\cdot}_M(I)) = 0$. Apply the functor $\operatorname{Hom}_R(k, \cdot)$ to this exact sequence we get the following exact sequence

$$0 \to \operatorname{Hom}_R(k, H^{t-1}_{\mathfrak{m}}(C^{\cdot}_M(I))) \to \operatorname{Hom}_R(k, H^d_{\mathfrak{m}}(H^c_I(M))) \to \operatorname{Hom}_R(k, H^t_{\mathfrak{m}}(M))$$

But $\mathrm{Hom}_R(k, H^d_{\mathfrak{m}}(H^c_I(M))) \to \mathrm{Hom}_R(k, H^t_{\mathfrak{m}}(M))$ is injective (by the above commutative diagram and the assumption on φ_2). It follows that $\mathrm{Hom}_R(k, H^{t-1}_{\mathfrak{m}}(C^{\cdot}_M(I))) = 0$. It is well-known that if X is an R-module with support in $V(\mathfrak{m})$. Then the socle of X is zero if and only if X is zero. This proves that $H^{t-1}_{\mathfrak{m}}(C^{\cdot}_M(I)) = 0$ because of $\mathrm{Supp}(H^{t-1}_{\mathfrak{m}}(C^{\cdot}_M(I))) \subset \{\mathfrak{m}\}$. Hence

$$\varphi_1 : H^d_{\mathfrak{m}}(H^c_I(M)) \to H^t_{\mathfrak{m}}(M)$$

is injective (by sequence 4). This completes the proof of the Theorem. □

In the next result we are going to prove the surjectivity of φ_1. In fact this result can be used to show that the natural homomorphism $\mathrm{Ext}^{t-c}_R(k, H^c_I(M)) \to \mathrm{Ext}^t_R(k, M)$ is non-zero. Note that the following version of φ_1 and φ_2 is already proved in case of a local Gorenstein ring, see [16, Theorem 4.4]. Here we will generalize it to Cohen-Macaulay modules.

Theorem 4.2. *With the previous notation assume that φ_2 is surjective (resp. injective) and $\mathrm{Ext}^i_R(k, H^c_I(M)) = 0$ for all $i > d = t - c$. Then φ_1 is surjective (resp. injective).*

Proof. Let $H = H^c_I(M)$ then by Remark 3.2 there are the natural homomorphisms

$$f_\alpha : \mathrm{Ext}^d_R(R/\mathfrak{m}^\alpha, H) \to \mathrm{Ext}^t_R(R/\mathfrak{m}^\alpha, M)$$

for all $\alpha \in \mathbb{N}$. We claim that f_α is surjective (resp. injective) for each $\alpha \in \mathbb{N}$. We prove our claim by induction on α. Let $\alpha = 1$ then, by assumption on φ_2, we are true. Note that there is a short exact sequence

$$0 \to \mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1} \to R/\mathfrak{m}^{\alpha+1} \to R/\mathfrak{m}^\alpha \to 0 \tag{5}$$

Since $\mathrm{Ext}^i_R(k, H) = 0$ for all $i > d$. So, in view of the above exact sequence 5, one can easily prove by induction that $\mathrm{Ext}^i_R(R/\mathfrak{m}^\alpha, H) = 0$ for all $i > d$ and for all $\alpha \in \mathbb{N}$. To this end note that $\mathfrak{m}^s/\mathfrak{m}^{s+1}$ is a finite dimensional k-vector space. Now apply the functor $\mathrm{Ext}_R(\cdot, H)$ to the exact sequence 5. Then the long exact sequence of cohomologies provides the following exact sequence

$$\mathrm{Ext}^d_R(R/\mathfrak{m}^{\alpha+1}, H) \to \mathrm{Ext}^d_R(\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}, H) \to \mathrm{Ext}^{d+1}_R(R/\mathfrak{m}^\alpha, H)$$

Since $\mathrm{Ext}^{d+1}_R(R/\mathfrak{m}^\alpha, H) = 0$ so the last sequence is right exact. Moreover note that $\mathrm{Ext}^i_R(R/\mathfrak{m}^\alpha, M) = 0$ for all $i < t = \mathrm{depth}_R(M)$ and for all $\alpha \in \mathbb{N}$ (see [3, Theorem 1.2.5]). So the vanishing of the Ext-modules and the exact sequence 5 induce the following commutative diagram with exact rows

$$\begin{array}{ccccccc} & \mathrm{Ext}^d_R(R/\mathfrak{m}^\alpha, H) & \to & \mathrm{Ext}^d_R(R/\mathfrak{m}^{\alpha+1}, H) & \to & \mathrm{Ext}^d_R(\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}, H) & \to 0 \\ & \downarrow f_\alpha & & \downarrow f_{\alpha+1} & & \downarrow f & \\ 0 \to & \mathrm{Ext}^t_R(R/\mathfrak{m}^\alpha, M) & \to & \mathrm{Ext}^t_R(R/\mathfrak{m}^{\alpha+1}, M) & \to & \mathrm{Ext}^t_R(\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}, M) & \end{array}$$

Since $\mathfrak{m}^s/\mathfrak{m}^{s+1}$ is a finite dimensional k-vector space. Then the natural homomorphism f is surjective (resp. injective) because of φ_2 is surjective (resp.

injective). It implies that f_α is surjective (resp. injective) for all $\alpha \in \mathbb{N}$ (by snake lemma and induction hypothesis). It completes the proof of the claim. So if φ_2 is surjective (resp. injective) then the natural homomorphisms

$$f_\alpha : \operatorname{Ext}^d_R(R/\mathfrak{m}^\alpha, H) \to \operatorname{Ext}^t_R(R/\mathfrak{m}^\alpha, M)$$

are surjective (resp. injective) for each $\alpha \in \mathbb{N}$. Since direct limit is an exact functor. So by passing to the direct limit it induces that

$$\varphi_1 : H^d_{\mathfrak{m}}(H^c_I(M)) \to H^t_{\mathfrak{m}}(M)$$

is surjective (resp. injective). □

Remark 4.3. Suppose that $0 \neq M$ is a Cohen-Macaulay R-module with $\operatorname{inj}\dim_R(H^c_I(M)) \leq d = t - c$. If φ_2 is surjective (resp. injective). Then φ_1 is surjective (resp. injective). It is clear from the proof of the last Theorem 4.2. The similar result is obtained in [16, Theorem 4.4] for a Gorenstein local ring.

Note that the following result is a generalization of [14, Corollary 4.6]. The proof given there depends on the derived category theory. Here we get the similar result as a consequence of our Theorem 4.2.

Corollary 4.4. *Fix the notation of Theorem 4.1. Then the following conditions hold:*

(1) *If φ_2 is an isomorphism and $\operatorname{Ext}^i_R(k, H^c_I(M)) = 0$ for all $i \neq d = t - c$. Then φ_1 an isomorphism and $H^i_{\mathfrak{m}}(H^c_I(M)) = 0$ for all $i \neq d$.*

(2) *If φ_1 is an isomorphism and $H^i_{\mathfrak{m}}(H^c_I(M)) = 0$ for all $i < d$. Then φ_2 is an isomorphism and $\operatorname{Ext}^i_R(k, H^c_I(M)) = 0$ for all $i < d$. Moreover there are isomorphisms*

$$\operatorname{Ext}^i_R(k, H^c_I(M)) \to \operatorname{Ext}^{i+c}_R(k, M)$$

for all $i > d$.

Proof. Firstly we prove the statement (1). By Theorems 4.2 and 4.1 we only need to prove that $H^i_{\mathfrak{m}}(H^c_I(M)) = 0$ for all $i > d$. To do this note that $\dim_R(H^c_I(M)) \leq d = t-c$ (see [4]). Then by Grothendieck's vanishing result, [2, Theorem 6.1.2], it follows that $H^i_{\mathfrak{m}}(H^c_I(M)) = 0$ for all $i > d$.

For the statement (2) we only check the isomorphisms (for vanishing result see Lemma 2.5 for $X = H^c_I(M)$ and $s = d$). But the isomorphisms follow from [14, Lemma 4.5]. □

At the end of this section we will relate the endomorphism rings of M and the local cohomology module $H^c_I(M)$. This concept is firstly studied by the author and Z. Zahid in [17, Theorem 1.1]. The next result is actually a Corollary of [17, Theorem 1.1]. Before this let us make some identifications. We will denote $\hat{R}_{\mathfrak{p}}^{IR_{\mathfrak{p}}}$ by the completion of $R_{\mathfrak{p}}$ with respect to the ideal $IR_{\mathfrak{p}}$ where $\mathfrak{p} \in V(I) \cap \operatorname{Supp}_R(M)$. Let $k(\mathfrak{p})$ stand for the the residue field of $R_{\mathfrak{p}}$. Moreover for $c = \operatorname{grade}(I, M)$ we set $h(\mathfrak{p}) := \dim(M_{\mathfrak{p}}) - c$.

Corollary 4.5. *(see [17, Theorem 1.1]) Let $0 \neq M$ be a maximal Cohen-Macaulay R-module. Let $I \subseteq R$ be an ideal of $c = \operatorname{grade}(I, M)$. Suppose that for all $\mathfrak{p} \in V(I) \cap \operatorname{Supp}_R(M)$ the natural homomorphism*

$$\operatorname{Ext}^{h(\mathfrak{p})}_{R_\mathfrak{p}}(k(\mathfrak{p}), H^c_{IR_\mathfrak{p}}(M_\mathfrak{p})) \to \operatorname{Ext}^{\dim(M_\mathfrak{p})}_{R_\mathfrak{p}}(k(\mathfrak{p}), (M_\mathfrak{p}))$$

is an isomorphism and $\operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), H^c_{IR_\mathfrak{p}}(M_\mathfrak{p})) = 0$ for all $i \neq h(\mathfrak{p})$. Then the natural homomorphism

$$\operatorname{Hom}_{\hat{R}^{IR_\mathfrak{p}}_\mathfrak{p}}(\hat{M}^{IR_\mathfrak{p}}_\mathfrak{p}, \hat{M}^{IR_\mathfrak{p}}_\mathfrak{p}) \to \operatorname{Hom}_{R_\mathfrak{p}}(H^c_{IR_\mathfrak{p}}(M_\mathfrak{p}), H^c_{IR_\mathfrak{p}}(M_\mathfrak{p}))$$

is an isomorphism for all $\mathfrak{p} \in V(I) \cap \operatorname{Supp}_R(M)$.

Proof. We claim that our assumption implies that $H^i_I(M) = 0$ for all $i \neq c$. To prove this we will use induction on $\dim_R(M/IM)$. Let $\dim_R(M/IM) = 0$ then it follows that $\operatorname{Supp}(H^i_I(M)) \subseteq V(\mathfrak{m})$ for all $i \in \mathbb{Z}$. So our assumption is true for $\mathfrak{p} = \mathfrak{m}$.

Then by Corollary 4.4 φ_1 is an isomorphism and $H^i_\mathfrak{m}(H^c_I(M)) = 0$ for all $i \neq t - c$. Note that the exact sequence 1 provides the following exact sequence

$$0 \to H^{t-1}_\mathfrak{m}(C^\cdot_M(I)) \to H^d_\mathfrak{m}(H^c_I(M)) \to H^t_\mathfrak{m}(M) \to H^t_\mathfrak{m}(C^\cdot_M(I)) \to 0,$$

isomorphisms $H^{i-c}_\mathfrak{m}(H^c_I(M)) \cong H^{i-1}_\mathfrak{m}(C^\cdot_M(I))$ for $i < t$ and the vanishing $H^i_\mathfrak{m}(C^\cdot_M(I)) = 0$ for $i > t$. Recall that $H^i_\mathfrak{m}(M) = 0$ for all $i \neq t$ since M is Cohen-Macaulay. It implies that $H^i_\mathfrak{m}(C^\cdot_M(I)) = 0$ for all $i \neq t, t-1$. Moreover φ_1 is an isomorphism so the last exact sequence provides that $H^i_\mathfrak{m}(C^\cdot_M(I)) = 0$ for all $i \in \mathbb{Z}$.

Since $\operatorname{Supp}_R(H^i(C^\cdot_M(I))) \subseteq V(\mathfrak{m})$. So by [14, Lemma 2.5] in view of definition of the truncation complex we have

$$0 = H^i_\mathfrak{m}(C^\cdot_M(I)) \cong H^i(C^\cdot_M(I)) \cong H^i_I(M)$$

for all $c < i \leq n$. This proves the claim for $\dim(M/IM) = 0$. Now let us assume that $\dim(M/IM) > 0$. Then it is easy to see that $\dim(M_\mathfrak{p}/IM_\mathfrak{p}) < \dim(M/IM)$ for all $\mathfrak{p} \in V(I) \cap \operatorname{Supp}_R(M) \setminus \{\mathfrak{m}\}$. By the induction hypothesis we have

$$H^i_{IR_\mathfrak{p}}(M_\mathfrak{p}) = 0$$

for all $i \neq c$ and for all $\mathfrak{p} \in V(I) \cap \operatorname{Supp}(M) \setminus \{\mathfrak{m}\}$. That is $\operatorname{Supp}(H^i_I(M)) \subseteq V(\mathfrak{m})$ for all $i \neq c$. Then by the similar arguments as we use above, for $\mathfrak{p} = \mathfrak{m}$, our claim is true. That is $H^i_I(M) = 0$ for all $i \neq c$. Since M is maximal Cohen-Macaulay so by Lemma 2.3 and [14, Propositios 2.7] it follows that

$$H^i_{IR_\mathfrak{p}}(M_\mathfrak{p}) = 0$$

for all $i \neq c = \operatorname{grade}(IR_\mathfrak{p}, M_\mathfrak{p})$ and for all $\mathfrak{p} \in V(I) \cap \operatorname{Supp}_R M$. Now the existence of the natural homomorphism

$$\operatorname{Hom}_{\hat{R}^{IR_\mathfrak{p}}_\mathfrak{p}}(\hat{M}^{IR_\mathfrak{p}}_\mathfrak{p}, \hat{M}^{IR_\mathfrak{p}}_\mathfrak{p}) \to \operatorname{Hom}_{R_\mathfrak{p}}(H^c_{IR_\mathfrak{p}}(M_\mathfrak{p}), H^c_{IR_\mathfrak{p}}(M_\mathfrak{p}))$$

was shown in [17, Theorem 1.1]. But $H^i_{IR_\mathfrak{p}}(M_\mathfrak{p}) = 0$ for all $i \neq c = \operatorname{grade}(IR_\mathfrak{p}, M_\mathfrak{p})$. Hence it proves our required isomorphism in view of [17, Theorem 1.1]. □

Remark 4.6. Note that it is unknown to us whether the last Corollary is true for only $\mathfrak{p} = \mathfrak{m}$, the maximal ideal. However it is true if we replace M by a complete local Gorenstein ring (see [16, Theorem 4.4]).

5. The homomorphisms φ_3, φ_6 and φ_7

In this section our intention is to study the homomorphisms φ_3, φ_6 and φ_7 of Propositions 3.1 and 3.3 in more details. In fact our investigation is useful to prove that the natural homomorphism $\operatorname{Tor}^R_c(k, H^c_I(M)) \to k \otimes_R M$ is non-zero. First of all note that the motivation of the next Proposition is the relation between the endomorphism rings of modules $H^c_I(M)$ and $H^c_J(M)$ such that $c = \operatorname{grade}(I, M) = \operatorname{grade}(J, M)$ and $J \subseteq I$. This was firstly introduced by P. Schenzel in case of $M = R$ a local Gorenstein ring (see [20, Theorem 1.2]). Moreover for an extension to modules we refer to [13, Theorem 1.1] and [17, Theorem 1.2]. Here we are interested to prove the similar result for Tor modules.

Proposition 5.1. *Let M be a non-zero finitely generated R-module such that $c = \operatorname{grade}(I, M) = \operatorname{grade}(J, M)$ where $J \subseteq I$ are two ideals. Then we have the following results:*

(1) *There is a natural homomorphism*
$$\operatorname{Tor}^R_c(k, H^c_I(M)) \to \operatorname{Tor}^R_c(k, H^c_J(M)).$$
(2) *Suppose that $\operatorname{Rad} IR_\mathfrak{p} = \operatorname{Rad} JR_\mathfrak{p}$ for all $\mathfrak{p} \in V(J) \cap \operatorname{Supp}_R(M)$ such that $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq c$. Then the above natural homomorphism is an isomorphism.*

Proof. Since $J \subset I$ it induces the following short exact sequence
$$0 \to I^\alpha/J^\alpha \to R/J^\alpha \to R/I^\alpha \to 0$$
for each integer $\alpha \geq 1$. Let $E^{\cdot}_R(M)$ be a minimal injective resolution of M. Then we have the exact sequence
$$0 \to \operatorname{Hom}_R(R/I^\alpha, E^{\cdot}_R(M)) \to \operatorname{Hom}_R(R/J^\alpha, E^{\cdot}_R(M)) \to \operatorname{Hom}_R(I^\alpha/J^\alpha, E^{\cdot}_R(M)) \to 0.$$
Then the cohomology sequence of this, at degree c, gives rise to the exact sequence
$$0 \to \operatorname{Ext}^c_R(R/I^\alpha, M) \to \operatorname{Ext}^c_R(R/J^\alpha, M) \to \operatorname{Ext}^c_R(I^\alpha/J^\alpha, M).$$
To this end note that $\operatorname{grade}(I^\alpha/J^\alpha, M) \geq c$ (see [13, Proposition 2.1]). Now the direct limit is an exact functor. So pass to the direct limit of this sequence we get the exact sequence
$$0 \to H^c_I(M) \to H^c_J(M) \xrightarrow{f} \varinjlim \operatorname{Ext}^c_R(I^\alpha/J^\alpha, M) \tag{6}$$

Let $N := \operatorname{Im} f$. Then this induces the short exact sequence

$$0 \to H^c_I(M) \to H^c_J(M) \to N \to 0$$

So for a minimal free resolution $F^{\cdot}_R(k)$ of k we have the following short exact sequence of complexes

$$0 \to H^c_I(M) \otimes_R F^{\cdot}_R(k) \to H^c_J(M) \otimes_R F^{\cdot}_R(k) \to N \otimes_R F^{\cdot}_R(k) \to 0$$

Hence from the long exact sequens of cohomologies there is the following natural homomorphism

$$\operatorname{Tor}^R_c(k, H^c_I(M)) \to \operatorname{Tor}^R_c(k, H^c_J(M))$$

it completes the proof of (1).

For the proof of (2) note that $\operatorname{Ext}^c_R(I^\alpha/J^\alpha, M) = 0$ for all $\alpha \geq 1$ under the additional assumption in the statement (2) (see the proof of [17, Theorem 4.1(b)]). Hence (2) is true by virtue of the exact sequence 6. □

Consequently our Proposition 5.1 gives us a characterization such that our natural homomorphism φ_3 becomes an isomorphism as follows:

Corollary 5.2. *Let $0 \neq M$ be a finitely generated R-module of $\dim_R(M) = t$. Let $J \subseteq I$ be two ideals and $H^i_J(M) = 0$ for all $i \neq c = \operatorname{grade}(J, M) = \operatorname{grade}(I, M)$. Suppose that $\operatorname{Rad} IR_{\mathfrak{p}} = \operatorname{Rad} JR_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p} \in V(J) \cap \operatorname{Supp}_R(M)$ with $\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq c$. Then the natural homomorphism*

$$\varphi_3 : \operatorname{Tor}^R_c(k, H^c_I(M)) \to k \otimes_R M$$

is an isomorphism.

Proof. It is a consequence of Proposition 5.1 and Corollary 3.5. □

Here we will give some examples related to the homomorphism φ_3.

Example 5.3. (1) Let $R = A/J$ where $A = F[|x, y, z, w|]$ be the formal power series ring over a field F and $J = (xz, yw) = (x, y) \cap (y, z) \cap (z, w) \cap (x, w)$. Further assume that $I = (x, y, z)R$ and $M = R/\mathfrak{p}_1 \oplus R/\mathfrak{p}_2$ where $\mathfrak{p}_1 = (x, y)R$, and $\mathfrak{p}_2 = (y, z)R$. Then R is a complete local Gorenstein ring with residue filed $k = \mathfrak{m}/J$. Here $\mathfrak{m} = (x, y, z, w)$. Moreover, by [3, Proposition 2.14]), $H^i_I(M) = 0$ for all $i \neq 1$. Then by Corollary 3.5 the natural homomorphism

$$\varphi_3 : \operatorname{Tor}^R_1(k, H^1_I(M)) \to k \otimes_R M.$$

is an isomorphism so that $\operatorname{Tor}^R_1(k, H^1_I(M)) \cong k^2$.

(2) Let $R := k[|x, y, z, w|]$ be the formal power series ring over a field k. Suppose that $I := (x) \cap (z, y) = (xz, xy)$ and $J := (xz)$. Then it is easy to see that $\operatorname{grade} J = \operatorname{grade} I = 1$ and $H^i_I(R) \neq 0$ for $i = 1, 2$ and zero otherwise. Moreover $H^i_J(R) = 0$ for all $i \neq 1$. Now there are only two minimal primes of height 1 containing J, that is (x) and (z). It follows that $\operatorname{Rad} IR_{\mathfrak{p}} = \operatorname{Rad} JR_{\mathfrak{p}}$

for all $\mathfrak{p} \in V(J)$ such that $\operatorname{depth}_{R_\mathfrak{p}}(R_\mathfrak{p}) \leq 1$. Then by Proposition 5.1 and Corollary 5.2 the natural homomorphisms

$$\operatorname{Tor}_1^R(k, H_I^1(R)) \to \operatorname{Tor}_1^R(k, H_J^1(R)).$$

$$\varphi_3 : \operatorname{Tor}_1^R(k, H_I^1(R)) \to k.$$

are isomorphisms.

In the following result we will give a characterization to check that the natural homomorphism $\varphi_3 : \operatorname{Tor}_c^R(k, H_I^c(M)) \to k \otimes_R M$ is non-zero. In fact in the parallel of Theorem 4.1 we are succeeded to derive the equivalence of surjectivity and injectivity of φ_3, φ_6 and φ_7 in terms of vanishing of Betti numbers of the module $H_I^c(M)$.

Theorem 5.4. *Let M be a non-zero finitely generated module over R and $\dim_R(M) = t$. Suppose that $\operatorname{grade}(I, M) = c$ for an ideal I. Then the following conditions are equivalent:*

(1) *φ_3 is surjective and $\operatorname{Tor}_i^R(k, H_I^c(M)) = 0$ for all $i < c$.*
(2) *φ_6 is injective and $H_\mathfrak{m}^i(D(H_I^c(M))) = 0$ for all $i < c$.*
(3) *φ_7 is injective and $H_\mathfrak{m}^i(D(H_I^c(M))) = 0$ for all $i < c$.*

Moreover if any of the equivalent conditions holds then $H_\mathfrak{m}^c(D(H_I^c(M))) \neq 0$.

Proof. Firstly we prove that the statement in (1) is equivalent to the statement in (3). Since $\operatorname{Ext}_R^i(k, D(H_I^c(M))) \cong D(\operatorname{Tor}_i^R(k, H_I^c(M)))$ for all $i \in \mathbb{Z}$ (by Hom-tensor Duality). Then it follows that the vanishing statement in (1) is equivalent to the fact that $\operatorname{Ext}_R^i(k, D(H_I^c(M))) = 0$ for all $i < c$. Then it proves the equivalence of the vanishing statements in (1) and (3) (see Lemma 2.5 for $X = D(H_I^c(M))$).

Note that there is a natural homomorphism $D(k\otimes_R M) \to \operatorname{Ext}_R^c(k, D(H_I^c(M)))$ (see Remark 3.2). So by virtue of Propositions 3.1 and 3.3 the natural homomorphism $M \to k \otimes_R M$ induces the commutative diagram

$$\begin{array}{ccc} D(k \otimes_R M) & \to & D(M) \\ \downarrow & & \downarrow \varphi_7 \\ \operatorname{Ext}_R^c(k, D(H_I^c(M))) & \to & H_\mathfrak{m}^c(D(H_I^c(M))) \end{array}$$

Then applying $\operatorname{Hom}_R(k, \cdot)$ to the above diagram it provides the following commutative diagram

$$\begin{array}{ccc} D(k \otimes_R M) & \to & \operatorname{Hom}_R(k, D(M)) \\ \downarrow & & \downarrow \\ \operatorname{Ext}_R^c(k, D(H_I^c(M))) & \to & \operatorname{Hom}_R(k, H_\mathfrak{m}^c(D(H_I^c(M)))) \end{array}$$

Then both of the horizontal homomorphisms are isomorphisms (by Hom-Tensor Duality and the equivalence of the vanishing statements, see Lemma 2.5). Suppose that φ_7 is injective then the homomorphism $\operatorname{Hom}_R(k, D(M)) \to \operatorname{Hom}_R(k, H_\mathfrak{m}^c(D(H_I^c(M))))$ is injective. By view of the commutative diagram $D(k \otimes_R$

$M) \to \operatorname{Ext}^c_R(k, D(H^c_I(M)))$ is injective. But this last homomorphism is the Matlis dual of φ_3. This proves the surjectivity of φ_3.

For the converse let φ_3 be surjective. Then by the above arguments the homomorphism $\operatorname{Hom}_R(k, D(M)) \to \operatorname{Hom}_R(k, H^c_{\mathfrak{m}}(D(H^c_I(M))))$ is injective. Now let $N := \ker \varphi_7$ then it induces the following exact sequence

$$0 \to N \to D(M) \to H^c_{\mathfrak{m}}(D(H^c_I(M))) \quad (7)$$

We claim that $N = 0$. Note that the last sequence induces the following exact sequence

$$0 \to \operatorname{Hom}_R(k, N) \to \operatorname{Hom}_R(k, D(M)) \to \operatorname{Hom}_R(k, H^c_{\mathfrak{m}}(D(H^c_I(M))))$$

Then we have $\operatorname{Hom}_R(k, N) = 0$ this proves the claim. Since the support of N is contained in $\{\mathfrak{m}\}$. Then the result follows from sequence 7. This finishes the proof of (1) $\Leftrightarrow$ (3).

Now we prove that (1) is equivalent to (2). Note that the equivalence of the vanishing statement is proved in (1) $\Leftrightarrow$ (3). Moreover there are natural homomorphisms

$$D(k \otimes_R M) \to D(M) \text{ and } D(M) \to H^c_{\mathfrak{m}}(D(H^c_I(M))).$$

But the first map is injective and $\operatorname{Hom}_R(M, k) \cong D(k \otimes_R M)$. So by composing the last two maps, in view of the equivalence of the statements (1) and (3), it follows that φ_6 is injective if φ_3 is surjective.

Conversely suppose that φ_6 is injective. Then by Propositions 3.1 and 3.3 we have the commutative diagram

$$\begin{array}{ccc} D(k \otimes_R M) & = & D(k \otimes_R M) \\ \downarrow & & \downarrow \varphi_6 \\ \operatorname{Ext}^c_R(k, D(H^c_I(M))) & \to & H^c_{\mathfrak{m}}(D(H^c_I(M))) \end{array}$$

To this end note that $D(k \otimes_R M) \cong \operatorname{Hom}_R(M, k)$. Then it induces the following commutative diagram

$$\begin{array}{ccc} D(k \otimes_R M) & \to & \operatorname{Hom}_R(k, D(k \otimes_R M)) \\ \downarrow & & \downarrow \\ \operatorname{Ext}^c_R(k, D(H^c_I(M))) & \to & \operatorname{Hom}_R(k, H^c_{\mathfrak{m}}(D(H^c_I(M)))) \end{array}$$

Then by the similar arguments as we used above one can easily prove that φ_3 is surjective. Moreover $H^c_{\mathfrak{m}}(D(H^c_I(M))) \neq 0$ by definition of φ_7 in view of the equivalence of the above statements. Recall that $M \neq 0$. This finishes the proof of the Theorem. □

For the next Theorem we will make some more notations. Note that in case of $M = R$ we have the following homomorphisms:

$$\psi_1 : \operatorname{Tor}^R_c(k, H^c_I(R)) \to k, \text{ and}$$

$$\psi_2 : E \to H^c_{\mathfrak{m}}(D(H^c_I(R)))$$

(see Propositions 3.1 and 3.3).

Theorem 5.5. *Let I be an ideal of R with* $\operatorname{grade}(I) = c$. *If the homomorphism ψ_1 is injective and $\operatorname{Tor}_i^R(k, H_I^c(R)) = 0$ for all $i < c$. Then the homomorphism ψ_2 is surjective and $H_{\mathfrak{m}}^i(D(H_I^c(R))) = 0$ for all $i < c$.*

Proof. We know that there are the natural homomorphisms

$$f_\alpha : \operatorname{Tor}_c^R(R/\mathfrak{m}^\alpha, H_I^c(R)) \to R/\mathfrak{m}^\alpha$$

for all $\alpha \in \mathbb{N}$ (see Remark 3.2). Moreover after the application of the functor $\operatorname{Tor}_i^R(\cdot, H_I^c(R))$ to the exact sequence 5 we get the exact sequence

$$\operatorname{Tor}_i^R(\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}, H_I^c(R)) \to \operatorname{Tor}_i^R(R/\mathfrak{m}^{\alpha+1}, H_I^c(R)) \to \operatorname{Tor}_i^R(R/\mathfrak{m}^\alpha, H_I^c(R))$$

for all $i \in \mathbb{Z}$. Then by induction on α , in view of vanishing of Tor modules, this proves that $\operatorname{Tor}_i^R(R/\mathfrak{m}^\alpha, H_I^c(R))) = 0$ for all $i < c$ and for all $\alpha \in \mathbb{N}$.

Now we show that f_α is injective for all $\alpha \in \mathbb{N}$. Clearly $f_1 = \psi_1$ is injective. Then the short exact sequence 5 induces the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & \operatorname{Tor}_c^R(\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}, H_I^c(R)) & \to & \operatorname{Tor}_c^R(R/\mathfrak{m}^{\alpha+1}, H_I^c(R)) & \to & \operatorname{Tor}_c^R(R/\mathfrak{m}^\alpha, H_I^c(R)) & \to & 0 \\ & & \downarrow f & & \downarrow f_{\alpha+1} & & \downarrow f_\alpha & & \\ 0 & \to & \mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1} & \to & R/\mathfrak{m}^{\alpha+1} & \to & R/\mathfrak{m}^\alpha & \to & 0 \end{array}$$

Note that the above row is exact because of $\operatorname{Tor}_i^R(R/\mathfrak{m}^\alpha, H_I^c(R)) = 0$ for all $i < c$ and for all $\alpha \in \mathbb{N}$. Then the natural homomorphism f is injective because of $\mathfrak{m}^\alpha/\mathfrak{m}^{\alpha+1}$ is a finite dimensional k-vector space. Hence by induction, in view of snake lemma, it implies that f_α is injective for all $\alpha \in \mathbb{N}$. Take the Matlis dual of f_α it induces that the surjective homomorphism

$$D(R/\mathfrak{m}^\alpha) \to \operatorname{Ext}_R^c(R/\mathfrak{m}^\alpha, D(H_I^c(R)))$$

for all $\alpha \in \mathbb{N}$. Now apply the direct limit to these maps. Since the direct limit is an exact functor so it implies that the homomorphism

$$\varinjlim D(R/\mathfrak{m}^\alpha) \to H_{\mathfrak{m}}^c(D(H_I^c(R)))$$

is surjective. Moreover there is an isomorphism

$$\varinjlim D(R/\mathfrak{m}^\alpha) \cong H_{\mathfrak{m}}^0(E).$$

But $H_{\mathfrak{m}}^0(E) \cong E$ because of $\operatorname{Supp}_R(E) \subseteq \{\mathfrak{m}\}$. Therefore it proves that ψ_2 is surjective. Moreover Theorem 5.4 implies that $H_{\mathfrak{m}}^i(D(H_I^c(R))) = 0$ for all $i < c$. □

Corollary 5.6. *With the previous notation if the homomorphism ψ_1 is an isomorphism and $\operatorname{Tor}_i^R(k, H_I^c(R)) = 0$ for all $i < c$. Then the homomorphism ψ_2 is an isomorphism and $H_{\mathfrak{m}}^i(D(H_I^c(R))) = 0$ for all $i < c$.*

Proof. This is immediately follows from Theorems 5.4 and 5.5. □

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