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GLOBAL CONVERGENCE OF AN INEXACT INTERIOR-POINT METHOD FOR CONVEX QUADRATIC SYMMETRIC CONE PROGRAMMING

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Abstract. In this paper, we propose a feasible interior-point method for convex quadratic programming over symmetric cones. The proposed algorithm relaxes the accuracy requirements in the solution of the Newton equation system, by using an inexact Newton direction. Furthermore, we obtain an acceptable level of error in the inexact algorithm on convex quadratic symmetric cone programming (CQSCP). We also prove that the iteration bound for the feasible short-step method is $O(\sqrt{n}\log\frac{1}{\varepsilon})$, and $O(n\log\frac{1}{\varepsilon})$ for the large-step method which coincide with the currently best known iteration bounds for CQSCPs.
Keywords: Convex quadratic symmetric cone programming, short- and large-step feasible interior-point method, inexact search directions, polynomial complexity.
MSC(2010): Primary: 90C05; Secondary: 90C51.

1. Introduction

The path-following interior-point methods (IPMs) are one of the most efficient numerical methods for various classes of optimization problems. A major advantage of IPMs in comparison with other methods is their polynomial complexity. IPMs for solving linear optimization (LO) problems were initiated by Karmarkar [19]. These methods could be naturally extended to obtain polynomial-time algorithms for conic optimization such as convex quadratic symmetric cone programming, semidefinite optimization (SDO) problems and second order cone optimization (SOCO) problems.
Jordan algebra initially created in quantum mechanics was first introduced by Jordan. Some Jordan algebras were proved to be an indispensable tool in the unified study of IPMs. The first work connecting Jordan algebras and

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optimization is due to Guler [12]. Alizadeh analyzed a primal-dual IPM for SDO problems in [1]. Faybusovich analyzed several IPMs for symmetric optimization using the Jordan algebra framework, and presented some short-step path-following IPMs in [7, 8]. Faybusovich and Arana [10] derived complexity estimates for a large-step primal-dual interior-point algorithm.

The interest in the use of iterative methods to solve the Newton equation system in IPMs has been growing over the last decade. In all of interior-point algorithms, we need to solve the Newton search direction system to obtain the exact search directions. Thus, in each iteration of a primal-dual IPM, most of the computational work is devoted to the computation of exact search directions by solving a linear system of equations. Even if one uses a direct or iterative method to solve the linear system exactly, the solution may not satisfy the linear equations due to rounding errors. However, finding an accurate solution of the Newton search directions system is hard and difficult in IPMs. It is well known that this difficulty can be remedied by relaxing the accuracy requirement in the solution of the Newton system. We will refer to this algorithm as an "inexact" feasible algorithm which determines the search directions only approximately at each iteration. This algorithm requires that the equations corresponding to the primal and dual feasibilities be satisfied exactly, but the equation corresponding to complementarity is relaxed. In order to guarantee the global polynomial convergence of the inexact feasible IPM, the inexactness in the search directions must be appropriately controlled.

The use of an inexact IPM was started in the 1980's when attempts were made to solve large LO problems. For LO and monotone linear complementarity problems (MLCPs), numerous papers have been devoted to the design and analysis of inexact IPMs. Bellavia [2] applied an inexact IPM to solve monotone nonlinear complementarity problems (NLCPs) and proved global and local super linear convergence of this method. Freund et al. [9] and Mizuno and Jarre [22] extended a very popular globally convergent infeasible path-following method for LO problems of Kojima et al. [18] to accommodate the inexact solution of Newton systems. There are relatively fewer papers on the convergence analysis of inexact IPMs for convex quadratic programming (CQP), the most recent ones are [4, 21]. For the computational aspects of inexact IPMs for LO and CQP, we refer the readers to [3, 5] and the references therein.

Recently, Gondzio [15] presented a new analysis for convergence of inexact feasible IPM on CQP. He proved that the iteration bound complexities of short- and long-step inexact feasible primal-dual algorithms for CQP are $O(\sqrt{n}\log\frac{1}{\varepsilon})$ and $O(n\log\frac{1}{\varepsilon})$, respectively. In this paper, we generalize the convergence analysis of the feasible IPM on CQP proposed by Gondzio [15] to CQSCP.

The paper is organized as follows. In Section 2, we provide some basic concepts on Euclidean Jordan algebra. In Sections 3, we introduce the primal-dual pair of CQSCPs and propose an IPM as a good approach to solve this class of optimization problems. Section 4 presents the analysis of two variants of an inexact feasible interior-point algorithm for CQSCP; feasible short-step inexact IPM and feasible long-step inexact IPM. We respectively get the complexity of feasible short- and long-step IPMs for CQSCP in Subsections 4.2 and 5.1. In Section 6, we provide some preliminary numerical experiments. Finally, the paper ends with some conclusions in Section 7.

2. **Preliminaries**

In this section, we briefly review and introduce Jordan algebras as well as some of their basic properties.

A Jordan algebra $\mathcal{J}$ is a finite dimensional vector space over the field of real or complex numbers endowed with a bilinear map $\circ : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ satisfying the following properties for all $x, y \in \mathcal{J}$:

(**i**): $x \circ y = y \circ x$,

(**ii**): $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$,

where $x^2 = x \circ x$. Moreover, $(\mathcal{J}, \circ)$ is called an Euclidean Jordan algebra (EJA) if there exists an inner product denoted by $\langle \cdot, \cdot \rangle$ such that $\langle x \circ y, z\rangle = \langle x, y \circ z\rangle$ for all $x, y, z \in \mathcal{J}$. A Jordan algebra has an identity element, if there exist a unique element $e \in \mathcal{J}$ such that $x \circ e = e \circ x = x$ for all $x \in \mathcal{J}$. The set $\mathcal{K} := \mathcal{K}(\mathcal{J}) = \{x^2 : x \in \mathcal{J}\}$ is called the cone of squares of EJA $(\mathcal{J}, \circ, \langle \cdot, \cdot \rangle)$ and $\text{int}(\mathcal{K})$ denotes the interior of $\mathcal{K}$. A cone is symmetric if and only if it is the cone of squares of an EJA. An element $c \in \mathcal{J}$ is said to be idempotent if $c^2 = c$. An idempotent c is primitive if it is nonzero and can not be expressed by sum of two other nonzero idempotents. A set of idempotents $\{c_1, c_2, ..., c_k\}$ is called a Jordan frame if $c_i \circ c_j = 0$ for any $i \neq j$, and $\sum_{i=1}^{k} c_i = e$. For any $x \in \mathcal{J}$, let l be the smallest positive integer such that $\{e, x, x^2, ..., x^l\}$ is linearly dependent, l is called the degree of x and is denoted by $\deg(x)$. The rank of $\mathcal{J}$, denoted by $\text{rank}\,(\mathcal{J})$, is defined as the maximum of $\deg(x)$ over all $x \in \mathcal{J}$.

Theorem 2.1. (Theorem III.1.2 in [6]) *Let $x \in \mathcal{J}$ and $\text{rank}\,(\mathcal{J}) = n$. Then, there exist unique real numbers $\lambda_1(x), \lambda_2(x), ..., \lambda_n(x)$ all distinct, and Jordan frame $\{c_1, c_2, ..., c_n\}$ such that $x = \sum_{i=1}^{n} \lambda_i(x) c_i$.*

Every $\lambda_i(x)$ is called an eigenvalue of x. We denote $\lambda_{\min}(x)$ $(\lambda_{\max}(x))$ as the minimal(maximal) eigenvalue of x. Also, we can define the following conceptions:

Inverse : $x^{-1} := \sum_{i=1}^{n} \lambda_i^{-1}(x) c_i$, wherever all $\lambda_i(x) \neq 0$,

Square root : $x^{\frac{1}{2}} := \sum_{i=1}^{n} \lambda_i^{\frac{1}{2}}(x) c_i$,

Squqre : $x^2 := \sum_{i=1}^{n} \lambda_i^2(x) c_i$,

Trace : $\mathbf{tr}(x) := \sum_{i=1}^{n} \lambda_i(x)$,

Determinant : $\mathbf{det}(x) := \prod_{i=1}^{n} \lambda_i(x)$.

Since "$\circ$" is a bilinear map, for every $x \in \mathcal{J}$, a linear operator $L(x)$ can be defined such that $L(x)y = x \circ y$ for all $y \in \mathcal{J}$. In particular, $L(x)e = x$ and

$L(x)x = x^2$. For each $x \in \mathcal{J}$, we define

$$Q_x := 2L(x)^2 - L(x^2),$$

where, $L(x)^2 = L(x)L(x)$. The map Q_x is called the quadratic representation of x. The quadratic representation is an essential concept in theory of Jordan algebras and plays an important role in convergence analysis of IPMs in symmetric optimization. For any $x, y \in \mathcal{J}$, x and y are said to be operator commute if $L(x)$ and $L(y)$ commute, i.e., $L(x)L(y) = L(y)L(x)$. We define the inner product of $x, y \in \mathcal{J}$ as $\langle x, y\rangle = \mathbf{tr}(x \circ y)$. The norm induced by this inner product is named as the Frobenius norm, which is given by

$$\|x\|_F := \sqrt{\langle x, x\rangle} = \sqrt{\mathbf{tr}(x^2)} = \sqrt{\sum_{i=1}^{n} \lambda_i^2(x)}.$$

We can also define some other norms such as 1-norm and 2-norm as follow:

$$\|x\|_1 := \sum_{i=1}^{n} |\lambda_i(x)|, \ \|x\|_2 := \max_i |\lambda_i(x)|,$$

where $\lambda_i(x)$ is the i-th eigenvalue of the vector x. Here, we list some results which are required in this paper.

Lemma 2.2. (Lemma 3.2 in [11]) *For $x, s \in int(\mathcal{K})$ there exists a unique $u \in int(\mathcal{K})$ such that $x = Q_u s$. Moreover,*

$$u = Q_{x^{\frac{1}{2}}}\left(Q_{x^{\frac{1}{2}}} s\right)^{-\frac{1}{2}} \left[= Q_{s^{-\frac{1}{2}}}\left(Q_{s^{\frac{1}{2}}} x\right)^{\frac{1}{2}}\right],$$

where the point u is called the scaling point of x and s.

Lemma 2.3. (Lemma 4.38 in [13]) *If $x \in int(\mathcal{K})$, then $x^{\frac{1}{2}}$ is well-defined and $Q_{x^{\frac{1}{2}}} = (Q_x)^{\frac{1}{2}}$.*

Lemma 2.4. (Lemma 4.52 in [13]) *Let $x, s \in \mathcal{J}$, then*

$$|\langle x, s\rangle| \le \|x\|_F \|s\|_F .$$

Lemma 2.5. (Lemma 14 in [16]) *Let $x, s \in \mathcal{J}$, then*

$$\begin{aligned} \lambda_{\min}(x+s) &\ge \lambda_{\min}(x) - \|s\|_F, \\ \lambda_{\max}(x+s) &\le \lambda_{\max}(x) + \|s\|_F. \end{aligned}$$

Lemma 2.6. (Lemma 2.15 in [14]) *If $x \circ s \in int(\mathcal{K})$, then $\boldsymbol{det}(x) \neq 0$.*

3. Interior-point methods for CQSCP

In this section, we introduce the primal-dual pair of CQSCPs and then we use the iterative methods to solve and obtain an ε-optimal solution of this class of optimization problems. In the following, we consider the primal CQSCP:

$$(CP)\qquad \begin{array}{rcl} \min\ \mathcal{F}(x) & := & \frac{1}{2}\langle x, \mathcal{H}(x)\rangle + \langle c, x\rangle \\ \mathcal{A}(x) & = & b, \\ x & \in & \mathcal{K}, \end{array}$$

where, $c \in \mathcal{J}$ and $b \in \mathbb{R}^m$ are given data, $\mathcal{A} : \mathcal{J} \longrightarrow \mathbb{R}^m$ is a linear map and $\mathcal{H}$ is a given self-adjoint positive semidefinite linear operator on $\mathcal{J}$. That is, $\langle \mathcal{H}(x), y\rangle = \langle x, \mathcal{H}(y)\rangle$ and $\langle \mathcal{H}(x), x\rangle \geq 0$ for $x, y \in \mathcal{J}$. The dual CQSCP is given by

$$(CD)\qquad \begin{array}{rcl} \max\ \mathcal{G}(x) & := & -\frac{1}{2}\langle x, \mathcal{H}(x)\rangle + b^T y \\ \mathcal{A}^T(y) + s & = & \bigtriangledown\mathcal{F}(x) = \mathcal{H}(x) + c, \\ s & \in & \mathcal{K}, \end{array}$$

where, $\mathcal{A}^T$ denotes the adjoint of $\mathcal{A}$. Throughout the paper, we assume that the problems (CP) and (CD) are strictly feasible, i.e., there exists (x, y, s) satisfying the linear constraints in (CP) and (CD) and $x, s \in \mathrm{int}(\mathcal{K})$. The primal problem (CP) includes the symmetric cone optimization (SCO) problems when $\mathcal{H} = 0$ and when $\mathcal{K}$ is the cone of symmetric positive semidefinite matrices, it includes SDO problems. In generic IPM, to find an ε-approximate optimal solution of (CP) and (CD) problems, the complementarity condition $x \circ s = 0$ will be perturbed to $x \circ s = \mu e$. In other words, we use the perturbed Karush-Kuhn-Tucker (KKT) optimality conditions for the problems (CP) and (CD) as follow:

$$(3.1)\qquad \begin{pmatrix} -\bigtriangledown\mathcal{F}(x) + \mathcal{A}^T(y) + s \\ \mathcal{A}(x) - b \\ x \circ s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mu e \end{pmatrix},\ x, s \in \mathcal{K},$$

where μ is the duality measure defined by

$$(3.2)\qquad \mu = \frac{1}{n}\sum_{i=1}^{n} \lambda_i(x \circ s) = \frac{\mathbf{tr}(x \circ s)}{n}.$$

System (3.1) has a unique solution denoted by $(x(\mu), y(\mu), s(\mu))$ for any $\mu > 0$. We call $x(\mu)$ and $(y(\mu), s(\mu))$ as the μ-centers of (CP) and (CD) problems, respectively. The set of all μ-centers is called the central path of (CP) and (CD). If $\mu \to 0$, then the limit of the central path exists and since the limit points satisfy the complementarity condition, they yield an ε-approximate optimal solution of (CP) and (CD) (Theorem 4.4 in [17]). According to

$$\begin{aligned}\mathcal{F}(x)-\mathcal{G}(x) &= \langle x,\mathcal{H}(x)\rangle+\langle c,x\rangle-b^Ty\\ &= \langle x,\mathcal{H}(x)\rangle+\langle A^Ty+s-\mathcal{H}(x),x\rangle-b^Ty\\ &= \langle y,Ax\rangle+\langle x,s\rangle-b^Ty=\langle x,s\rangle=\mathbf{tr}(x\circ s)=n\mu,\end{aligned}$$

the duality gap is equal to the complementarity gap and by reducing the barrier parameter μ, IPMs converge to optimality. In order to solve system (3.1), we apply Newton's method to find an approximate solution of CQSCP. That is, we compute the Newton search direction $(\Delta x,\Delta y,\Delta s)$ and make a step in this direction to obtain the new iterate $(x(\alpha),y(\alpha),s(\alpha))$ with $\mu(\alpha)\le\mu$. The reduction of the barrier term μ is enforced by using the parameter $\sigma\in(0,1)$. Note that, linearizing the third equation in system (3.1) may not lead to an element in $\mathcal{J}$. Thus, it is necessary to symmetrize this equation before linearizing it. This difficulty can be remedied by using Lemma 28 in [16]. That is, given an invertible $p\in\mathcal{K}$, we have

$$x\circ s=\mu e \;\Leftrightarrow\; Q_px\circ Q_{p^{-1}}s=\mu e. \tag{3.3}$$

Now, by replacing the third equation in (3.1) by $Q_px\circ Q_{p^{-1}}s=\mu e$, and then applying Newton's method, we obtain the system

$$\begin{aligned}\mathcal{A}\Delta x &= 0,\\ -\mathcal{H}\Delta x+\mathcal{A}^T\Delta y+\Delta s &= 0,\\ Q_px\circ Q_{p^{-1}}\Delta s+Q_{p^{-1}}s\circ Q_p\Delta x &= \xi,\end{aligned} \tag{3.4}$$

where

$$\xi:=\sigma\mu e-Q_px\ o\ Q_{p^{-1}}s. \tag{3.5}$$

We denote $C(x,s)$ as a subclass of the Monteiro-Zhang family of search directions such that the scaled elements are operator commute, i.e.,

$$C(x,s)=\{p|\ p \text{ nonsingular},\ Q_px \text{ and } Q_{p^{-1}}s \text{ operator commute}\}.$$

Some of the best-known choices of the scaled element p has been suggested by different authors. Among them, the scaled element $p=u^{-\frac{1}{2}}$, where u is defined as in Lemma 2.2, leads to the Nesterov-Todd (NT) directions and we use the NT directions in our analysis. Most of iterative methods such as IPMs solve system (3.4) exactly. The word "exact" here means that at each iteration, the search direction $(\Delta x,\Delta y,\Delta s)$ is computed exactly from system (3.4). In contrast, in this paper, we analyze the method that allows system (3.4) to be solved inexactly. In following, we define

$$\begin{gathered}w=Q_{x^{\frac{1}{2}}}s,\ \bar{w}=Q_{\bar{x}^{\frac{1}{2}}}\underline{s},\ \bar{x}=Q_px,\ \underline{s}=Q_{p^{-1}}s,\\ \underline{\mathcal{A}}=\mathcal{A}Q_{p^{-1}},\ \bar{\mathcal{H}}=Q_{p^{-1}}\mathcal{H}Q_{p^{-1}}.\end{gathered} \tag{3.6}$$

With this notations, the Newton inexact system can be defined as follows:

$$
\begin{aligned}
\underline{\mathcal{A}}\overline{\Delta x} &= 0, \\
-\bar{\mathcal{H}}\overline{\Delta x} + \underline{\mathcal{A}}^T \Delta y + \underline{\Delta s} &= 0, \qquad (3.7)\\
L(\bar{x})\underline{\Delta s} + L(\underline{s})\overline{\Delta x} &= \sigma\mu e - \bar{x} \circ \underline{s} + r,
\end{aligned}
$$

where $\overline{\Delta x} = Q_p \Delta x$, $\underline{\Delta s} = Q_{p^{-1}} \Delta s$ and the third equation admits an error term r.

4. **Convergence analysis of inexact IPMs**

In this section, we prove the global convergence of feasible inexact IPM. To this end, we define two neighborhoods of the central path. By controlling the proximity to the central path, we show that all generated iterates belong to these neighborhoods. Finally, in both cases, we prove the convergence of the inexact IPM and derive the complexity results. In our analysis of inexact primal-dual feasible IPM, we consider small and large neighborhoods induced by using the Frobenius and 2-norm for some $\theta \in (0, 1)$ and $\gamma \in (0, 1)$, respectively, as follow:

$$
\mathcal{N}_F(\theta) = \Big\{(x, y, s) \in \mathcal{F}^0 \ : \ \|w - \mu e\|_F \leq \theta\mu\Big\}, \qquad (4.1)
$$

$$
\mathcal{N}_2(\gamma) = \Big\{(x, y, s) \in \mathcal{F}^0 \ : \ \gamma\mu \leq \lambda_j(w) \leq \frac{1}{\gamma}\mu\Big\}, \qquad (4.2)
$$

where

$$
\mathcal{F}^0 = \{(x, y, s) \mid \mathcal{A}(x) = b, \ \mathcal{A}^T(y) + s - \mathcal{H}(x) = c, \ (x, s) \in \mathrm{int}(\mathcal{K}) \times \mathrm{int}(\mathcal{K})\}.
$$

The short-step methods have the best theoretical complexity in comparison with the large-step methods. The large-step methods, unlike their poor theoretical complexity, lead to efficient algorithms in practice. In the following, we investigate the convergence behavior of the both short- and large-step methods for CQSCP using the inexact Newton directions. Finally, by following the general scheme presented by Wright [24], we conclude the best and worst-case complexity results. First, we state some elementary lemmas which will be used in our analysis.

Lemma 4.1. *The neighborhoods defined in (4.1) and (4.2) are scaling invariant, i.e., (x, s) is in the neighborhoods iff $(\bar{x}, \underline{s})$ is.*

Proof. Due to Lemma 21 in [16], the vectors w and $\bar{w} = Q_{\bar{x}^{\frac{1}{2}}} \underline{s}$ have the same eigenvalues. On the other hand, we can rewrite the two neighborhoods $\mathcal{N}_F(\theta)$ and $\mathcal{N}_2(\gamma)$ in term of eigenvalues of w. Thus the result follows. □

Lemma 4.2. *Let $x, s \in int(\mathcal{K})$. If x and s are operator commute, then $w = x \circ s$.*

Proof. Let x and s be operator commute. Then, $x^{\frac{1}{2}}$ and s are also operator commute. This implies

$$\begin{aligned} w = Q_{x^{\frac{1}{2}}} s &= \Big[2L(x^{\frac{1}{2}})^2 - L(x)\Big] s = 2L(x^{\frac{1}{2}})^2 L(s) e - x \circ s \\ &= 2L(s) L(x^{\frac{1}{2}})^2 e - x \circ s = 2L(s) x - x \circ s = x \circ s. \end{aligned}$$

This completes the proof. □

Corollary 4.3. *Let $x, s \in int(\mathcal{K})$ and $\bar{x}$ and $\underline{s}$ be as defined in (3.6). Then, $\bar{w} = \bar{x} \circ \underline{s}$.*

Proof. By Lemma 2.2 in [20], if $x, s \in \mathrm{int}(\mathcal{K})$ then we have $\bar{x}, \underline{s} \in \mathrm{int}(\mathcal{K})$. Moreover, $\bar{x}$ and $\underline{s}$ are operator commute. Now, the result follows by Lemma 4.2. □

In what follows, we use the following notations:

$$\begin{aligned} x(\alpha) &= x + \alpha \Delta x, \ s(\alpha) = s + \alpha \Delta s, \ \bar{x}(\alpha) = \bar{x} + \alpha \overline{\Delta x}, \\ \underline{s}(\alpha) &= \underline{s} + \alpha \underline{\Delta s}, \ \bar{\mu}(\alpha) = \mu(\alpha) = \frac{\langle \bar{x}(\alpha), \underline{s}(\alpha) \rangle}{n} = \frac{\langle x(\alpha), s(\alpha) \rangle}{n}. \end{aligned}$$

Lemma 4.4. *One has*

$$\langle \overline{\Delta x}, \underline{\Delta s} \rangle \geq 0, \tag{4.3}$$

$$\mu(\alpha) \leq (1 - \alpha(1-\sigma))\mu + \alpha \frac{\|r\|_F}{\sqrt{n}} + \alpha^2 \frac{\|\overline{\Delta x} \circ \underline{\Delta s}\|_F}{\sqrt{n}}. \tag{4.4}$$

Proof. Feasibility of (x, y, s) implies that

$$\begin{aligned} \langle \overline{\Delta x}, \underline{\Delta s} \rangle = \mathbf{tr}(\overline{\Delta x} \circ \underline{\Delta s}) &= \mathbf{tr}(\Delta x \circ \Delta s) = \mathbf{tr}\Big[\Delta x \circ (\mathcal{H} \Delta x - \mathcal{A}^T \Delta y)\Big] \\ &= \mathbf{tr}(\Delta x \circ \mathcal{H} \Delta x) - \mathbf{tr}(\Delta x \circ \mathcal{A}^T \Delta y) \\ &= \langle \Delta x, \mathcal{H} \Delta x \rangle - \langle \Delta x, \mathcal{A}^T \Delta y \rangle \\ &= \langle \Delta x, \mathcal{H} \Delta x \rangle - \langle \mathcal{A} \Delta x, \Delta y \rangle = \langle \Delta x, \mathcal{H} \Delta x \rangle. \end{aligned}$$

Now, the positive semidefinite property of the operator $\mathcal{H}$ concludes the first claim. For the second claim, we observe

$$\begin{aligned} \mathbf{tr}(\bar{x}(\alpha) \circ \underline{s}(\alpha)) &= \mathbf{tr}\Big[(\bar{x} + \alpha \overline{\Delta x}) \circ (\underline{s} + \alpha \underline{\Delta s})\Big] \\ &= \mathbf{tr}(\bar{x} \circ \underline{s}) + \alpha \mathbf{tr}(\bar{x} \circ \underline{\Delta s} + \underline{s} \circ \overline{\Delta x}) + \alpha^2 \mathbf{tr}(\overline{\Delta x} \circ \underline{\Delta s}) \\ &= \mathbf{tr}(\bar{x} \circ \underline{s}) + \alpha \mathbf{tr}(\sigma \mu e - \bar{x} \circ \underline{s} + r) + \alpha^2 \mathbf{tr}(\overline{\Delta x} \circ \underline{\Delta s}) \\ &= (1 - \alpha(1-\sigma)) \mathbf{tr}(\mu e) + \alpha \mathbf{tr}(r) + \alpha^2 \mathbf{tr}(\overline{\Delta x} \circ \underline{\Delta s}), \end{aligned}$$

which implies

$$\mu(\alpha)=\bar{\mu}(\alpha)=(1-\alpha(1-\sigma))\mu+\frac{\alpha\mathbf{tr}(r)}{n}+\frac{\alpha^2\mathbf{tr}(\overline{\Delta x}\circ\underline{\Delta s})}{n} \le (1-\alpha(1-\sigma))\mu+\alpha\frac{\|r\|_F}{\sqrt{n}}+\alpha^2\frac{\left\|\overline{\Delta x}\circ\underline{\Delta s}\right\|_F}{\sqrt{n}}, \tag{4.5}$$

where the last inequality follows from this fact that $\mathbf{tr}(x)\le\sqrt{n}\,\|x\|_F$, $\forall x\in\mathcal{J}$. This completes the proof. $\square$

According to (4.4), it is obvious that the complementarity gap at the new point is reduced in comparison with that at the previous iteration if and only if the terms $\left\|\overline{\Delta x}\circ\underline{\Delta s}\right\|_F$ and $\|r\|_F$ are kept small enough in comparison with $\alpha(1-\sigma)\mu$. However, assuming $\delta\in(0,1)$, we set $r=\delta\xi$, and control the barrier reduction parameter σ and the step size α such that the duality gap in the current iteration be noticeably smaller than the one in previous iteration. The following lemma plays an important role in our analysis. For proof and more details see Lemma 33 in [16].

Lemma 4.5. *Let $x,s\in\mathcal{J}$ and G be a positive definite matrix which is symmetric with respect to the scalar product $\langle\cdot,\cdot\rangle$. Then,*

$$\|x\|_F\,\|s\|_F\le\frac{1}{2}\sqrt{\boldsymbol{cond(G)}}\left(\left\|G^{\frac{1}{2}}x\right\|_F^2+\left\|G^{\frac{-1}{2}}s\right\|_F^2\right), \tag{4.6}$$

where $\boldsymbol{cond(G)}=\frac{\lambda_{\max}(G)}{\lambda_{\min}(G)}$.

In order to ensure that the duality gap in the current iteration is noticeably smaller than the one in the previous iteration, we need to control and obtain an upper bound for the term $\left\|\overline{\Delta x}\circ\underline{\Delta s}\right\|_F$ in (4.4). The following lemmas focus on this goal.

Lemma 4.6. *Let $\theta\in(0,1)$. If $(x,y,s)\in N_F(\theta)$, then the inexact Newton direction $(\overline{\Delta x},\Delta y,\underline{\Delta s})$ satisfies*

$$\left\|\overline{\Delta x}\circ\underline{\Delta s}\right\|_F\le\frac{1}{2}\sqrt{\boldsymbol{cond(G)}}(1+\delta)^2\frac{\|\sigma\mu e-\bar{w}\|_F^2}{(1-\theta)\mu}. \tag{4.7}$$

Proof. Let $G=L(\underline{s})^{-1}L(\bar{x})$. Since $\bar{x}$ and $\underline{s}$ are operator commute, G is a symmetric positive definite matrix and

$$\Big(L(\bar{x})L(\underline{s})\Big)^{-\frac{1}{2}}L(\bar{x})=L(\underline{s})^{-\frac{1}{2}}L(\bar{x})^{\frac{1}{2}}=G^{\frac{1}{2}},$$

$$\Big(L(\bar{x})L(\underline{s})\Big)^{-\frac{1}{2}}L(\underline{s})=L(\underline{s})^{\frac{1}{2}}L(\bar{x})^{-\frac{1}{2}}=G^{-\frac{1}{2}}.$$

Multiplying the last equation in (3.7) by $\Big(L(\bar{x})L(\underline{s})\Big)^{-\frac{1}{2}}$, we obtain

$$G^{\frac{1}{2}}\underline{\Delta s}+G^{-\frac{1}{2}}\overline{\Delta x}=\sigma\mu\Big(L(\bar{x})L(\underline{s})\Big)^{-\frac{1}{2}}e+\Big(L(\bar{x})L(\underline{s})\Big)^{-\frac{1}{2}}r-G^{\frac{1}{2}}\underline{s}. \tag{4.8}$$

Using Lemmas 4.4 and 4.5, we have

$$\begin{aligned}
\left\|\overline{\Delta x}\circ\underline{\Delta s}\right\|_F &\leq \left\|\overline{\Delta x}\right\|_F\left\|\underline{\Delta s}\right\|_F\\
&\leq \frac{1}{2}\sqrt{\mathbf{cond(G)}}\Big(\left\|G^{\frac{-1}{2}}\overline{\Delta x}\right\|_F^2+\left\|G^{\frac{1}{2}}\underline{\Delta s}\right\|_F^2\Big)\\
&= \frac{1}{2}\sqrt{\mathbf{cond(G)}}\Big(\left\|G^{\frac{-1}{2}}\overline{\Delta x}+G^{\frac{1}{2}}\underline{\Delta s}\right\|_F^2-2\mathbf{tr}(\overline{\Delta x}\circ\underline{\Delta s})\Big)\\
&\leq \frac{1}{2}\sqrt{\mathbf{cond(G)}}\left\|G^{\frac{-1}{2}}\overline{\Delta x}+G^{\frac{1}{2}}\underline{\Delta s}\right\|_F^2.
\end{aligned}$$

Now, by using (4.8), $r=\delta\xi$ and $\xi=\sigma\mu e-\overline{x}\circ\underline{s}$, it follows that

$$\begin{aligned}
\left\|\overline{\Delta x}\circ\underline{\Delta s}\right\|_F \leq{}& \frac{1}{2}\sqrt{\mathbf{cond(G)}}\left\|\sigma\mu\left(L(\bar{x})L(\underline{s})\right)^{-\frac{1}{2}}e+\left(L(\bar{x})L(\underline{s})\right)^{-\frac{1}{2}}r-G^{\frac{1}{2}}\underline{s}\right\|_F^2\\
={}& \frac{1}{2}\sqrt{\mathbf{cond(G)}}(1+\delta)^2\left\|\sigma\mu\left(L(\bar{x})L(\underline{s})\right)^{-\frac{1}{2}}e-G^{\frac{1}{2}}\underline{s}\right\|_F^2\\
={}& \frac{1}{2}\sqrt{\mathbf{cond(G)}}(1+\delta)^2\Big[\sigma^2\mu^2\langle\bar{x}^{-1},\underline{s}^{-1}\rangle+\langle\bar{x},\underline{s}\rangle-2\sigma\mu\langle e,e\rangle\Big]\\
={}& \frac{1}{2}\sqrt{\mathbf{cond(G)}}(1+\delta)^2\Big[\sigma^2\mu^2\mathbf{tr}(\bar{w}^{-1})+\mathbf{tr}(\bar{w})-2\sigma\mu\mathbf{tr}(e)\Big].
\end{aligned}$$

Finally, by some simple calculations, we have

$$\begin{aligned}
\left\|\overline{\Delta x}\circ\underline{\Delta s}\right\|_F &\leq \frac{1}{2}\sqrt{\mathbf{cond(G)}}(1+\delta)^2\sum_{i=1}^n\Big(\sigma^2\mu^2\left(\lambda_i(\bar{w})\right)^{-1}+\lambda_i(\bar{w})-2\sigma\mu\Big)\\
&= \frac{1}{2}\sqrt{\mathbf{cond(G)}}(1+\delta)^2\sum_{i=1}^n\frac{\Big(\sigma^2\mu^2+\lambda_i(\bar{w})^2-2\sigma\mu\lambda_i(\bar{w})\Big)}{\lambda_i(\bar{w})}\\
&\leq \frac{1}{2}\sqrt{\mathbf{cond(G)}}(1+\delta)^2\frac{\|\sigma\mu e-\bar{w}\|_F^2}{\lambda_{\min}(\bar{w})}\\
&\leq \frac{1}{2}\sqrt{\mathbf{cond(G)}}(1+\delta)^2\frac{\|\sigma\mu e-\bar{w}\|_F^2}{(1-\theta)\mu},
\end{aligned}$$

where the last inequality follows from the definition of $\mathcal{N}_F(\theta)$ in (4.1). This follows the desired result. □

To proceed our analysis, we need to obtain some upper bounds for **cond(G)** and the numerator in (4.7).

Lemma 4.7. *For the Nesterov-Todd method, the condition number of G is always 1.*

Proof. In the Nesterov-Todd method, p is chosen as Lemma 2.2. This choice concludes $\bar{x} = \underline{s}$. Hence, $G = I$ and this follows the result. □

Lemma 4.8. *Let (x, y, s) be the current iterate in $\mathcal{N}_F(\theta)$ and $\sigma = 1 - \frac{\beta}{\sqrt{n}}$ be defined as a barrier reduction parameter for some $\beta \in (0, 1)$. Then,*

$$\|\sigma\mu e - \bar{w}\|_F^2 \leq (\theta^2 + \beta^2)\mu^2. \tag{4.9}$$

Proof. Due to (3.2) and Corollary 4.3, we have

$$\mu = \frac{1}{n}\mathbf{tr}\,(x \circ s) = \frac{1}{n}\mathbf{tr}\,(\bar{x} \circ \underline{s}) = \frac{1}{n}\mathbf{tr}\,(\bar{w}), \tag{4.10}$$

which implies $\mathbf{tr}(\bar{w}) = n\mu$ or equivalently $\mathbf{tr}(\bar{w} - \mu e) = 0$. Using this, we obtain

$$\begin{aligned}
\|\xi\|_F^2 = \|\bar{w} - \sigma\mu e\|_F^2 &= \|(\bar{w} - \mu e) + (1 - \sigma)\mu e\|_F^2 \\
&= \|\bar{w} - \mu e\|_F^2 + 2(1 - \sigma)\mu\mathbf{tr}(\bar{w} - \mu e) \\
&\quad + \mu^2(1 - \sigma)^2 \|e\|_F^2 \\
&\leq \theta^2\mu^2 + n\mu^2(1 - \sigma)^2 = (\theta^2 + \beta^2)\mu^2,
\end{aligned} \tag{4.11}$$

where the inequality follows from assumption $(x, y, s) \in \mathcal{N}_F(\theta)$ by using Lemma 4.1. This completes the proof. □

Now, assuming the current iterate $(x, y, s) \in \mathcal{N}_F(\theta)$, using Lemma 4.7 and substituting (4.9) in (4.7), we obtain

$$\left\|\overline{\Delta x} \circ \underline{\Delta s}\right\|_F \leq (1 + \delta)^2 \frac{(\theta^2 + \beta^2)}{1 - \theta}\mu. \tag{4.12}$$

4.1. **Values for θ, β and δ.** In this subsection, we obtain some values for the parameters θ, β and δ, then we prove that with these values the new iterate $\Big(x(\alpha), y(\alpha), s(\alpha)\Big)$ is feasible and well-defined. In other words, we confirm with an appropriate designation of these parameters if $(x, y, s) \in \mathcal{N}_F(\theta)$, the new generated point $\Big(x(\alpha), y(\alpha), s(\alpha)\Big)$ belongs to the $\mathcal{N}_F(\theta)$. Equivalently, due to Lemma 4.1, it is enough to show that if $(\bar{x}, y, \underline{s}) \in \mathcal{N}_F(\theta)$, then $(\bar{x}(\alpha), y(\alpha), \underline{s}(\alpha)) \in \mathcal{N}_F(\theta)$.
Using the last equation of the inexact system (3.7) and the equation (4.5), we

obtain

$$\begin{aligned}
\bar{x}(\alpha)\circ\underline{s}(\alpha)-\bar{\mu}(\alpha)e &= (\bar{x}+\alpha\overline{\Delta x})\circ(\underline{s}+\alpha\underline{\Delta s})-\mu(\alpha)e\\
&= \bar{x}\circ\underline{s}+\alpha(\bar{x}\circ\underline{\Delta s}+\underline{s}\circ\overline{\Delta x})+\alpha^2\overline{\Delta x}\circ\underline{\Delta s}-\mu(\alpha)e\\
&= (1-\alpha)\bar{x}\circ\underline{s}+\alpha\sigma\mu e+\alpha r+\alpha^2\overline{\Delta x}\circ\underline{\Delta s}\\
&- (1-\alpha)\mu e-\alpha\sigma\mu e-\alpha\frac{\mathbf{tr}(r)}{n}e-\alpha^2\frac{\mathbf{tr}(\overline{\Delta x}\circ\underline{\Delta s})}{n}e\\
&= (1-\alpha)(\bar{x}\circ\underline{s}-\mu e)+\alpha(r-\frac{\mathbf{tr}(r)}{n}e)\\
&\quad +\alpha^2\left(\overline{\Delta x}\circ\underline{\Delta s}-\frac{\mathbf{tr}(\overline{\Delta x}\circ\underline{\Delta s})e}{n}\right).
\end{aligned}$$

This implies

$$\begin{aligned}
\|\bar{x}(\alpha)\circ\underline{s}(\alpha)-\bar{\mu}(\alpha)e\|_F &\le (1-\alpha)\|\bar{x}\circ\underline{s}-\mu e\|_F+\alpha\left\|r-\frac{\mathbf{tr}(r)}{n}e\right\|_F\\
&\quad +\alpha^2\left\|\overline{\Delta x}\circ\underline{\Delta s}-\frac{\mathbf{tr}(\overline{\Delta x}\circ\underline{\Delta s})}{n}e\right\|_F. \qquad (4.13)
\end{aligned}$$

Lemma 4.9. *Let (x,y,s) be the current iterate belongs to the neighborhood $\mathcal{N}_F(\theta)$ and $(\overline{\Delta x},\Delta y,\underline{\Delta s})$ be the solution of system (3.7). If $\theta=\beta=0.1$ and $\delta=0.3$, then the inexact feasible short-step method is well-defined. Particulary, by starting from $(x,y,s)\in\mathcal{N}_F(\theta)$, after a Newton step the new generated point $(x(\alpha),y(\alpha),s(\alpha))$ belongs to $\mathcal{N}_F(\theta)$.*

Proof. Clearly, using the inexact Newton search direction system (3.7), for some appropriate constants θ,β and δ, the new iterate $(x(\alpha),y(\alpha),s(\alpha))=(x,y,s)+\alpha(\Delta x,\Delta y,\Delta s)$ for $\alpha\in(0,1]$ is feasible for primal and dual problems. On the other hand, using $\|e\|_F=\sqrt{n}$, we have

$$\begin{aligned}
\left\|r-\frac{\mathbf{tr}(r)}{n}e\right\|_F^2 &= \|r\|_F^2+\frac{\mathbf{tr}(r)^2}{n^2}\|e\|_F^2-2\frac{\mathbf{tr}(r)}{n}\langle e,r\rangle\\
&= \|r\|_F^2+\frac{\mathbf{tr}(r)^2}{n}-2\frac{\mathbf{tr}(r)^2}{n}\le\|r\|_F^2. \qquad (4.14)
\end{aligned}$$

We also have

$$\begin{aligned}
\left\|\overline{\Delta x}\circ\underline{\Delta s}-\frac{\mathbf{tr}(\overline{\Delta x}\circ\underline{\Delta s})}{n}\,e\right\|_F^2 &= \left\|\overline{\Delta x}\circ\underline{\Delta s}\right\|_F^2+\frac{\mathbf{tr}(\overline{\Delta x}\circ\underline{\Delta s})^2}{n^2}\|e\|_F^2\\
&\quad -2\frac{\mathbf{tr}(\overline{\Delta x}\circ\underline{\Delta s})}{n}\langle\overline{\Delta x}\circ\underline{\Delta s},e\rangle\le\left\|\overline{\Delta x}\circ\underline{\Delta s}\right\|_F^2. \qquad (4.15)
\end{aligned}$$

Substituting (4.14) and (4.15) into (4.13) and using the definition of $\mathcal{N}_F(\theta)$, $r = \delta\xi$, (4.11) and (4.12), we conclude that

$$
\begin{aligned}
\|\bar{x}(\alpha) \circ \underline{s}(\alpha) - \bar{\mu}(\alpha)e\|_F &\leq (1-\alpha)\theta\mu + \alpha\delta\mu\sqrt{\theta^2+\beta^2} \\
&\quad +\alpha^2 \frac{(1+\delta)^2(\theta^2+\beta^2)}{1-\theta}\mu. \qquad (4.16)
\end{aligned}
$$

The choice of the parameters θ and β as $\theta = \beta$ guarantees that

$$
\frac{(1+\delta)^2(\theta^2+\beta^2)}{1-\theta} = \frac{2(1+\delta)^2}{1-\theta}\theta^2, \qquad (4.17)
$$

$$
\sqrt{\theta^2+\beta^2} = \sqrt{2}\theta. \qquad (4.18)
$$

Choosing $\theta = 0.1$ and substituting in (4.17), we have

$$
\frac{(1+\delta)^2(\theta^2+\beta^2)}{1-\theta} = \frac{2(1+\delta)^2}{9}\theta. \qquad (4.19)
$$

Substituting (4.18) and (4.19) into (4.16), we have

$$
\|\bar{x}(\alpha) \circ \underline{s}(\alpha) - \bar{\mu}(\alpha)e\|_F \leq (1-\alpha)\theta\mu + \sqrt{2}\alpha\delta\theta\mu + 2\alpha^2 \tfrac{(1+\delta)^2}{9}\theta\mu. \qquad (4.20)
$$

Using (4.5) and (4.20), the inequality $\|\bar{x}(\alpha) \circ \underline{s}(\alpha) - \bar{\mu}(\alpha)e\|_F \leq \theta\bar{\mu}(\alpha)$ will be satisfied if the following holds

$$
\begin{aligned}
(1-\alpha)\theta\mu + \sqrt{2}\alpha\delta\theta\mu + 2\alpha^2\frac{(1+\delta)^2}{9}\theta\mu &\leq \theta\Big(\big(1-\alpha(1-\sigma)\big)\mu + \frac{\alpha}{n}\mathbf{tr}(r) \\
&\quad + \frac{\alpha^2}{n}\mathbf{tr}\big(\overline{\Delta x} \circ \underline{\Delta s}\big)\Big).
\end{aligned}
$$

Removing the same terms appeared on both sides, dividing both sides by $\alpha\theta$ and using the nonnegative property of the term $\mathbf{tr}\left(\overline{\Delta x} \circ \underline{\Delta s}\right)$, the latter inequality simplifies to

$$
2\alpha\mu\frac{(1+\delta)^2}{9} + \sqrt{2}\delta\mu \leq \sigma\mu + \frac{\mathbf{tr}(r)}{n}.
$$

From Lemma 2.4, $r = \delta\xi$, (4.11) and (4.18), we obtain

$$
\left|\frac{\mathbf{tr}(r)}{n}\right| \leq \frac{\delta}{\sqrt{n}}\sqrt{\theta^2+\beta^2}\mu = \sqrt{2}\frac{\delta}{\sqrt{n}}\theta\mu. \qquad (4.21)
$$

Therefore, to ensure that $\|\bar{x}(\alpha) \circ \underline{s}(\alpha) - \bar{\mu}(\alpha)e\|_F \leq \theta\bar{\mu}(\alpha)$ for any $\alpha \in (0,1]$, it suffices to choose δ such that

$$
2\mu\frac{(1+\delta)^2}{9} + \sqrt{2}\delta\mu \leq \sigma\mu - \sqrt{2}\frac{\delta}{\sqrt{n}}\theta\mu,
$$

and this simplifies to

$$
\sqrt{2}\delta(1+\frac{\theta}{\sqrt{n}}) + 2\frac{(1+\delta)^2}{9} \leq \sigma = 1 - \frac{\beta}{\sqrt{n}}.
$$

The left hand side of this inequality is an increasing function of δ and we can easily check that this inequality holds for $\theta = \beta = 0.1, \delta = 0.3$ and for any $n \geq 2$. However, using Lemma 30 in [16] and the above discussion, we obtain $\|\bar{w}(\alpha) - \bar{\mu}(\alpha)e\|_F \leq \|\bar{x}(\alpha) \circ \underline{s}(\alpha) - \bar{\mu}(\alpha)e\|_F \leq \theta\bar{\mu}(\alpha)$ and therefore, using the definition of $\mathcal{N}_F(\theta)$, we have, by simple calculations, $\bar{x}(\alpha) \circ \underline{s}(\alpha) \in \text{int}(\mathcal{K})$. Due to Lemma 2.6, it is clear $\mathbf{det}(\bar{x}(\alpha)) \neq 0$ and $\mathbf{det}(\underline{s}(\alpha)) \neq 0$. Furthermore, since $\bar{x} \in \text{int}(\mathcal{K})$ and $\underline{s} \in \text{int}(\mathcal{K})$, by continuity, it follows that $\bar{x}(\alpha) \in \text{int}(\mathcal{K})$ and $\underline{s}(\alpha) \in \text{int}(\mathcal{K})$ for $\alpha \in (0, 1]$. The proof is completed. □

4.2. **Iteration bound.** As we show, by starting from an initial feasible solution in $\mathcal{N}_F(\theta)$, Lemma 4.9 guarantees that for any $\alpha \in (0, 1]$ and specific values of parameters δ, θ and β, the new generated point $(x(\alpha), y(\alpha), s(\alpha))$ also belongs to $\mathcal{N}_F(\theta)$. In this subsection, we will ask for an aggressive reduction of the barrier parameter μ from one iteration to another. We will set $\alpha = 1$ and take the full Newton step by the inexact Newton system (3.7). Setting $\alpha = 1$ in (4.4), we conclude that

$$\begin{aligned}
\mu(1) = \bar{\mu} \quad &\leq \quad (1 - (1 - \sigma))\mu + \frac{\|r\|_F}{\sqrt{n}} + \frac{\left\|\overline{\Delta x} \circ \underline{\Delta s}\right\|_F}{\sqrt{n}} \\
&= \quad (1 - (1 - \sigma))\mu + \delta\frac{\|\xi\|_F}{\sqrt{n}} + \frac{\left\|\overline{\Delta x} \circ \underline{\Delta s}\right\|_F}{\sqrt{n}}.
\end{aligned}$$

Now, by substituting $\theta = \beta = 0.1$ and $\delta = 0.3$ in the right hand sides of (4.11) and (4.12) and using $\sigma = 1 - \frac{\beta}{\sqrt{n}}$, we obtain an upper bound for $\bar{\mu}$ as follows:

$$\bar{\mu} \leq (1 - \frac{\beta}{\sqrt{n}})\mu + \frac{\sqrt{2}\delta\beta}{\sqrt{n}}\mu + \frac{0.3756\beta}{\sqrt{n}}\mu \leq (1 - \frac{\eta}{\sqrt{n}})\mu, \tag{4.22}$$

where $\beta(1 - \sqrt{2}\delta - 0.3756) \geq 0.02$ and we set $\eta = 0.02$ in (4.22). We show that in each iteration the barrier parameter μ can be reduced by the factor $\frac{\eta}{\sqrt{n}}$. Finally, the complexity result for the inexact short-step feasible interior-point method, which is a straightforward application of Theorem 3.2 in Wright [24], can be stated as follows.

Theorem 4.10. *Given $\varepsilon > 0$, suppose that the feasible initial starting point $(x^0, y^0, s^0) \in \mathcal{N}_F(0.1)$ satisfies $\boldsymbol{tr}(x^0 \circ s^0) = n\mu^0$, where $\mu^0 \leq \frac{1}{\varepsilon^k}$ for some positive constant k. Then, there exists an index L with $L = O\left(\sqrt{n}\log(\frac{1}{\varepsilon})\right)$ such that $\mu^{\bar{l}} \leq \varepsilon$, $\forall \bar{l} \geq L$.*

5. Analysis of the inexact large-step method

The algorithm that we presented in the previous section is rather slow. This is due to the fact that the barrier reduction parameter σ, which is used in the right hand side of the third equation of system 3.4, is rather small. In practice one is tempted to accelerate the algorithm by taking larger values of σ. So, in this section, we consider the case where σ is some small (but fixed) constant in the interval $(0, 1)$ which leads to an efficient algorithm, namely large-step interior-point algorithm. In this section, we briefly prove the convergence of this method when it is used in inexact IPMs. As we have mentioned before, we assume that in large-step methods the current iterate $(x, y, s) \in \mathcal{N}_2(\gamma)$. Similar to the analysis of short-step method, we also accept the error term $r = \delta\xi$ with $\xi = \sigma\mu e - \bar{w}$ and $\delta \in (0, 1)$. According to the definition of $\mathcal{N}_2(\gamma)$, we can obtain an upper bound for the $\|\xi\|_2$ as follows:

$$\begin{aligned} \|\xi\|_2 = \|\bar{w} - \sigma\mu e\|_2 &= \|\bar{w} - \mu e + (1-\sigma)\mu e\|_2 \\ &\leq \|\bar{w} - \mu e\|_2 + (1-\sigma)\mu \\ &\leq \max\{(\frac{1}{\gamma} - 1)\mu, (1-\gamma)\mu\} + (1-\sigma)\mu = (\frac{1}{\gamma} - \sigma)\mu, \end{aligned} \tag{5.1}$$

where the last equality follows from $\gamma \in (0, 1)$. Similar to the analysis of short-step method, we need to obtain an upper bound for the term $\left\|\overline{\Delta x} \circ \underline{\Delta s}\right\|_F$. This is a key result of the next lemma.

Lemma 5.1. *If the current iterate $(x, y, s) \in \mathcal{N}_2(\gamma)$, then the inexact scaled Newton search direction $(\overline{\Delta x}, \Delta y, \underline{\Delta s})$ satisfies*

$$\left\|\overline{\Delta x} \circ \underline{\Delta s}\right\|_F \leq n\frac{(1+\delta)^2}{\gamma}(\frac{1}{\gamma} - \sigma)^2\mu, \tag{5.2}$$

$$\boldsymbol{tr}(\overline{\Delta x} \circ \underline{\Delta s}) \leq n\frac{(1+\delta)^2}{\gamma}(\frac{1}{\gamma} - \sigma)^2\mu. \tag{5.3}$$

Proof. According to the definition of $\mathcal{N}_2(\gamma)$, in the similar way to the proof of Lemma 4.6, we have

$$\begin{aligned} \left\|\overline{\Delta x} \circ \underline{\Delta s}\right\|_F &\leq \frac{1}{2}\sqrt{\mathbf{cond(G)}}(1+\delta)^2\frac{\|\sigma\mu e - \bar{w}\|_F^2}{\lambda_{\min}(\bar{w})} \\ &\leq \frac{1}{2}\sqrt{\mathbf{cond(G)}}(1+\delta)^2 n\frac{\|\sigma\mu e - \bar{w}\|_2^2}{\gamma\mu} \\ &\leq n\frac{(1+\delta)^2}{\gamma}(\frac{1}{\gamma} - \sigma)^2\mu, \end{aligned}$$

where the last inequality follows from Lemma 4.7 and (5.1). This implies inequality (5.2). To prove (5.3), using Lemmas 2.4, 4.4 and 4.5, we have

$$\begin{aligned}
\mathbf{tr}\left(\overline{\Delta x}\circ\underline{\Delta s}\right) &\leq \left\|\overline{\Delta x}\right\|_F \left\|\underline{\Delta s}\right\|_F \\
&\leq \frac{1}{2}\sqrt{\mathbf{cond(G)}}\left(\left\|G^{\frac{-1}{2}}\overline{\Delta x}\right\|_F^2 + \left\|G^{\frac{1}{2}}\underline{\Delta s}\right\|_F^2\right) \\
&= \frac{1}{2}\sqrt{\mathbf{cond(G)}}\left(\left\|G^{\frac{-1}{2}}\overline{\Delta x} + G^{\frac{1}{2}}\underline{\Delta s}\right\|_F^2 - 2\mathbf{tr}\left(\overline{\Delta x}\circ\underline{\Delta s}\right)\right) \\
&\leq \frac{1}{2}\sqrt{\mathbf{cond(G)}}\left\|G^{\frac{-1}{2}}\overline{\Delta x} + G^{\frac{1}{2}}\underline{\Delta s}\right\|_F^2.
\end{aligned}$$

Thus, in the same way as the proof of Lemma 4.6, we have

$$\begin{aligned}
\mathbf{tr}\left(\overline{\Delta x}\circ\underline{\Delta s}\right) &\leq \frac{1}{2}\sqrt{\mathbf{cond(G)}}(1+\delta)^2\frac{\|\sigma\mu e-\bar{w}\|_F^2}{\lambda_{\min}(\bar{w})} \\
&\leq \frac{1}{2}\sqrt{\mathbf{cond(G)}}(1+\delta)^2 n\frac{\|\sigma\mu e-\bar{w}\|_2^2}{\gamma\mu} \\
&\leq n\frac{(1+\delta)^2}{\gamma}(\frac{1}{\gamma}-\sigma)^2\mu,
\end{aligned}$$

where the last inequality follows from Lemma 4.7 and (5.1). This concludes the result. □

Now, we are ready to conclude some conditions which ensure the inexact large-step feasible IPM is well-defined. In other words, we want to demonstrate that by starting from the current iterate (x,y,s) in $\mathcal{N}_2(\gamma)$, under certain conditions, the generated iterate $(x(\alpha),y(\alpha),s(\alpha))$ belongs to $\mathcal{N}_2(\gamma)$.

Lemma 5.2. *Let (x,y,s) be the current iterate in the $\mathcal{N}_2(\gamma)$ and $\left(\overline{\Delta x},\Delta y,\underline{\Delta s}\right)$ be the solution of system (3.7). The inexact large-step feasible IPM is well-defined if the step size $\alpha\in(0,\,1]$ satisfies the following conditions:*

$$\alpha(\gamma+n)\frac{(1+\delta)^2}{\gamma}(\frac{1}{\gamma}-\sigma)^2 \leq \sigma(1-\gamma)-\delta(1+\gamma)(\frac{1}{\gamma}-\sigma), \tag{5.4}$$

$$\alpha n\frac{(1+\delta)^2}{\gamma}(\frac{1}{\gamma}-\sigma)^2 + \delta\frac{1}{\gamma}\left(\frac{1}{\gamma}-\sigma\right) + (\sigma-\gamma)\delta \leq (\frac{1}{\gamma}-1)\sigma. \tag{5.5}$$

Proof. According to the definition of $\mathcal{N}_2(\gamma)$,

$$(x(\alpha),y(\alpha),s(\alpha))\in\mathcal{N}_2(\gamma) \;\Leftrightarrow\; \gamma\mu(\alpha)\leq\lambda_j(w(\alpha))\leq\frac{1}{\gamma}\mu(\alpha), \tag{5.6}$$

or equivalently, due to Lemma 4.1, $(\bar{x}(\alpha),y(\alpha),\underline{s}(\alpha))\in\mathcal{N}_2(\gamma)$ if and only if

$$\gamma\bar{\mu}(\alpha)\leq\lambda_{\min}(\bar{w}(\alpha)) \text{ and } \lambda_{\max}(\bar{w}(\alpha))\leq\frac{1}{\gamma}\bar{\mu}(\alpha), \tag{5.7}$$

and $(\bar{x}(\alpha), y(\alpha), \underline{s}(\alpha)) \in \mathcal{F}^0$. Using Lemma 2.20 in [14] and equation (4.5), we deduce that $\gamma\bar{\mu}(\alpha) \leq \lambda_{\min}(\bar{w}(\alpha))$ if

$$\gamma\left((1-\alpha)\mu + \alpha\sigma\mu + \alpha\frac{\mathbf{tr}(r)}{n} + \alpha^2\frac{\mathbf{tr}\left(\overline{\Delta x}\circ\underline{\Delta s}\right)}{n}\right) \leq \lambda_{\min}(\bar{x}(\alpha)\circ\underline{s}(\alpha)).$$

On the other hand, due to $r = \delta\xi$, Lemmas 2.5 and 5.1, we have

$$\begin{aligned}
\lambda_{\min}(\bar{x}(\alpha)\circ\underline{s}(\alpha)) &= \alpha\sigma\mu + \lambda_{\min}\Big[(1-\alpha)\bar{x}\circ\underline{s} + \alpha r + \alpha^2\left(\overline{\Delta x}\circ\underline{\Delta s}\right)\Big] \\
&= \alpha\sigma\mu + \lambda_{\min}\left((1-\alpha(1+\delta))\bar{x}\circ\underline{s} + \alpha\delta\sigma\mu e + \alpha^2\left(\overline{\Delta x}\circ\underline{\Delta s}\right)\right) \\
&= (1+\delta)\alpha\sigma\mu + \lambda_{\min}\left((1-\alpha(1+\delta))\bar{x}\circ\underline{s} + \alpha^2\left(\overline{\Delta x}\circ\underline{\Delta s}\right)\right) \\
&\geq (1+\delta)\alpha\sigma\mu + (1-\alpha(1+\delta))\lambda_{\min}(\bar{x}\circ\underline{s}) - \alpha^2\left\|\overline{\Delta x}\circ\underline{\Delta s}\right\|_F \\
&= (1+\delta)\alpha\sigma\mu + (1-\alpha)\lambda_{\min}(\bar{x}\circ\underline{s}) - \alpha\delta\lambda_{\min}(\bar{x}\circ\underline{s}) \\
&\quad -\alpha^2\left\|\overline{\Delta x}\circ\underline{\Delta s}\right\|_F \\
&\geq (1+\delta)\alpha\sigma\mu + (1-\alpha)\gamma\mu - \frac{1}{\gamma}\alpha\delta\mu - \alpha^2 n\frac{(1+\delta)^2}{\gamma}(\frac{1}{\gamma}-\sigma)^2\mu.
\end{aligned}$$

However, $\gamma\bar{\mu}(\alpha) \leq \lambda_{\min}(\bar{w}(\alpha))$ if

$$\begin{aligned}
\gamma\left((1-\alpha)\mu + \alpha\sigma\mu + \alpha\frac{\mathbf{tr}(r)}{n} + \alpha^2\frac{\mathbf{tr}\left(\overline{\Delta x}\circ\underline{\Delta s}\right)}{n}\right) &\leq (1+\delta)\alpha\sigma\mu + (1-\alpha)\gamma\mu \\
&\quad -\frac{1}{\gamma}\alpha\delta\mu - \alpha^2 n\frac{(1+\delta)^2}{\gamma}(\frac{1}{\gamma}-\sigma)^2\mu.
\end{aligned}$$

Canceling the identical terms from both sides and dividing both sides by α, we conclude a tighter version of the later inequality as follows:

$$\begin{aligned}
&\alpha\left(\gamma\frac{\mathbf{tr}\left(\overline{\Delta x}\circ\underline{\Delta s}\right)}{n} + n\frac{(1+\delta)^2}{\gamma}(\frac{1}{\gamma}-\sigma)^2\mu\right) \\
&\leq (1+\delta)\sigma\mu - \frac{\delta\mu}{\gamma} - \gamma\sigma\mu - \gamma\frac{\mathbf{tr}(r)}{n} = (1-\gamma)\sigma\mu + (\sigma - \frac{1}{\gamma})\delta\mu - \gamma\frac{\mathbf{tr}(r)}{n}.
\end{aligned}$$

Using (5.3), we have

$$\gamma\frac{\mathbf{tr}\left(\overline{\Delta x}\circ\underline{\Delta s}\right)}{n} + n\frac{(1+\delta)^2}{\gamma}(\frac{1}{\gamma}-\sigma)^2\mu \leq (\gamma+n)\frac{(1+\delta)^2}{\gamma}(\frac{1}{\gamma}-\sigma)^2\mu,$$

on the other hand, using (5.1), from

$$\|r\|_2 = \delta\,\|\xi\|_2 \leq \delta(\frac{1}{\gamma}-\sigma)\mu, \tag{5.8}$$

$$|\mathbf{tr}(r)| \leq n\,\|r\|_2 \leq n\delta(\frac{1}{\gamma}-\sigma)\mu, \tag{5.9}$$

we obtain

$$(1-\gamma)\sigma\mu + (\sigma - \frac{1}{\gamma})\delta\mu - \gamma\frac{\mathbf{tr}(r)}{n} \geq (1-\gamma)\sigma\mu + (1+\gamma)(\sigma - \frac{1}{\gamma})\delta\mu.$$

Finally, $\gamma\bar{\mu}(\alpha) \leq \lambda_{\min}(\bar{w}(\alpha))$ if

$$\alpha\left((\gamma+n)\frac{(1+\delta)^2}{\gamma}(\frac{1}{\gamma}-\sigma)^2\mu\right) \leq (1-\gamma)\sigma\mu + (1+\gamma)(\sigma - \frac{1}{\gamma})\delta\mu.$$

This follows the first part of the lemma. To complete our proof, it is enough to obtain some conditions in which $\lambda_{\max}(\bar{w}(\alpha)) \leq \frac{1}{\gamma}\bar{\mu}(\alpha)$. Using equation (4.5) and this fact that $\lambda_{\max}(\bar{w}(\alpha)) \leq \lambda_{\max}(\bar{x}(\alpha)\circ \underline{s}(\alpha))$ (see [16]), we deduce that $\lambda_{\max}(\bar{w}(\alpha)) \leq \gamma\bar{\mu}(\alpha)$ if

$$\lambda_{\max}(\bar{x}(\alpha)\circ\underline{s}(\alpha)) \leq \gamma\left((1-\alpha)\mu + \alpha\sigma\mu + \alpha\frac{\mathbf{tr}(r)}{n} + \alpha^2\frac{\mathbf{tr}\left(\overline{\Delta x}\circ\underline{\Delta s}\right)}{n}\right).$$

Moreover, using $r = \delta\xi$, Lemmas 2.5 and 5.1, we have

$$\begin{aligned}
\lambda_{\max}(\bar{x}(\alpha)\circ\underline{s}(\alpha)) &= \alpha\sigma\mu + \lambda_{\max}\Big[(1-\alpha)\bar{x}\circ\underline{s} + \alpha r + \alpha^2\left(\overline{\Delta x}\circ\underline{\Delta s}\right)\Big] \\
&= \alpha\sigma\mu + \lambda_{\max}\left((1-\alpha(1+\delta))\bar{x}\circ\underline{s} + \alpha\delta\sigma\mu e + \alpha^2\left(\overline{\Delta x}\circ\underline{\Delta s}\right)\right) \\
&= (1+\delta)\alpha\sigma\mu + \lambda_{\max}\left((1-\alpha(1+\delta))\bar{x}\circ\underline{s} + \alpha^2\left(\overline{\Delta x}\circ\underline{\Delta s}\right)\right) \\
&\leq (1+\delta)\alpha\sigma\mu + (1-\alpha(1+\delta))\lambda_{\max}(\bar{x}\circ\underline{s}) + \alpha^2\left\|\overline{\Delta x}\circ\underline{\Delta s}\right\|_F \\
&= (1+\delta)\alpha\sigma\mu + (1-\alpha)\lambda_{\max}(\bar{x}\circ\underline{s}) - \alpha\delta\lambda_{\max}(\bar{x}\circ\underline{s}) \\
&\quad + \alpha^2\left\|\overline{\Delta x}\circ\underline{\Delta s}\right\|_F \\
&\leq (1+\delta)\alpha\sigma\mu + (1-\alpha)\frac{1}{\gamma}\mu - \alpha\delta\gamma\mu + \alpha^2 n\frac{(1+\delta)^2}{\gamma}(\frac{1}{\gamma}-\sigma)^2\mu.
\end{aligned}$$

Then, $\lambda_{\max}(\bar{w}(\alpha)) \leq \frac{1}{\gamma}\bar{\mu}(\alpha)$ if

$$\begin{gathered}
(1+\delta)\alpha\sigma\mu + (1-\alpha)\frac{1}{\gamma}\mu - \alpha\delta\gamma\mu + \alpha^2 n\frac{(1+\delta)^2}{\gamma}(\frac{1}{\gamma}-\sigma)^2\mu \leq \\
\frac{1}{\gamma}\left((1-\alpha)\mu + \alpha\sigma\mu + \alpha\frac{\mathbf{tr}(r)}{n} + \alpha^2\frac{\mathbf{tr}\left(\overline{\Delta x}\circ\underline{\Delta s}\right)}{n}\right).
\end{gathered}$$

Canceling the identical terms from both sides, dividing both sides by α and simplifying the later inequality, we obtain the following inequality

$$\begin{aligned}
&(1+\delta)\sigma\mu - \delta\gamma\mu + \alpha n\frac{(1+\delta)^2}{\gamma}(\frac{1}{\gamma}-\sigma)^2\mu \\
&\qquad \leq \frac{1}{\gamma}\left(\sigma\mu + \frac{\mathbf{tr}(r)}{n} + \alpha\frac{\mathbf{tr}\left(\overline{\Delta x}\circ\underline{\Delta s}\right)}{n}\right),
\end{aligned}$$

or equivalently

$$\alpha\left(n\frac{(1+\delta)^2}{\gamma}(\frac{1}{\gamma}-\sigma)^2\mu-\frac{\mathbf{tr}\left(\overline{\Delta x}\circ\underline{\Delta s}\right)}{\gamma n}\right)-\frac{\mathbf{tr}(r)}{\gamma n}\leq\frac{\sigma\mu}{\gamma}+\delta\gamma\mu-(1+\delta)\sigma\mu$$

$$=(\frac{1}{\gamma}-1)\sigma\mu+(\gamma-\sigma)\delta\mu.$$

Now, using (4.3) and (5.9), the inequality $\lambda_{\max}(\bar{w}(\alpha))\leq\frac{1}{\gamma}\bar{\mu}(\alpha)$ holds if

$$\alpha n\frac{(1+\delta)^2}{\gamma}(\frac{1}{\gamma}-\sigma)^2\mu+\delta\frac{1}{\gamma}\left(\frac{1}{\gamma}-\sigma\right)\mu\leq(\frac{1}{\gamma}-1)\sigma\mu+(\gamma-\sigma)\delta\mu.$$

Dividing both sides of the later inequality by μ concludes the inequality (5.5). To end the proof, it suffices to show that the new generated point $(\bar{x}(\alpha),y(\alpha),\underline{s}(\alpha))\in\mathcal{F}^0$. To this end, using the inexact Newton system (3.7), it is clear that the new generated point $(\bar{x}(\alpha),y(\alpha),\underline{s}(\alpha))$ is primal-dual feasible. As we have proved, for the specific value of the step size α, we have $\gamma\bar{\mu}(\alpha)\leq\lambda_j\left(\bar{x}(\alpha)\circ\underline{s}(\alpha)\right)\leq\frac{1}{\gamma}\bar{\mu}(\alpha)$, which implies, by simple calculations, $\bar{x}(\alpha)\circ\underline{s}(\alpha)\in$ *int*$(\mathcal{K})$. Therefore, due to Lemma 2.6, we have $\mathbf{det}(\bar{x}(\alpha))\neq 0$ and $\mathbf{det}(\underline{s}(\alpha))\neq 0$. Furthermore, since $\bar{x}\in\text{int}(\mathcal{K})$ and $\underline{s}\in\text{int}(\mathcal{K})$, by continuity, it follows that both $\bar{x}(\alpha)\in\text{int}(\mathcal{K})$ and $\underline{s}(\alpha)\in\text{int}(\mathcal{K})$, which completes the proof. $\square$

As we have mentioned, each iteration of IPMs takes a step along the search directions and causes a reduction of the barrier parameter μ. This leads to a reduction in the duality gap. In order to complete the analysis, we need to obtain some conditions that guarantee the reduction of the duality gap after updating the barrier parameter μ. This fact is the main goal of the following lemma.

Lemma 5.3. *Let (x,y,s) be the current iterate in $\mathcal{N}_2(\gamma)$ and $\left(\overline{\Delta x},\Delta y,\underline{\Delta s}\right)$ be the solution of system (3.7). If the step size $\alpha\in(0,1]$ satisfies*

$$\sigma+\delta(\frac{1}{\gamma}-\sigma)+\alpha\frac{(1+\delta)^2}{\gamma}(\frac{1}{\gamma}-\sigma)^2\leq 0.9, \tag{5.10}$$

then

$$\mu(\alpha)\leq(1-0.1\alpha)\mu. \tag{5.11}$$

Proof. Substituting (4.5) into (5.11) and dividing the obtained inequality by $\alpha,$ we deduce (5.11) only if

$$\sigma\mu+\frac{\mathbf{tr}(r)}{n}+\alpha\frac{\mathbf{tr}\left(\overline{\Delta x}\circ\underline{\Delta s}\right)}{n}\leq 0.9\mu.$$

Substituting (5.3) and (5.9) in the above inequality, we conclude (5.10). This completes the proof. $\square$

5.1. **Complexity bound.** In the previous section, we proved that under conditions (5.4), (5.5) and (5.10) the inexact large-step feasible IPM is well-defined. Now, it remains to set appropriate values for the parameters γ, σ and δ to guarantee that all these conditions hold. To this end, we set $\gamma = 0.5, \sigma = 0.5$ and $\delta = 0.05$ which are called the proximity constant, the barrier parameter and the level of error, respectively. These choices guarantee that all three conditions (5.4), (5.5) and (5.10) are satisfied by $\alpha = \frac{1}{50n}$, for any $n \geq 2$. However, with this value of α, the inequality (5.11) gives $\mu(\alpha) \leq (1 - \frac{\eta}{n})\mu$ where $\eta = 0.002$. The following theorem, which is a straightforward application of Theorem 3.2 in Wright [24], concludes the complexity result.

Theorem 5.4. *Given $\varepsilon > 0$, suppose that a feasible starting point $(x^0, y^0, s^0) \in \mathcal{N}_2(0.5)$ satisfies $\boldsymbol{tr}\left(x^0 \circ s^0\right) = n\mu^0$, where $\mu^0 \leq \frac{1}{\varepsilon^k}$ for some positive constant k. Then, there exists an index L with $L = O\left(n \log(\frac{1}{\varepsilon})\right)$ such that $\mu^{\bar{l}} \leq \varepsilon$, $\forall \bar{l} \geq L$.*

6. Numerical results

In this section, we report the computational performance of the proposed inexact short- and large-step feasible IPMs for CQO problems and CQSDO problems, which are two important classes of CQSCP.

Example 6.1. Consider the primal problem of CQO in the standard form

$$\min \{\frac{1}{2}x^T Hx + c^T x \ : \ Ax = b \ x \geq 0\},$$

and its dual problem

$$\max \{-\frac{1}{2}x^T Hx + b^T y \ : \ A^T y - Hx + s = c, \ s \geq 0\},$$

with the following data [23]:

$$A = \begin{bmatrix} 2 & 3 & 8 & 2 & 5 & 6 & 0 & 2 \\ 8 & 4 & 1 & 1 & 2 & 5 & 1 & 3 \\ 4 & 3 & 6 & 5 & 3 & 1 & 2 & 2 \\ 5 & 2 & 5 & 8 & 1 & 2 & 6 & 3 \\ 6 & 1 & 4 & 3 & 10 & 1 & 4 & 7 \\ 4 & 2 & 1 & 8 & 5 & 3 & 2 & 3 \end{bmatrix}, H = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix},$$

$$b = \begin{bmatrix} 2.6292 \\ 2.3475 \\ 2.4414 \\ 3.0048 \\ 3.3804 \\ 2.6292 \end{bmatrix}, \ c = \begin{bmatrix} 6.9998 \\ 5.9999 \\ 0.9997 \\ 8.0004 \\ 3.9999 \\ 1.9997 \\ 2.9999 \\ 4.9998 \end{bmatrix}.$$

Assuming the allocated values of the parameters δ, β and σ in subsections 4.2 and 5.1 and considering the accuracy parameter $\varepsilon = 10^{-5}$, we respectively need 234 and 155 iterations to reach an ε-approximate optimal solution of the CQO problem by using the inexact short- and large-step methods.

Example 6.2. Consider the primal problem of CQSDO in the standard form:

$$\min \{\frac{1}{2}X \bullet H(X) + C \bullet X \ : \ A_i \bullet X = b_i \ i = 1, 2, ..., m, \ X \succeq 0\},$$

and its dual problem

$$\max \{-\frac{1}{2}X \bullet H(X) + b^T y \ : \ \sum_{i=1}^{n} y_i A_i - H(X) + S = C, \ S \succeq 0\},$$

with the following data [23], where $C \in \mathcal{S}^n$ (S^n is the vector space of symmetric matrices) and $b \in \mathbb{R}^m$. The notations " $\succeq$ " and "$\bullet$" respectively denote the positive semidefinite matrices and the inner product of symmetric matrices, $A_i \in \mathcal{S}^n$ are linearly independent matrices and $H : \mathcal{S}^n \longrightarrow \mathcal{S}^n$ is a self-adjoint positive semidefinite linear operator on $\mathcal{S}^n$, i.e., for any $M, N \in \mathcal{S}^n$, $H(M) \bullet N = M \bullet H(N)$ and $H(M) \bullet M \geq 0$.

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & -1 \\ 0 & -1 & 1 & -1 & -2 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 0 & -2 & 2 & 0 \\ 0 & 2 & 1 & 0 & 2 \\ -2 & 1 & -2 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 2 & 2 & -1 & -1 & 1 \\ 2 & 0 & 2 & 1 & 1 \\ -1 & 2 & 0 & 1 & 0 \\ -1 & 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 & -2 \end{bmatrix}, \ C = \begin{bmatrix} 2 & 3 & -3 & 1 & 1 \\ 3 & 4 & 3 & 1 & 2 \\ -3 & 3 & -2 & 1 & 2 \\ 1 & 1 & 1 & -4 & -1 \\ 1 & 2 & 2 & -1 & -2 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ b = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}.$$

We solve this example by using both the inexact short- and long-step IPMs. For both algorithms, the parameters δ, β, σ and γ are assumed as described in subsections 4.2 and 5.1 and the accuracy parameter ε is set to 10^{-5}. For the inexact short- and long-step feasible algorithms we need 222 and 184 iterations to reach our accuracy, respectively.

7. Conclusions and remarks

In this paper, we presented an extension of Gondzio's algorithm [15] on CQP to CQSCP. In fact, we used the inexact feasible IPMs on CQSCP to drive certain conditions which short- and large-step inexact feasible primal-dual algorithms would be well-defined. By using an elegant analysis, we proved that the complexities of the inexact short- and large-step feasible IPM are $O(\sqrt{n}\log(\frac{1}{\varepsilon}))$ and $O(n\log(\frac{1}{\varepsilon}))$, respectively, which coincide with the best-known iteration bounds for solving CQSCP by exact feasible IPMs.

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