ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 42 (2016), No. 6, pp. 1387–1401

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 6, pp. 1387–1401
Online ISSN: 1735-8515

QUASILINEAR SCHRÖDINGER EQUATIONS INVOLVING CRITICAL EXPONENTS IN $\mathbf{R}^2$

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(Communicated by Asadollah Aghajani)

Abstract. We study the existence of soliton solutions for a class of quasilinear elliptic equation in $\mathbf{R}^2$ with critical exponential growth. This model has been proposed in the self-channeling of a high-power ultra short laser in matter.

Keywords: Schrödinger equations, mountain pass theorem, Soliton solutions, citical exponents.
MSC(2010): Primary: 35J60; Secondary: 35J20.

1. Introduction

We study the existence of solution for the following quasilinear Schrödinger equation

$$-\triangle u+V(x)u-[\triangle(1+u^2)^{\frac{1}{2}}]\frac{u}{2(1+u^2)^{\frac{1}{2}}}=h(u),\quad x\in\mathbf{R}^2. \tag{1.1}$$

These equations are related to the existence of standing wave solutions for quasilinear Schrödinger equations of the form:

$$iz_t=-\triangle z+W(x)z-h(|z|^2)z-\triangle l(|z|^2)l'(|z|^2)z,\ x\in\mathbf{R}^{\mathbf{N}},\ \mathbf{N}\geq 2, \tag{1.2}$$

where W is a given potential, l and h are real functions. Quasilinear equations of the form (1.2) have been established in several areas of physics corresponding to various types of l, see [5,6,8,12,19] for physical backgrounds. The superfluid film equation in plasma physics has this structure for $l(s)=s$ [9]. In this case, the first existence results are due to [18]. Subsequently a general existence result was derived in [13]. In [13], the authors make a change of variable and reduce the quasilinear problem to semilinear one and Orlicz space framework was used to prove the existence of positive solutions via the Mountain pass theorem. The same method of changing of variables was also used in [3,16,17],

Article electronically published on December 18, 2016.
Received: 12 March 2015, Accepted: 7 September 2015.
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but the usual Sobolev space $H^1(\mathbf{R^N})$ framework was used as the working space. Precisely, they first make the changing of unknown variables $v = f^{-1}(u)$, where f is defined by ODE:

$$f'(t) = \frac{1}{\sqrt{1+2f^2(t)}}, \quad t \in [0, +\infty),$$

and $f(t) = -f(-t)$, $t \in (-\infty, 0]$. In the case $l(s) = (1+s)^{\frac{1}{2}}$, Eq.(1.2) models the sel-channeling of a high-power ultra short laser in matter [10]. In this case, few results are known. In [1], the authors proved global existence and uniqueness of small solutions in transverse space dimensions 2 and 3, and local existence without any smallness condition in transverse space of dimension 1. In [24], the authors proved the existence of nontrivial solution with $\mathbf{N} \geq 3$. In this paper, we will extend this result to the case $\mathbf{N} = 2$ by using a change of variables due to [22].

Let Ω be a bounded domain in $\mathbf{R}^2$, the Trudinger-Moser inequality [14, 21] asserts that

$$\exp(\alpha|u|^2) \in L^1(\Omega), \quad \forall u \in H_0^1(\Omega), \ \forall \alpha > 0.$$

and

$$\sup_{\|u\|_{H_0^1} \leq 1} \int_\Omega \exp(\alpha|u|^2)dx \leq C, \quad \forall \alpha \leq 4\pi,$$

where $\Omega \subset \mathbf{R}^2$ is a bounded smooth domain. Subsequently, [2] proved a version of Trudinger-Moser inequality in whole space, namely,

$$\exp(\alpha|u|^2) - 1 \in L^1(\mathbf{R}^2), \quad \forall u \in H^1(\mathbf{R}^2), \quad \forall \alpha > 0.$$

Moreover, if $\alpha < 4\pi$ and $|u|_{L^2(\mathbf{R}^2)} \leq C$, there exist a constant $C_2 = C_2(C, \alpha)$ such that

$$\sup_{\|\nabla u\|_{L^2(\mathbf{R}^2)} \leq 1} \int_{\mathbf{R}^2} (\exp(\alpha|u|^2) - 1)dx \leq C_2. \tag{1.3}$$

The main purpose of this paper is to obtain standing wave solutions for quasilinear Schrödinger type problems (1.1) and h satisfies the following growth critical condition:

$(c)_{\alpha_0}$ there exists $\alpha_0 > 0$ such that

$$\lim_{t\to\infty} \frac{|h(t)|}{\exp(\alpha t^2)} = \begin{cases} 0 & \forall \alpha > \alpha_0, \\ +\infty & \forall \alpha < \alpha_0. \end{cases}$$

Before stating the main result, we assume that the potential function $V : \mathbf{R}^2 \to \mathbf{R}$ is continuous and satisfies the following conditions

- (V_0) $V(x) \geq V_0 > 0$, for all $x \in \mathbf{R}^2$.
- (V_1) $\lim_{|x|\to\infty} V(x) = V_\infty$ and $V(x) \leq V_\infty < \infty$, with $V(x) \not\equiv V_\infty$, for all $x \in \mathbf{R}^2$.

The nonlinearity $h:\mathbb{R}^2\to\mathbb{R}$ is Hölder continuous and satisfies the following conditions

(H_1) $h(t)=o(t)$ as $t\to 0$.

(H_2) $h(t)$ has at most critical growth at $+\infty$, there exists $\beta_0>0$,

$$\lim_{t\to+\infty}\frac{th(t)}{\exp(\alpha_0 t^2)}\geq\beta_0>0,$$

where α_0 is given by condition $(c)_{\alpha_0}$.

(H_3) The Ambrosetti-Rabinowitz type growth condition: There exists $\mu>2$ such that

$$0\leq\mu g(t)H(t)=\mu g(t)\int_0^t h(s)ds\leq G(t)h(t),\quad t>0.$$

Obviously $h(t)=\begin{cases}2\alpha_0(\frac{3}{2})^{\frac{\mu}{2}}(\exp\alpha_0-1)^{-1}t\exp(\alpha_0t^2)\text{if } 0<t\leq 1,\\ 2\alpha_0(\frac{3}{2})^{\frac{\mu}{2}}(\exp\alpha_0-1)^{-1}\exp(\alpha_0t^2)\ \text{if } t>1,\end{cases}$ satisfy H_1,H_2,H_3 conditions.

Our main result is the following:

Theorem 1.1. *Assume that $V(x)$ verifies $(V_0)-(V_1)$ and $h(t)$ satisfies $(H_1)-(H_3)$ and $(c)_{\alpha_0}$. Then Eq.(1.1) has a positive solution.*

In this paper, C denotes positive (possibly different) constant, $L^p(\mathbf{R^N})$ denotes the usual Lebesgue space with norm $|u|_p=(\int_{\mathbf{R^N}}|u|^pdx)^{\frac{1}{p}}$, $1\leq p<\infty$, $H^1(\mathbf{R^N})$ denotes the Sobolev space with norm $\|u\|=(\int_{\mathbf{R^N}}(|\nabla u|^2+V(x)u^2)dx)^{\frac{1}{2}}$.

2. Preliminaries

We note that the solutions of (1.1) are the critical points of the following functional

$$(2.1)\quad I(u)=\frac{1}{2}\int_{\mathbf{R}^2}[1+\frac{u^2}{2(1+u^2)}]|\nabla u|^2dx+\frac{1}{2}\int_{\mathbf{R}^2}V(x)u^2dx-\int_{\mathbf{R}^2}H(u)dx,$$

where $H(u)=\int_0^u h(s)ds$. Since the functional $I(u)$ may not be well defined in the usual Sobolev spaces $H^1(\mathbf{R}^2)$. We make a change of variables as $v=G(u)=\int_0^u g(t)dt$, where $g(t)=\sqrt{1+\frac{t^2}{2(1+t^2)}}$, see[23]. Since $g(t)$ is monotonous with respect to $|t|$, the inverse function $G^{-1}(t)$ of $G(t)$ exists. Then we get

$$(2.2)\quad J(v)=\frac{1}{2}\int_{\mathbf{R}^2}|\nabla v|^2dx+\frac{1}{2}\int_{\mathbf{R}^2}V(x)|G^{-1}(v)|^2dx-\int_{\mathbf{R}^2}H(G^{-1}(v))dx.$$

Note that since $\lim_{t\to 0}G^{-1}(t)/t=1$ and $\lim_{t\to\infty}|G^{-1}(t)|/t=\sqrt{\frac{2}{3}}$, we see that $J(v)$ is well defined in $H^1(\mathbf{R}^2)$ and $J(v)\in C^1$.

If u is a solution of (1.1), then it should satisfy

$$\int_{\mathbf{R}^2}[(1+\frac{u^2}{2(1+u^2)})\nabla u\nabla\varphi+V(x)u\varphi-h(u)\varphi]dx=0,\quad \forall\varphi\in C_0^\infty(\mathbf{R}^2). \tag{2.3}$$

We show that (2.3) is equivalent to

$$\begin{aligned}J'(v)\psi=&\int_{\mathbf{R}^2}\Big[\nabla v\nabla\psi+V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\psi-\frac{h(G^{-1}(v))}{g(G^{-1}(v))}\psi\Big]dx\\ =&0,\quad \forall\psi\in C_0^\infty(\mathbf{R}^2).\end{aligned} \tag{2.4}$$

Indeed, if we choose $\varphi=\frac{1}{g(u)}\psi$ in (2.3), then we immediately get (2.4). On the other hand, since $u=G^{-1}(v)$, if we let $\psi=g(u)\varphi$ in (2.4), we get (2.3).

Therefore, in order to find the nontrivial solutions of (1.1), it suffices to study the existence of the nontrivial solutions of the following equations

$$-\Delta v+V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}-\frac{h(G^{-1}(v))}{g(G^{-1}(v))}=0. \tag{2.5}$$

We define $-\Delta v=K(x,v)$, where

$$K(x,v)=-V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}+\frac{h(G^{-1}(v))}{g(G^{-1}(v))} \tag{2.6}$$

Before we close this section, we collect some properties of the change of variables.

Lemma 2.1. (1) $\sqrt{\frac{2}{3}}t\le|G^{-1}(t)|\le t$, *for all* $t\ge 0$*;*

(2) $|(G^{-1}(t))'|\le 1$, *for all* $t\in\boldsymbol{R}$*;*

(3) $\lim\limits_{t\to 0}\frac{|G^{-1}(t)|}{t}=1$*;*

(4) $\lim\limits_{t\to\infty}\frac{|G^{-1}(t)|}{t}=\sqrt{\frac{2}{3}}$*;*

(5) $\sqrt{\frac{2}{3}}G^{-1}(t)\le t(G^{-1}(t))'\le G^{-1}(t)$ *for all* $t\ge 0$*;*

(6) $\frac{tg'(t)}{g(t)}\le 5-2\sqrt{6}$ *for all* $t\in\boldsymbol{R}$.

Proof. (1) Since $[G^{-1}(t)-\frac{1}{g(0)}t]'=\frac{1}{g(G^{-1}(t))}-\frac{1}{g(0)}\le 0$ and $[G^{-1}(t)-\frac{1}{g(\infty)}t]'=\frac{1}{g(G^{-1}(t))}-\frac{1}{g(\infty)}\ge 0$, so $\frac{1}{g(\infty)}t\le G^{-1}(t)\le\frac{1}{g(0)}t$, for $t\ge 0$, that is $\frac{1}{g(\infty)}t=\sqrt{\frac{2}{3}}t\le G^{-1}(t)\le\frac{1}{g(0)}t=t$, for $t\ge 0$, which proves (1).

Since $\lim\limits_{t\to 0}\frac{G^{-1}(t)}{t}=((G^{-1}(t))'|_{t=0}=\frac{1}{g(G^{-1}(0))}=1$ and $g(t)$ is increasing, so properties (2) and (3) are clear.

For (4), the result is clear since $g(t)$ is an increasing bounded function.

For (5), since g is a increasing function, then $G(t)\le g(t)t$, which implies that $t(G^{-1}(t))'\le G^{-1}(t)$. On the other hand, by (1) and $\sqrt{\frac{2}{3}}\le(G^{-1}(t))'\le 1$, we get $\sqrt{\frac{2}{3}}G^{-1}(t)\le t(G^{-1}(t))'$.

Since

$$\frac{t}{g(t)}g'(t) = \frac{t^2}{2(1+t^2)^2 g^2(t)} = \frac{t^2}{2+5t^2+3t^4}$$
$$= \frac{1}{\frac{2}{t^2}+5+3t^2} \leq 5-2\sqrt{6},$$

which proves (6). □

3. Mountain pass geometry

In this section we establish the geometric hypotheses of the mountain pass theorem.

Lemma 3.1. *There exist $\rho_0, a_0 > 0$ such that $J(v) \geq a_0$ for all $\|v\| = \rho_0$.*

Proof. Let

$$Q(x,t) := -\frac{1}{2}V(x)|G^{-1}(t)|^2 + H(G^{-1}(t)).$$

Then, by Lemma 2.1 and (H_2), (H_3), for $\epsilon > 0$ sufficiently small, given $\alpha > \alpha_0$, there exists a constants $C_\epsilon > 0$ and $p > 2$ such that

$$\lim_{t\to 0}\frac{Q(x,t)}{t^2} = -\frac{1}{2}V(x), \tag{3.1}$$

$$\lim_{t\to\infty}\frac{Q(x,t)}{t^{p+1}(\exp(\alpha t^2)-1)} = 0. \tag{3.2}$$

$$Q(x,t) \leq (-\frac{1}{2}V(x)+\epsilon)t^2 + C_\epsilon(\exp(\alpha t^2)-1)t^{p+1}. \tag{3.3}$$

By Trudinger-Moser inequality

$$\int_{\mathbf{R}^2}(\exp(\alpha t^2)-1)dx \leq C, \tag{3.4}$$

for every $q > 1$ close to one, it follows from the above inequality and Hölder inequality that

(3.5)
$$\int_{\mathbf{R}^2}|v|^{p+1}(\exp(\alpha v^2)-1)dx \leq (\int_{\mathbf{R}^2}|v|^{q'(p+1)}dx)^{\frac{1}{q'}}(\int_{\mathbf{R}^2}(\exp(\alpha v^2)-1)^q)^{\frac{1}{q}}$$
$$\leq C(\int_{\mathbf{R}^2}|v|^{q'(p+1)}dx)^{\frac{1}{q'}}$$
$$\leq C\|v\|^{p+1}.$$

where $\frac{1}{q}+\frac{1}{q'}=1$. Then, we have

$$\begin{aligned}J(v) &= \frac{1}{2}\int_{\mathbf{R}^2}|\nabla v|^2dx+\frac{1}{2}\int_{\mathbf{R}^2}V(x)|G^{-1}(v)|^2dx-\int_{\mathbf{R}^2}H(G^{-1}(v))dx\\ &\geq \frac{1}{2}\int_{\mathbf{R}^2}|\nabla v|^2dx+\frac{1}{2}\int_{\mathbf{R}^2}V(x)v^2dx-\frac{\varepsilon}{2}\int_{\mathbf{R}^2}v^2dx-C\int_{\mathbf{R}^2}(|v|^{p+1}\exp(\alpha|v|^2)-1)dx\\ &\geq C\|v\|^2-C\|v\|^{p+1},\end{aligned}$$

which implies the result since $2<p+1$. Thus, by choosing $\rho_0>0$, a_0 small, such that $J(v)\geq a_0$, if $\|v\|=\rho_0$. □

Lemma 3.2. *There exists $v\in H^1(\boldsymbol{R}^2)$ such that $J(v)<0$.*

Proof. Given $\varphi\in C_0^\infty(\mathbf{R}^2,[0,1])$ with $supp\varphi=\bar{B}_1$. We will prove that $J(t\varphi)\to -\infty$ as $t\to\infty$, which will prove the result if we take $v=t\varphi$ with t large enough. Since $G^{-1}(v)\leq\frac{1}{g(0)}v$, by (H_3), we have

$$\begin{aligned}J(t\varphi) &\leq \frac{1}{2}t^2\int_{\mathbf{R}^2}|\nabla\varphi|^2dx+\frac{1}{2}t^2C\int_{\mathbf{R}^2}V_\infty|\varphi|^2dx-\int_{\mathbf{R}^2}H(G^{-1}(t\varphi))dx\\ &\leq \frac{1}{2}t^2\int_{\mathbf{R}^2}|\nabla\varphi|^2dx+\frac{1}{2}t^2C\int_{\mathbf{R}^2}V_\infty|\varphi|^2dx-Ct^\mu\int_{\mathbf{R}^2}\varphi^\mu dx.\end{aligned}$$

We get the result since $\mu>2$. □

4. **Existence**

In consequence of Lemma 3.1 and 3.2 and of Ambrosetti-Rabinowitz Mountain Pass Theorem [20], see also [4,7,11], for the constant

$$c=\inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}J(\gamma(t))>0,$$

where $\Gamma=\Big\{\gamma\in C([0,1],H^1(\mathbf{R^N})):\gamma(0)=0,\gamma(1)\neq 0,J(\gamma(1))<0\Big\}$, there exists a Palais-Smale sequence at level c, that is, $J(v_n)\to c$ and $J'(v_n)\to 0$ as $n\to\infty$.

Lemma 4.1. *The Palais-Smale sequence $\{v_n\}$ for J is bounded.*

Proof. Since $\{v_n\}\subset H^1(\mathbf{R}^2)$ satisfies

$$\begin{aligned}J(v_n)=&\frac{1}{2}\int_{\mathbf{R}^2}|\nabla v_n|^2dx+\frac{1}{2}\int_{\mathbf{R}^2}V(x)|G^{-1}(v_n)|^2dx\\ &-\int_{\mathbf{R}^2}H(G^{-1}(v_n))dx\to c,\end{aligned}\tag{4.1}$$

and for any $\psi\in C_0^\infty(\mathbf{R}^2)$,

$$\begin{aligned}J'(v_n)\psi=&\int_{\mathbf{R}^2}\Big[\nabla v_n\nabla\psi+V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\psi-\frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}\psi\Big]dx\\ =&\,o(1)\|\psi\|.\end{aligned}\tag{4.2}$$

Since $C_0^\infty(\mathbf{R}^2)$ is dense in $H^1(\mathbf{R}^2)$, by choosing $\psi = v_n$ in (4.2), we deduce that

$$J'(v_n)v_n = \int_{\mathbf{R}^2} \Big[|\nabla v_n|^2 + V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n - \frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n\Big]dx \tag{4.3}$$
$$= o(1)\|v_n\|.$$

By (4.1) and (4.3), using (H_3), we get

$$\begin{aligned}\mu J(v_n) - J'(v_n)v_n &= \frac{\mu-2}{2}\int_{\mathbf{R}^2}|\nabla v_n|^2dx \\ &+ \int_{\mathbf{R}^2} V(x)G^{-1}(v_n)\Big[\frac{1}{2}\mu G^{-1}(v_n) - \frac{1}{g(G^{-1}(v_n))}v_n\Big]dx \\ &- \int_{\mathbf{R}^2}\Big[\mu H(G^{-1}(v_n)) - \frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n\Big]dx \\ &\ge \frac{\mu-2}{2}\Big[\int_{\mathbf{R}^2}|\nabla v_n|^2dx + \int_{\mathbf{R}^2}V(x)|G^{-1}(v_n)|^2dx\Big].\end{aligned}$$

Since $G(t) \le g(t)t$, so $G^{-1}(v_n) \ge \frac{v_n}{g(G^{-1}(v_n))} \ge \sqrt{\frac{2}{3}}v_n$. Hence,

$$\mu J(v_n) - J'(v_n)v_n \ge \frac{\mu-2}{2}\Big[\int_{\mathbf{R}^2}|\nabla v_n|^2dx + \frac{2}{3}\int_{\mathbf{R}^2}V(x){v_n}^2dx\Big],$$

which implies the result. □

From Lemma 4.1, there exists $v \in H^1(\mathbf{R}^2)$ such that $v_n \rightharpoonup v$ weakly in $H^1(\mathbf{R}^2)$ and $J'(v)\psi = 0$ for every $\psi \in C_0^\infty(\mathbf{R}^2)$, that is v a weak solution. In fact, recalling the definition of the function K given by (2.6), it suffices to prove that:

$$\int_{\mathbf{R}^2} K(x, v_n)\psi \to \int_{\mathbf{R}^2} K(x, v)\psi, \quad \forall \psi \in C_0^\infty(\mathbf{R}^2).$$

In order to verify this convergence, given $\psi \in C_0^\infty(\mathbf{R}^2)$, we denote by Ω the support set of ψ. Since $\{v_n\}$ is bounded in $H^1(\mathbf{R}^2)$, we may take a subsequence denoted again by v_n such that:

$$v_n \rightharpoonup v \text{ in } H^1(\mathbf{R}^2);\ v_n \to v \text{ in } L^q(\Omega),\ \forall\, q \ge 1;\ v_n(x) \to v(x)\ a.e. \text{in } \Omega.$$

Moreover, from preceding paragraphs, we know the sequence $\{\int K(x, v_n)\psi v_n\}$ is bounded. Then, invoking Lemma 2.1, we have

$$\int_{\mathbf{R}^2} K(x, v_n)\psi = \int_\Omega K(x, v_n)\psi \to \int_\Omega K(x, v)\psi = \int_{\mathbf{R}^2} K(x, v)\psi.$$

Hence, v is a weak solution of (1).

In order to show v is nontrivial, we will estimate the minimax level obtained by the Mountain Pass theorem. First, we introduce some notations and facts. Let V_∞ be given by condition (V_1). Consider the Sobolev space $H^1(\mathbf{R}^2)$ endowed with the equivalent norm:

$$\|v\| = (\int |\nabla v|^2 + V_\infty v^2)^{1/2}, \quad \forall v \in H^1(\mathbf{R}^2).$$

We define the functional $I_\infty : H^1(\mathbf{R}^2) \to \mathbf{R}$ given by:

$$I_\infty(v) = \frac{1}{2}\int (|\nabla v|^2 + V_\infty v^2) - \int H(G^{-1}(v)).$$

Working with the analogue of J, the function I_∞ is well defined and belongs to $C^1(H^1(\mathbf{R}^2), \mathbf{R})$.

Now, we take β_0 given by (H_2) and let $r > 0$ be such that

$$\beta_0 > \frac{2\sqrt{6}}{\alpha_1 r^2}, \tag{4.4}$$

where $\alpha_1 = \alpha_0\delta,\ 0 < \delta < \sqrt{\frac{2}{3}}$.

We consider the Moser sequence [14] defined by:

$$\widetilde{M_n}(x,r) \equiv \widetilde{M_n} = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{1/2} & \text{if} |x| \le \frac{r}{n}, \\ (\log(r/|x|))/(\log n)^{1/2} & \text{if} \frac{r}{n} \le |x| \le r, \\ 0 & \text{if} |x| > r, \end{cases}$$

which satisfies: $\widetilde{M_n} \in H^1(\mathbf{R}^2)$ and $\|\widetilde{M_n}\|^2 = 1 + O((\log n)^{-1})$, as $n \to \infty$. Moreover, let $M_n(x,r) \equiv M_n = \widetilde{M_n}/\|\widetilde{M_n}\|$, it is not difficult to see that $M_n^2(x,r) \equiv M_n^2 = (2\pi)^{-1}\log n + d_n$, where d_n is a bounded real sequence.

Thus we have the following estimate, whose proof is based on the argument used in [15] Lemma 5.

Proposition 4.2. *Suppose $h(t)$ satisfies $(c)_{\alpha_0}$ and $(H_1) - (H_3)$. Then, there exists $n \in \boldsymbol{N}$ such that:* $\max\{I_\infty(tM_n) : t \ge 0\} < C^* \equiv \frac{4\pi}{\alpha_1}$, *where* $\alpha_1 = \alpha_0\delta$ *and* $0 < \delta < \sqrt{\frac{2}{3}}$.

Proof. By contradiction, suppose that for all n we have

$$\max\{I_\infty(tM_n) : t \ge 0\} \ge C^*.$$

Thus, there exists $t_n > 0$ such that

$$I_\infty(t_n M_n) = \max\{I_\infty(tM_n) : t \ge 0\}.$$

By the definition of I_∞ and M_n, we have

$$I_\infty(t_n M_n) = \frac{t_n^2}{2} - \int_{\mathbf{R}^\mathbf{N}} H(G^{-1}(t_n M_n)) \ge C^*.$$

Since $H > 0$, we get $t_n^2 \geq 2C^*$. On the other hand, by $\frac{d}{dt}I_\infty(tM_n)|_{t=t_n} = 0$. We have

$$(4.5)\qquad \begin{aligned} t_n^2 &= \int_{\mathbf{R}^2} t_n M_n h(G^{-1}(t_n M_n))[G^{-1}(t_n M_n)]' dx \\ &= \int_{|x|\leq r} t_n M_n h(G^{-1}(t_n M_n))[G^{-1}(t_n M_n)]' dx. \end{aligned}$$

By (H_2), given $\epsilon > 0$ there exists $R_\epsilon > 0$ such that for all $t \geq R_\epsilon$ and for all $|x| \leq r, th(t) \geq (\beta_0 - \epsilon)\exp(\alpha_0 t^2)$. Since $M_n \to +\infty$ as $n \to \infty$ and t_n is bounded below by a positive constant, i.e. $t_n \geq c$, $M_n(x) \geq R_\epsilon$, as $n \to \infty$.

Since $\sqrt{\frac{2}{3}}G^{-1}(t) \leq t(G^{-1}(t))'$, then

$$\begin{aligned} t_n M_n h(G^{-1}(t_n M_n))[G^{-1}(t_n M_n)]' &\geq \sqrt{\frac{2}{3}} G^{-1}(t_n M_n) h(G^{-1}(t_n M_n)) \\ &\geq \sqrt{\frac{2}{3}}(\beta_0 - \epsilon)\exp(\alpha_0 (G^{-1}(t_n M_n))^2). \end{aligned}$$

On the other hand, since $\lim\limits_{t\to\infty} \frac{|G^{-1}(t)|}{t} = \sqrt{\frac{2}{3}}$, then $G^{-1}(t) > \delta t$, for $\delta < \sqrt{\frac{2}{3}}$, as $t \to +\infty (t > R_\epsilon)$. So $t_n M_n h(G^{-1}(t_n M_n))[G^{-1}(t_n M_n)]' \geq \sqrt{\frac{2}{3}}(\beta_0 - \epsilon)\exp(\alpha_0 \delta^2 {t_n}^2 {M_n}^2)$.

Let $\alpha_1 = \alpha_0 \delta^2$, then

$$(4.6)\qquad \begin{aligned} t_n^2 &\geq \sqrt{\frac{2}{3}} \int_{\mathbf{R}^2} \exp(\alpha_1 (t_n M_n)^2) \\ &\geq \sqrt{\frac{2}{3}}(\beta_0 - \epsilon) \int_{|x|\leq \frac{r}{n}} \exp(\alpha_1 (t_n M_n)^2). \\ &\geq \sqrt{\frac{2}{3}}(\beta_0 - \epsilon)\pi(\frac{r}{n})^2 exp[(\alpha_1 t_n^2 (2\pi)^{-1} \log n) + \alpha_1 t_n^2 d_n]. \end{aligned}$$

Thus

$$1 \geq \sqrt{\frac{2}{3}}(\beta_0 - \epsilon)\pi r^2 exp[(\alpha_1 t_n^2 (2\pi)^{-1} \log n) + \alpha_1 t_n^2 d_n - 2\log n - 2\log t_n],$$

which implies that t_n is bounded.

By $t_n^2 \geq 2C^*$ and

$$t_n^2 \geq \sqrt{\frac{2}{3}}(\beta_0 - \epsilon)\pi r^2 exp[(\alpha_1 t_n^2 (2\pi)^{-1} - 2)\log n + \alpha_1 t_n^2 d_n]$$

it follows that:

$$t_n^2 \to \frac{4\pi}{\alpha_1}.$$

Now, let

$$A_n = \{x : t_n M_n \geq R_\epsilon, |x| \leq r\}.$$

$$B_n = \{x : t_n M_n < R_\epsilon, |x| \leq r\}.$$

Then

$$\begin{aligned}
t_n^2 =& \int_{|x|\leq r} t_n M_n h(G^{-1}(t_n M_n))[G^{-1}(t_n M_n)]^{'} \\
=& \int_{A_n \cup B_n} t_n M_n h(G^{-1}(t_n M_n))[G^{-1}(t_n M_n)]^{'} \\
\geq& \sqrt{\frac{2}{3}}(\beta_0 - \epsilon) \int_{|x|\leq r} \exp[\alpha_1 (t_n M_n)^2] - \sqrt{\frac{2}{3}}(\beta_0 - \epsilon) \int_{B_n} exp[\alpha_1 (t_n M_n)^2] \\
+& \int_{B_n} t_n M_n h(G^{-1}(t_n M_n))[G^{-1}(t_n M_n)]^{'}.
\end{aligned}$$

Since $M_n \to 0$ a.e. in B_n, then by the Lebesgue Dominated Convergence Theorem,

$$\int_{B_n} t_n M_n h(G^{-1}(t_n M_n))(G^{-1}(t_n M_n))^{'} \to o$$

and

$$\int_{B_n} exp(\alpha_1 (t_n M_n)^2) \to \pi r^2,$$

as $n \to \infty$. On the other hand, since $t_n^2 \geq \frac{4\pi}{\alpha_1}$, then

$$\begin{aligned}
\int_{|x|\leq r} exp[\alpha_1 (t_n M_n)^2] \geq& \int_{|x|\leq r} exp[4\pi (M_n)^2] \\
=& (\int_{|x|\leq \frac{r}{n}} + \int_{\frac{r}{n}\leq |x|\leq r}) exp[4\pi (M_n)^2].
\end{aligned}$$

Now,

$$\begin{aligned}
\int_{|x|\leq \frac{r}{n}} exp[4\pi (M_n)^2] =& \int_{|x|\leq \frac{r}{n}} exp[2\log n + 4\pi d_n] \\
=& \pi (\frac{r}{n})^2 n^2 exp(4\pi d_n) \\
=& \pi r^2 exp(4\pi d_n),
\end{aligned}$$

and $\int_{\frac{r}{n}\le|x|\le r} exp[4\pi(M_n)^2]$, let $t = re^{-||\widetilde{M_n}||(\log n)^{\frac{1}{2}}s}$, thus

$$\begin{aligned}
&\int_{\frac{r}{n}\le|x|\le r}\exp[4\pi(M_n)^2]\\
&\qquad=\int_{\frac{r}{n}\le|x|\le r}\exp\{\frac{4\pi}{||\widetilde{M_n}||^2}[\frac{1}{2\pi}\frac{(\log\frac{r}{|x|})^2}{\log n}]\}\\
&\qquad=2\pi\int_{\frac{r}{n}}^{r} t\exp\{2\frac{(\log\frac{r}{t})^2}{\log n||\widetilde{M_n}||^2}\}\\
&\qquad=2\pi\int_0^{(\log n)^{\frac{1}{2}}||\widetilde{M_n}||^{-1}} re^{-||\widetilde{M_n}||(\log n)^{\frac{1}{2}}s}e^{2s^2}r||\widetilde{M_n}||(\log n)^{\frac{1}{2}}e^{-||\widetilde{M_n}||(\log n)^{\frac{1}{2}}s}ds\\
&\qquad\ge 2\pi r^2\int_0^{(\log n)^{\frac{1}{2}}||\widetilde{M_n}||^{-1}} e^{-2||\widetilde{M_n}||(\log n)^{\frac{1}{2}}s}||\widetilde{M_n}||(\log n)^{\frac{1}{2}}ds\\
&\qquad=\frac{2\pi r^2}{-2}[e^{-2||\widetilde{M_n}||(\log n)^{\frac{1}{2}}s}|_0^{(\log n)^{\frac{1}{2}}||\widetilde{M_n}||^{-1}}]\\
&\qquad=-\pi r^2[e^{-2\log n}-1]\to\pi r^2.
\end{aligned}$$

Thus

$$t_n^2 \ge \sqrt{\frac{2}{3}}(\beta_0-\epsilon)[\pi r^2 exp(4\pi d_n)+\pi r^2]-\sqrt{\frac{2}{3}}(\beta_0-\epsilon)\pi r^2$$

Since $d_n \to d_0 > 0$, then we get

$$t_n^2 \ge \sqrt{\frac{2}{3}}(\beta_0-\epsilon)\pi r^2.$$

So $\frac{4\pi}{\alpha_1} \ge \sqrt{\frac{2}{3}}(\beta_0-\epsilon)\pi r^2$. Hence, we gain $(\beta_0-\epsilon) \le \frac{2\sqrt{6}}{\alpha_1 r^2}$, which contrary to (4.4). Thus Proposition 4.2 is proved.

The following lemma shows that the Cerami sequence $\{v_n\}$ has a nonvanishing behaviour.

Lemma 4.3. *There exist positive constants a and R, and a sequence $(y_n) \subset \boldsymbol{R}^2$ such that*

$$\lim_{n\to\infty}\int_{B_{R(y_n)}}[G^{-1}(v_n)]^2 \ge a > 0, \tag{4.7}$$

where $B_R(x)$ denotes a ball of radius R centred at the point x.

Proof. Suppose by contradiction that (4.7) does not occur. Then

$$\lim_{n\to\infty}\sup_{y\in\mathbf{R}^2}\int_{B_{R(y_n)}}[G^{-1}(v_n)]^2 = 0. \tag{4.8}$$

From (4.8) and applying a Lions compactness lemma [11] we obtain as $n \to \infty$,

$$G^{-1}(v_n) \to 0, \text{in} L^q(\mathbf{R}^2), \forall q \in (2, \infty). \tag{4.9}$$

Then, we can show the crucial part of this proof, which is the following:

$$\int_{R^2} h(G^{-1}(v_n))G^{-1}(v_n) \to 0, \tag{4.10}$$

as $n \to \infty$. To prove such convergence, we start arguing as in the proof of Lemma 4.1. Thus, $J(v_n) \to C_3$. So $J(v_n) = \frac{2}{3}||v_n|| - \int_{R^2} H(G^{-1}(v_n))$ and

$$J'(v_n)g(G^{-1}(v_n))G^{-1}(v_n) = \int_{\mathbf{R}^2} [|Dv_n|^2 + \frac{G^{-1}(v_n)g'(G^{-1}(v_n))}{g(G^{-1}(v_n))}|Dv_n|^2 \\ + V(x)(G^{-1}(v_n))^2 - h(G^{-1}(v_n))G^{-1}(v_n)].$$

Thus

$$|Dv_n|^2 \leq \frac{C_3}{\zeta}, \tag{4.11}$$

for some ζ.

From lemma 2.1 (4), there exist $\sigma > \sqrt{\frac{2}{3}}$ and $\mathbf{R} > 0$ such that

$$G^{-1}(t) < \sigma t, \forall t > \mathbf{R}. \tag{4.12}$$

Now, we take $\alpha > \alpha_0$. From (H_1) and $(c)_{\alpha_0}$, given $\epsilon > 0$, there exists a positive constant $C = C(\epsilon, \alpha, q)$ such that:

$$h(t) \leq \epsilon t + C(\exp(\alpha t^2) - 1)t^3, \forall t \geq 0. \tag{4.13}$$

Thus, using (4.11),(4.12) and (4.13), we get for every $n \geq n_0$

$$\begin{aligned}
0 \leq & \int_{\mathbf{R}^2} h(G^{-1}(v_n))G^{-1}(v_n) \\
\leq & \epsilon \int_{\mathbf{R}^2} [G^{-1}(v_n)]^2 + C \int_{\mathbf{R}^2} (\exp(\alpha[G^{-1}(v_n)]^2) - 1)G^{-1}(v_n)^4 \\
= & \epsilon \int_{\mathbf{R}^2} [G^{-1}(v_n)]^2 + \\
& C\{\int_{\{x;|v_n(x)| \leq R\}} + \int_{\{x;|v_n(x)| \geq R\}}\}(\exp(\alpha[G^{-1}(v_n)]^2) - 1)G^{-1}(v_n)^4 \\
\leq & \epsilon \int_{\mathbf{R}^2} [G^{-1}(v_n)]^2 + \widetilde{C} \int_{\mathbf{R}^2} [G^{-1}(v_n)]^2 \\
& + C(\int_{\mathbf{R}^2} (\exp(\alpha r[G^{-1}(v_n)]^2) - 1))^{1/r} (\int_{\mathbf{R}^2} G^{-1}(v_n)^{4r'})^{1/r'}
\end{aligned}$$

$$\leq \epsilon \int_{\mathbf{R}^2} [G^{-1}(v_n)]^2 + \widetilde{C} \int_{\mathbf{R}^2} [G^{-1}(v_n)]^2$$
$$+C(\int_{\mathbf{R}^2} [\exp(\alpha r \sigma^2 \frac{C_3}{\zeta} (\frac{v_n}{|Dv_n|})^2 - 1])^{1/r} (\int_{\mathbf{R}^2} G^{-1}(v_n)^{4r'})^{1/r'},$$

where r satisfies (4.4) and $1/r + 1/r' = 1$. By Proposition 4.2, we may take $\alpha > \alpha_0$ such that $\alpha r \sigma^2 C_3 < 4\pi\zeta$. From (1.3), we get the last integral is bounded uniformly. Hence, from (4.9), we conclude that (4.10) holds. Now, we are ready to conclude the proof of Lemma 4.3. Taking again $w_n = G^{-1}(v_n)/[G^{-1}(v_n)]'$. We have

$$\begin{aligned} o(1) =& J'(v_n) w_n \\ =& \int_{\mathbf{R}^2} (1 + \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} g'(G^{-1}(v_n))) |\nabla v_n|^2 dx \\ &+ \int_{\mathbf{R}^2} (V(x)[G^{-1}(v_n)]^2 - h(G^{-1}(v_n)) G^{-1}(v_n)) dx \\ \geq& \int_{\mathbf{R}^2} |\nabla v_n|^2 dx + \int_{\mathbf{R}^2} (V(x)[G^{-1}(v_n)]^2 - h(G^{-1}(v_n)) G^{-1}(v_n)) dx. \end{aligned}$$

Then from (4.10), we conclude that

$$\int |\nabla v_n|^2 + \int V(x)[G^{-1}(v_n)]^2 \to 0, \text{as} \quad n \to \infty. \tag{4.14}$$

On the other hand, by (H_3) and (4.10), we also have

$$\int_{\mathbf{R}^2} H(G^{-1}(v_n)) \to 0, \text{as} \quad n \to \infty. \tag{4.15}$$

By combing (4.14) and (4.15), we get a contradiction because

$$0 < c_0 = \lim_{n \to \infty} J(v_n) = 0.$$

The proof of Lemma 4.3 is complete. □

Now, we consider the functional at infinity J_∞ associated with J. We define $J_\infty : H^1(\mathbf{R}^2) \to \mathbf{R}$ by:

$$J_\infty(v) = \frac{1}{2} \int_{\mathbf{R}^2} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbf{R}^2} V_\infty [G^{-1}(v)]^2 dx - \int_{\mathbf{R}^2} H(G^{-1}(v)) dx.$$

Lemma 4.4. *The Cerami sequence $\{v_n\}$ is a (PS) sequence for J_∞ at level c_0.*

Proof. From (V_1), given $\epsilon > 0$ there exists $R > 0$ such that

$$|V(x) - V_\infty| < \epsilon, \forall |x| \geq R.$$

Thus,

$$\begin{aligned}&|J_\infty(v_n) - J(v_n)|\\&= \frac{1}{2}\int_{B_{R(0)}} |V_\infty - V(x)|[G^{-1}(v_n)]^2dx + \frac{1}{2}\int_{\mathbf{R}^2\setminus B_{R(0)}} |V_\infty - V(x)|[G^{-1}(v_n)]^2dx\\&\leq \frac{1}{2}|V_\infty - V(x)|_\infty \int_{B_{R(0)}} [G^{-1}(v_n)]^2dx + \frac{1}{2}\epsilon\int_{\mathbf{R}^2\setminus B_{R(0)}} [G^{-1}(v_n)]^2dx\\&\leq o(1), \quad \text{as} \quad n\to\infty,\end{aligned}$$

where in the last inequality we made use that:

$$\int_{B_{R(0)}} [G^{-1}(v_n)]^2 \to 0, \text{as} \quad n\to\infty,$$

since $G^{-1}(v_n) \in H^1(\mathbf{R}^2)$ and the embedding $H^1(\mathbf{R}^2)$ into $L^q(\mathbf{R}^2), q > 1$,is locally compact and $v_n \rightharpoonup v \equiv 0$ weakly in $H^1(\mathbf{R}^2)$.
Therefore,

$$J_\infty(v_n) \to c_0, \text{as} \quad n\to\infty.$$

Similarly,

$$\begin{aligned}&\sup_{\|\psi\|<1} |(J'_\infty(v_n) - J'(v_n), \psi)|\\&= \sup_{\|\psi\|<1} |\int_{\mathbf{R}^2} (V_\infty - V(x))G^{-1}(v_n)[G^{-1}(v_n)]'\psi| = o(1), \text{as} \quad n\to\infty.\end{aligned}$$

Hence $J'_\infty(v_n) \to 0$, as $n\to\infty$. This proves Lemma 4.4.

Finally, by [17] Theorem 1.1 is proved. □

Acknowledgements

The authors would like to thank the anonymous referees for their valuable comments and suggestions which substantially improved the manuscript. The research is supported by SRF of GZNU(2016) and NSF of China (11461014).

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