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SIMPLE GROUPS WITH THE SAME PRIME GRAPH AS $D_n(q)$

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ABSTRACT. Vasil'ev posed Problem 16.26 in [The Kourovka Notebook: Unsolved Problems in Group Theory, 16th ed., Sobolev Inst. Math., Novosibirsk (2006).] as follows:
Does there exist a positive integer k such that there are no k pairwise nonisomorphic nonabelian finite simple groups with the same graphs of primes? Conjecture: $k = 5$. In [Zvezdina, On nonabelian simple groups having the same prime graph as an alternating group, *Siberian Math. J.*, 2013] the above conjecture is positively answered when one of these pairwise nonisomorphic groups is an alternating group.

In this paper, we continue this work and determine all nonabelian simple groups, which have the same prime graph as the nonabelian simple group $D_n(q)$.
Keywords: Prime graph, simple group, Vasil'ev conjecture.
MSC(2010): Primary: 20D05; Secondary: 20D60.

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. The *spectrum* of a finite group G which is denoted by $\omega(G)$ is the set of its element orders. We construct the *prime graph* of G which is denoted by $\Gamma(G)$ as follows: the vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge (we write $p \sim q$) if and only if G contains an element of order pq. Let $s(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_i(G)$, $i = 1, ..., s(G)$, be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$ we always suppose that $2 \in \pi_1(G)$. The connected components of the prime graph of nonabelian simple groups with disconnected prime graph are listed in [13]. In graph theory a subset of vertices of a graph is called an independent set if its vertices are pairwise

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non-adjacent. Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise non-adjacent in $\Gamma(G)$. In other words, if $\rho(G)$ is some independent set with the maximal number of vertices in $\Gamma(G)$, then $t(G) = |\rho(G)|$. Similarly if $p \in \pi(G)$, then let $\rho(p, G)$ be some independent set with the maximal number of vertices in $\Gamma(G)$ containing p and $t(p, G) = |\rho(p, G)|$. In [11, Tables 2-9], independent sets also independence numbers for all simple groups are listed.

Hagie in [6] determined finite groups G satisfying $\Gamma(G) = \Gamma(S)$, where S is a sporadic simple group. The same problem is considered for some finite simple groups (see [1–3, 8, 14]).

A. V. Vasil'ev formulated the following problem in [9]:

Problem 16.26. *Does there exist a positive integer k such that there are no k pairwise nonisomorphic nonabelian finite simple groups with the same graphs of primes? Conjecture: $k = 5$.*

In [16] this problem solved when one of these pairwise nonisomorphic groups is an alternating group. The conjecture is true in this case.

In the rest of this paper, we denote by r_i and u_i, a primitive prime divisor of $q^i - 1$ and $q'^i - 1$, respectively. Also we consider $R_i(q)$ and $U_i(q')$ as the set of all primitive prime divisors of $q^i - 1$ and $q'^i - 1$, respectively.

In this paper, we continue this work and we determine all nonabelian simple groups, with the same prime graph as $D_n(q)$ for $n > 4$. In fact we prove the following theorem:

Main Theorem: *Let $G = D_n(q)$, where $n > 4$ and $q = p^\alpha$, and also S be a simple group. Then the prime graphs of G and S coincide if and only if one of the following holds:*

(1) $S = D_n(q)$.

(2) $S = D_n(q')$, where $q' = p'^\beta$ and n is even, $p \neq p'$, $p \equiv 1 \pmod 4$ and $p' \equiv 1 \pmod 4$. Moreover, $R_1(q) = U_1(q')$, $R_2(q) = U_2(q')$, $R_{n-1}(q) = U_{n-1}(q')$, $R_{2(n-1)}(q) = U_{2(n-1)}(q')$, $\{p\} \cup R_4(q) = \{p'\} \cup U_4(q')$.

(3) $S = C_{n-1}(q')$ or $S = B_{n-1}(q')$, where $q' = p'^\beta$, $n \equiv 0 \pmod 4$, $p \neq p'$, $p \equiv 1 \pmod 4$ and $p' \equiv 1 \pmod 4$. Moreover, $R_1(q) = U_1(q')$, $R_2(q) = U_2(q')$, $R_{2(n-1)}(q) = U_{2n'}(q')$, $R_{n-1}(q) = U_{n'}(q')$ and $\{p\} \cup R_4(q) = \{p'\} \cup U_4(q')$.

Consequently, we get that the conjecture is true when n is odd. As we can see in (2) and (3), if n is even, then under some conditions $D_n(q)$ may have the same prime graph as $D_n(q')$, $C_{n-1}(q')$ and $B_{n-1}(q')$. Also we conjecture that we cannot find any q and q' such that satisfying in all conditions in parts (2) and (3) of the above theorem. We note that for the proof of the main theorem we use Remark 3.1 and so $n = 4$ is still open.

In this paper, we use the classification of finite simple groups, all groups are finite and by simple groups we mean nonabelian simple groups. Also for a natural number n and a prime number p, we denote by n_p, the p-part of n, i.e. $n_p = p^\alpha$, such that $p^\alpha \mid n$, but $p^{\alpha+1} \nmid n$.

2. Preliminary results

Remark 2.1. ([10]) Let p be a prime number and $(q,p)=1$. Let $k\geq 1$ be the smallest positive integer such that $q^k\equiv 1 \pmod p$. Then k is called *the order of q with respect to p* and we denote it by $\mathrm{ord}_p(q)$. Obviously by the Fermat's little theorem it follows that $\mathrm{ord}_p(q)|(p-1)$. Also if $q^n\equiv 1 \pmod p$, then $\mathrm{ord}_p(q)|n$. Similarly if $m>1$ is an integer and $(q,m)=1$, we can define $\mathrm{ord}_m(q)$. If a is odd, then $\mathrm{ord}_a(q)$ is denoted by $e(a,q)$, too.
When q is odd, let $e(2,q)=1$ if $q\equiv 1 \pmod 4$ and $e(2,q)=2$ if $q\equiv -1 \pmod 4$.

Lemma 2.2. ([5, Remark 1]) *The equation $p^m-q^n=1$, where $m,n>1$ has only one solution, namely $3^2-2^3=1$.*

Lemma 2.3. ([5, 7]) *Except the relations $(239)^2-2(13)^4=-1$ and $(3)^5-2(11)^2=1$ every solution of the equation*

$$p^m-2q^n=\pm 1; \quad p,q \quad \textit{prime}; \quad m,n>1,$$

has exponents $m=n=2$; i.e. it comes from a unit $p-q2^{1/2}$ of the quadratic field $\mathbb{Q}(2^{1/2})$ for which the coefficients p,q are primes.

Lemma 2.4. (Zsigmondy's Theorem) ([15]) *Let p be a prime and let n be a positive integer. Then one of the following holds:*
(i) there is a primitive prime p' for p^n-1, that is , $p'\mid(p^n-1)$ but $p'\nmid(p^m-1)$, for every $1\leq m<n$, (usually p' is denoted by r_n)
(ii) $p=2$, $n=1$ or 6,
(iii) p is a Mersenne prime and $n=2$.

We denote by $D_n^+(q)$ the simple group $D_n(q)$, and by $D_n^-(q)$ the simple group ${}^2D_n(q)$.

Lemma 2.5. ([12, Proposition 2.5]) *Let $G=D_n^\varepsilon(q)$ be a finite simple group of Lie type over a field of characteristic p. Define*

$$\eta(m)=\begin{cases} m & \textit{if } m \textit{ is odd,}\\ m/2 & \textit{otherwise.}\end{cases}$$

Let r and s be odd primes and $r,s\in\pi(G)\setminus\{p\}$. Put $k=e(r,q)$ and $l=e(s,q)$, and $1\leq\eta(k)\leq\eta(l)$. Then r and s are non-adjacent if and only if $2\eta(k)+2\eta(l)>2n-(1-\varepsilon(-1)^{k+l})$, and k,l satisfy:

$$l/k \quad \textit{is not an odd natural number,}$$

and if $\varepsilon=+$, then the chain of equalities:

$$n=l=2\eta(l)=2\eta(k)=2k$$

is not true.

Lemma 2.6. ([12, Proposition 2.4]) *Let G be a simple group of Lie type, $B_n(q)$ or $C_n(q)$ over a field of characteristic p. Define*

$$\eta(m) = \begin{cases} m & \text{if } m \text{ is odd,} \\ m/2 & \text{otherwise.} \end{cases}$$

Let r, s be odd primes with $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r, q)$ and $l = e(s, q)$, and suppose that $1 \le \eta(k) \le \eta(l)$. Then r and s are non-adjacent if and only if $\eta(k) + \eta(l) > n$, and k, l satisfy:

$$l/k \quad \text{is not an odd natural number.}$$

Lemma 2.7. ([11, Proposition 2.1]) *Let $G = A_{n-1}(q)$ be a finite simple group of Lie type over a field of characteristic p. Let r and s be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r, q)$ and $l = e(s, q)$, and suppose that $2 \le k \le l$. Then r and s are non-adjacent if and only if $k + l > n$, and k does not divide l.*

Lemma 2.8. ([11, Proposition 2.2]) *Let $G = {}^2A_{n'-1}(q)$ be a finite simple group of Lie type over a field of characteristic p. Define*

$$\nu(m) = \begin{cases} m & \text{if } m \equiv 0 \pmod 4; \\ m/2 & \text{if } m \equiv 2 \pmod 4; \\ 2m & \text{if } m \equiv 1 \pmod 4. \end{cases}$$

Let r and s be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r, q)$ and $l = e(s, q)$, and suppose that $2 \le \nu(k) \le \nu(l)$. Then r and s are non-adjacent if and only if $\nu(k) + \nu(l) > n$, and $\nu(k)$ does not divide $\nu(l)$.

Lemma 2.9. *Let p be a prime number and there exist natural numbers α and β such that $q = p^\alpha$ and $q' = p^\beta$. If $r_n = u_m$, then $n\alpha = m\beta$.*

Proof. Let x be a primitive prime divisor of $p^{n\alpha} - 1$. Then x is a primitive prime divisor of $q^n - 1$. Since $r_n = u_m$, so $x \mid (p^{m\beta} - 1)$. Therefore, $n\alpha \le m\beta$. Now let y be a primitive prime divisor of $p^{m\beta} - 1$. So similarly to the above we get $y \mid (p^{n\alpha} - 1)$. Hence $m\beta \le n\alpha$. Consequently, $n\alpha = m\beta$. □

3. Prime graph of simple classical Lie type groups

In the rest of this section we denote by r_i a primitive prime divisor of $q^i - 1$.

Remark 3.1. Let $G = D_n(q)$, where $q = p^\alpha$ and $n > 4$. By [11, Tables 4, 6] and Lemma 2.5, we know that:

Condition	$\rho(p, G)$	$\rho(r_1, G)$	$\rho(r_2, G)$
n is odd	$\{p, r_n, r_{2(n-1)}\}$	$\{r_1, r_{2(n-1)}\}$	$\{r_2, r_n\}$
n is even	$\{p, r_{n-1}, r_{2(n-1)}\}$	$\{r_1, r_{2(n-1)}\}$	$\{r_2, r_{n-1}\}$

- Let n be odd.
  Let $r_k \nsim r_i$. We consider the following two cases:
  (1) *Let $k > 2$ be a fixed odd number.* Hence $2k + 2\eta(i) > 2n -$

$(1-(-1)^{i+k})$. Suppose $A = \{2(n-1), 2(n-2), \ldots, 2(n-k)\}$ and $B = \{n, n-2, \ldots, n-k+1\}$. Therefore, $i \in A \cup B$. Since $k \geq 3$, so $t(r_k, G) \geq 4$.
(2) *Let $k > 2$ be a fixed even number.* Hence $k + 2\eta(i) > 2n - (1-(-1)^{i+k})$. Define $A = \{2(n-1), 2(n-2), \ldots, 2(n-k/2+1)\}$ and $B = \{n, n-2, \ldots, n-k/2+a\}$, where if $k \equiv 0 \pmod 4$, then $a = 0$ and otherwise, $a = 1$. Therefore, $i \in A \cup B$. For $k = 4$, if $n \equiv 1 \pmod 4$, then $\rho(r_4, G) = \{r_4, r_{n-2}, r_n, r_{2(n-1)}\}$ and otherwise, $\rho(r_4, G) = \{r_4, r_{n-2}, r_n\}$. If $k \geq 6$, then $t(r_k, G) \geq 4$.

- ∘ Let n be even.
Let $r_k \nsim r_i$. We consider the following two cases:
(1) *Let $k > 2$ be a fixed odd number.* Hence $2k + 2\eta(i) > 2n - (1-(-1)^{i+k})$. Suppose $A = \{2(n-1), 2(n-2), \ldots, 2(n-k)\}$ and $B = \{n-1, n-3, \ldots, n-k+2\}$. Therefore, $i \in A \cup B$. Since $k \geq 3$, so $t(r_k, G) \geq 4$.
(2) *Let $k > 2$ be a fixed even number.* Hence $k + 2\eta(i) > 2n - (1-(-1)^{i+k})$. Define $A = \{2(n-1), 2(n-2), \ldots, 2(n-k/2+1)\}$ and $B = \{n-1, n-3, \ldots, n-k/2+a\}$, where if $k \equiv 0 \pmod 4$, then $a = 1$ and otherwise, $a = 0$. Therefore, $i \in A \cup B$. Similar to the above, if $k = 4$, then $\rho(r_4, G) = \{r_4, r_{n-1}, r_{2(n-1)}\}$ and otherwise, $t(r_k, G) \geq 4$.

Also by [11, Table 6], we know that:

Table 1. 2-independence numbers for group $D_n(q)$

G	conditions	$t(2,G)$	$\rho(2,G)$
$D_n(q)$	$n \equiv 0 \pmod 2$, $n \geq 4$, $q \equiv 3 \pmod 4$	2	$\{2, r_{n-1}\}$
	$n \equiv 0 \pmod 2$, $n \geq 4$, $q \equiv 1 \pmod 4$	2	$\{2, r_{2(n-1)}\}$
	$n \equiv 1 \pmod 2$, $n > 4$, $q \equiv 3 \pmod 4$	2	$\{2, r_n\}$
	$n \equiv 1 \pmod 2$, $n > 4$, $q \equiv 1 \pmod 8$	2	$\{2, r_{2(n-1)}\}$
	$n \equiv 1 \pmod 2$, $n > 4$, $q \equiv 5 \pmod 8$	3	$\{2, r_n, r_{2(n-1)}\}$

Remark 3.2. Let $G = {}^2D_n(q)$, where $q = p^\alpha$ and $n \geq 4$. By [11, Tables 4, 6] and Lemma 2.5, we know that:

Condition	$\rho(p,G)$	$\rho(r_1,G)$	$\rho(r_2,G)$
n is odd	$\{p, r_{2(n-1)}, r_{2n}\}$	$\{r_1, r_{2n}\}$	$\{r_2, r_{2(n-1)}\}$
n is even	$\{p, r_{n-1}, r_{2(n-1)}, r_{2n}\}$	$\{r_1, r_{2n}\}$	$\{r_2, r_{2n}\}$

- Let n be odd.
Let $r_k \nsim r_i$. We consider the following two cases:
(1) *Let $k > 2$ be a fixed odd number.* Hence $2k + 2\eta(i) > 2n - (1+(-1)^{i+k})$. We define $A = \{2n, 2(n-1), \ldots, 2(n-k+1)\}$ and $B = \{n-2, n-4, \ldots, n-k+1\}$. Therefore, $i \in A \cup B$. Since $k \geq 3$, so r_k is not adjacent to r_{2n}, $r_{2(n-1)}$ and $r_{2(n-2)}$. Moreover, $\{r_{2(n-2)}, r_{2(n-1)}, r_{2n}\}$

is an independent set. So $\{r_k, r_{2(n-2)}, r_{2(n-1)}, r_{2n}\} \subseteq \rho(r_k, G)$. Therefore, $t(r_k, G) \geq 4$.
(2) *Let $k > 2$ be a fixed even number.* Hence $k + 2\eta(i) > 2n - (1 + (-1)^{i+k})$. Define $A = \{2n, 2(n-1), \ldots, 2(n-k/2)\}$ and $B = \{n-2, n-4, \ldots, n-k/2+a\}$, where if $k \equiv 0 \pmod 4$, then $a = 2$ and otherwise, $a = 1$. Therefore, $i \in A \cup B$. For $k = 4$, if $n \equiv 1 \pmod 4$, then $\rho(r_4, G) = \{r_4, r_{2(n-2)}, r_{2(n-1)}, r_{2n}\}$ and otherwise, $\rho(r_4, G) = \{r_4, r_{2(n-2)}, r_{2n}\}$. If $k \geq 6$, then $t(r_k, G) \geq 5$.

◦ Let n be even.
Let $r_k \nsim r_i$. We consider the following two cases:
(1) *Let $k > 2$ be a fixed odd number.* Hence $2k + 2\eta(i) > 2n - (1 + (-1)^{i+k})$. Suppose $A = \{2n, 2(n-1), \ldots, 2(n-k+1)\}$ and $B = \{n-1, n-3, \ldots, n-k\}$. Therefore, $i \in A \cup B$. Since $k \geq 3$, so $t(r_k, G) \geq 5$.
(2) *Let $k > 2$ be a fixed even number.* Hence $k + 2\eta(i) > 2n - (1 + (-1)^{i+k})$. Define $A = \{2n, 2(n-1), \ldots, 2(n-k/2)\}$ and $B = \{n-1, n-3, \ldots, n-k/2+a\}$, where if $k \equiv 0 \pmod 4$, then $a = 1$ and otherwise, $a = 2$. Therefore, $i \in A \cup B$. Let $k = 4$, if $n \equiv 0 \pmod 4$, then $\rho(r_4, G) = \{r_4, r_{n-1}, r_{2(n-1)}, r_{2n}\}$ and otherwise, $\rho(r_4, G) = \{r_4, r_{n-1}, r_{2(n-2)}, r_{2(n-1)}\}$. Also if $k \geq 6$, then $t(r_k, G) \geq 5$.

Remark 3.3. Let $G = C_n(q)$ or $G = B_n(q)$, where $q = p^\alpha$ and $n > 3$. By [11, Tables 4, 6] and Lemma 2.5, we know that:

Condition	$\rho(p, G)$	$\rho(r_1, G)$	$\rho(r_2, G)$
n is odd	$\{p, r_n, r_{2n}\}$	$\{r_1, r_{2n}\}$	$\{r_2, r_n\}$
n is even	$\{p, r_{2n}\}$	$\{r_1, r_{2n}\}$	$\{r_2, r_{2n}\}$

• Let n be odd.
Let $r_k \nsim r_i$. We consider the following two cases:
(1) *Let $k > 2$ be a fixed odd number.* Hence $k + \eta(i) > n$. Suppose $A = \{2n, 2(n-1), \ldots, 2(n-k+1)\}$ and $B = \{n, n-2, \ldots, n-k+1\}$. Therefore, $i \in A \cup B$. Since $k \geq 3$, so $t(r_k, G) \geq 4$.
(2) *Let $k > 2$ be a fixed even number.* Hence $k/2 + \eta(i) > n$. Define $A = \{2n, 2(n-1), 2(n-2), \ldots, 2(n-k/2+1)\}$ and $B = \{n, n-2, \ldots, n-k/2+a\}$, where if $k \equiv 0 \pmod 4$, then $a = 2$ and otherwise, $a = 1$. Therefore, $i \in A \cup B$. If $k \neq 4$, then $t(r_k, G) \geq 4$. Let $k = 4$, if $n \equiv 1 \pmod 4$, then $t(r_4, G) = 4$ otherwise, $\rho(r_4, G) = \{r_4, r_n, r_{2n}\}$.

◦ Let n be even.
Let $r_k \nsim r_i$. We consider the following two cases:
(1) *Let $k > 2$ be a fixed odd number.* Hence $k + \eta(i) > n$. Suppose $A = \{2n, 2(n-1), \ldots, 2(n-k+1)\}$ and $B = \{n-1, n-3, \ldots, n-k+2\}$. Therefore, $i \in A \cup B$. Since $k \geq 3$, so $t(r_k, G) \geq 4$.
(2) *Let $k > 2$ be a fixed even number.* Hence $k/2 + \eta(i) > n$. Define

$A = \{2n, 2(n-1), \ldots, 2(n-k/2+1)\}$ and $B = \{n-1, n-3, \ldots, n-k/2+a\}$, where if $k \equiv 0 \pmod 4$, then $a = 1$ and otherwise, $a = 2$. Therefore, $i \in A \cup B$. Let $k = 4$, if $n \equiv 0 \pmod 4$, then $\rho(r_4, G) = \{r_4, r_{n-1}, r_{2(n-1)}, r_{2n}\}$ otherwise, $\rho(r_4, G) = \{r_4, r_{n-1}, r_{2(n-1)}\}$. Let $k \geq 6$, so $t(r_k, G) \geq 4$.

Remark 3.4. Let $G = A_{n-1}(q)$.

By [11, Proposition 3.1], we have $t(p, G) = 3$ and also by [11, Proposition 4.1], we know that $2 \leq t(r_1, G) \leq 3$. Let $r_k \nsim r_i$, where $k \neq 1$ is a fixed number, hence $i \in \{n, n-1, \ldots, n-k+1\}$. Therefore, by Lemma 2.7, we have $t(r_2, G) = 2$ and $t(r_3, G) = 3$. Let $k \geq 4$, so $t(r_k, G) \geq 4$.

Remark 3.5. Let $G = {}^2A_{n-1}(q)$.

By [11, Proposition 3.1], we have $t(p, G) = 3$ and also by [11, Proposition 4.2], we know that $2 \leq t(r_2, G) \leq 3$. Let $r_k \nsim r_i$, where $k \neq 2$ is a fixed number, hence $i \in \{n, n-1, \ldots, n-k+1\}$. Therefore, by Lemma 2.8, we have $t(r_1, G) = 2$ and $t(r_6, G) = 3$. Let $\nu(k) \geq 4$, so $t(r_k, G) \geq 4$.

4. Proof of the main theorem

We know that $t(p, G) = 3$ and $t(r_1, G) = t(r_2, G) = 2$ and for every $r_i \in \pi(G)$, where $i \notin \{1, 2\}$, we have $t(r_i, G) > 2$, by Remark 3.1. Now we consider each possibility for S in the following lemmas:

Lemma 4.1. *If $S = D_{n'}(q')$, where $q' = p'^{\beta}$ and $n' \geq 4$ such that $\Gamma(S) = \Gamma(G)$, then either $S = G$ or $S = D_n(q')$, where n is even, $p \neq p'$, $p \equiv 1 \pmod 4$ and $p' \equiv 1 \pmod 4$. Also we have $R_1(q) = U_1(q')$, $R_2(q) = U_2(q')$, $R_{2(n-1)}(q) = U_{2(n-1)}(q')$, $R_{n-1}(q) = U_{n-1}(q')$ and $\{p\} \cup R_4(q) = \{p'\} \cup U_4(q')$.*

Proof. We get that $t(S) = t(G)$, where $t(S)$ is equal to $[(3n'+1)/4]$ or $(3n'+3)/4$ and $t(G)$ is equal to $[(3n+1)/4]$ or $(3n+3)/4$. Also $t(2, S) = t(2, G)$ and for every $r \in \pi(G)$, we have $t(r, G) = t(r, S)$. So $[(3n'+1)/4] = [(3n+1)/4]$, $(3n'+3)/4 = [(3n+1)/4]$, $[(3n'+1)/4] = (3n+3)/4$ or $(3n'+3)/4 = (3n+3)/4$. Therefore, $n' > 4$ and also $n = n'$, $n+1 = n'$ or $n'+1 = n$. We know that $t(p, S) = 3$, $t(u_1, S) = t(u_2, S) = 2$ and for every $u_i \in \pi(S)$, where $i \notin \{1, 2\}$, we have $t(u_i, S) > 2$, by Remark 3.1. Therefore, $R_1(q) \cup R_2(q) = U_1(q') \cup U_2(q')$ or in other words $\pi(q^2 - 1) = \pi(q'^2 - 1)$.

Case (1). Let n be odd.

(1-1) Let $n = n' + 1$. We know that $p' = 2$ if and only if $p = 2$, since $\pi(q^2 - 1) = \pi(q'^2 - 1)$.

Let $p' = 2$. In this case, we know that $\rho(2, S) = \{2, u_{n'-1}, u_{2(n'-1)}\}$ and $\rho(2, G) = \{2, r_n, r_{2(n-1)}\}$. Therefore, $R_n(q) \cup R_{2(n-1)}(q) = U_{n'-1}(q') \cup U_{2(n'-1)}(q')$. Consequently, r_n and $r_{2(n-1)}$ are some primitive prime divisors of $q'^{n'-1} - 1$ and $q'^{2(n'-1)} - 1$, say $u_{n'-1}$, $u_{2(n'-1)}$. In the sequel of this paper for simplicity we write $\{r_n, r_{2(n-1)}\} \leftrightarrow \{u_{n'-1}, u_{2(n'-1)}\}$ to illustrate these relations. By Remark 3.1, we know that $p = 2$ is the only vertex in $\Gamma(G)$, which is adjacent to all vertices except $r_{2(n-1)}$ and r_n. On the other hand, $p' = 2$ and u_4 are the only vertices in $\Gamma(S)$, which are adjacent to all vertices except $u_{2(n'-1)}$ and $u_{n'-1}$. Consequently, $\{2\} = \{2\} \cup U_4(q')$, which is a contradiction.

Hence, $p' \neq 2$ and so $t(2, S) = 2$, since n' is even. Also we know that $\rho(2, S) = \{2, u_{n'-1}\}$ or $\rho(2, S) = \{2, u_{2(n'-1)}\}$. Therefore, $t(2, G) = 2$ and since n is odd $\rho(2, G) = \{2, r_n\}$ or $\rho(2, G) = \{2, r_{2(n-1)}\}$. Now we consider the following two cases:

• Let $\rho(2, S) = \{2, u_{n'-1}\}$. We consider the following two cases:

(I) Let $\rho(2, G) = \{2, r_n\}$. Therefore, $R_n(q) = U_{n'-1}(q')$. We know that $r_1 \sim r_n \nsim r_2$ in $\Gamma(G)$, and $u_1 \sim u_{n'-1} \nsim u_2$ in $\Gamma(S)$. Consequently, $R_2(q) = U_2(q')$ and $R_1(q) = U_1(q')$. Moreover, we know that u_1 is adjacent to all vertices except $u_{2(n'-1)}$ and also r_1 is adjacent to all vertices except $r_{2(n-1)}$, which implies that $R_{2(n-1)}(q) = U_{2(n'-1)}(q')$. Consequently, $R_n(q) \cup R_{2(n-1)}(q) = U_{n'-1}(q') \cup U_{2(n'-1)}(q')$. Therefore, $\{r_n, r_{2(n-1)}\} \leftrightarrow \{u_{n'-1}, u_{2(n'-1)}\}$. By Remark 3.1, we know that p is the only vertex in $\Gamma(G)$, which is adjacent to all vertices except $r_{2(n-1)}$ and r_n, and similarly p' and u_4 are the only vertices in $\Gamma(S)$, which are adjacent to all vertices except $u_{2(n'-1)}$ and $u_{n'-1}$. Consequently, $\{p\} = \{p'\} \cup U_4(q')$, which is a contradiction.

(II) Let $\rho(2, G) = \{2, r_{2(n-1)}\}$. Therefore, $R_{2(n-1)}(q) = U_{n'-1}(q')$. We know that $r_1 \nsim r_{2(n-1)} \sim r_2$, and $u_1 \sim u_{n'-1} \nsim u_2$. Consequently, $R_1(q) = U_2(q')$ and $R_2(q) = U_1(q')$. Moreover, we know that u_1 is adjacent to all vertices except $u_{2(n'-1)}$ in $\Gamma(S)$ and also r_2 is adjacent to all vertices except r_n in $\Gamma(G)$, which implies that $R_n(q) = U_{2(n'-1)}(q')$. Similarly, $\{r_n, r_{2(n-1)}\} \leftrightarrow \{u_{n'-1}, u_{2(n'-1)}\}$. By Remark 3.1, we know that p is the only vertex in $\Gamma(G)$, which is adjacent to all vertices except $r_{2(n-1)}$ and r_n, and similarly p' and u_4 are the only vertices in $\Gamma(S)$, which are adjacent to all vertices except $u_{2(n'-1)}$ and $u_{n'-1}$. Consequently, $\{p\} = \{p'\} \cup U_4(q')$, which is a contradiction.

• Let $\rho(2, S) = \{2, u_{2(n'-1)}\}$. Similar to the above we get a contradiction.

(1-2) Let $n' = n + 1$. We know that $p' = 2$ if and only if $p = 2$. Let $p' = 2$. By 2-independent sets of S and G we know that $\{r_n, r_{2(n-1)}\} \leftrightarrow \{u_{n'-1}, u_{2(n'-1)}\}$. Completely similar to the above we get $\{2\} = \{2\} \cup U_4(q)$, which is a contradiction. So $p' \neq 2$ and $p \neq 2$, hence $t(2, S) = t(2, G) = 2$. We consider the following two cases:

• Let $\rho(2, S) = \{2, u_{n'-1}\}$. Also let $\rho(2, G) = \{2, r_n\}$. It follows that $R_n(q) = U_{n'-1}(q')$. We know that $r_1 \sim r_n \nsim r_2$, and $u_1 \sim u_{n'-1} \nsim u_2$. Consequently, $R_1(q) = U_1(q')$. Similar to the above we get $R_{2(n-1)}(q) = U_{2(n'-1)}(q')$ and so $\{r_n, r_{2(n-1)}\} \leftrightarrow \{u_{n'-1}, u_{2(n'-1)}\}$. Similarly, $\{p\} = \{p'\} \cup U_4(q)$, which is a contradiction. If $\rho(2, G) = \{2, r_{2(n-1)}\}$, then we get a contradiction similarly.

• Let $\rho(2, S) = \{2, u_{2(n'-1)}\}$. Similar to the above we get a contradiction.

(1-3) Let $n = n'$. We know that $p' = 2$ if and only if $p = 2$. Let $p = 2$. Since $\pi(S) = \pi(G)$, so $\alpha = \beta$, by Lemma 2.4. Therefore, $S = G$. Let $p \neq 2$.

▶ If $t(2,G) = t(2,S) = 3$, then $\rho(2,S) = \{2, u_n, u_{2(n-1)}\}$ and $\rho(2,G) = \{2, r_n, r_{2(n-1)}\}$. Therefore, $R_n(q) \cup R_{2(n-1)}(q) = U_n(q') \cup U_{2(n-1)}(q')$. Consequently, $\{r_n, r_{2(n-1)}\} \leftrightarrow \{u_n, u_{2(n-1)}\}$. By Remark 3.1, we know that p and 2 are the only vertices in $\Gamma(G)$, which are adjacent to all vertices except $r_{2(n-1)}$ and r_n, and similarly p' and 2 are the only vertices in $\Gamma(S)$, which are adjacent to all vertices except $u_{2(n-1)}$ and u_n. Consequently, $\{p, 2\} = \{p', 2\}$ and so $p = p'$. Similarly to the above we have $S = G$.

▶ So let $t(2,G) = t(2,S) = 2$. We consider the following two cases:

• Let $\rho(2,G) = \{2, r_n\}$, hence $q \equiv 3 \pmod 4$. We consider the following two cases:

(I) Let $\rho(2,S) = \{2, u_n\}$ so $R_n(q) = U_n(q')$. We know that $r_1 \sim r_n \nsim r_2$, and $u_1 \sim u_n \nsim u_2$. Consequently, $R_1(q) = U_1(q')$ and $R_2(q) = U_2(q')$. Moreover, we know that u_1 is adjacent to all vertices except $u_{2(n-1)}$ and also r_1 is adjacent to all vertices except $r_{2(n-1)}$, which implies that $R_{2(n-1)}(q) = U_{2(n-1)}(q')$. Consequently, $R_n(q) \cup R_{2(n-1)}(q) = U_n(q') \cup U_{2(n-1)}(q')$. Therefore, $\{r_n, r_{2(n-1)}\} \leftrightarrow \{u_n, u_{2(n-1)}\}$. By Remark 3.1, we know that p is the only vertex in $\Gamma(G)$, which is adjacent to all vertices except $r_{2(n-1)}$ and r_n, and similarly p' is the only vertex in $\Gamma(S)$, which is adjacent to all vertices except $u_{2(n-1)}$ and u_n. Consequently, $p = p'$. Since $\pi(S) = \pi(G)$, so $\alpha = \beta$, by Lemma 2.4, it follows that $S = G$.

(II) Let $\rho(2,S) = \{2, u_{2(n-1)}\}$. Hence $q' \equiv 1 \pmod 8$, so $R_n(q) = U_{2(n-1)}(q')$. Similarly, $\{r_{2(n-1)}, r_n\} \leftrightarrow \{u_{2(n-1)}, u_n\}$, and by Remark 3.1, $p = p'$ and so $\alpha = \beta$, which is a contradiction, since $q \equiv 3 \pmod 4$.

• Let $\rho(2,G) = \{2, r_{2(n-1)}\}$. Similarly we get $S = G$.

Case (2). Let n be even.

According to the above proof, it is enough to consider $n = n'$. We know that $p' = 2$ if and only if $p = 2$. Let $p = 2$. Since $\pi(S) = \pi(G)$, so $\alpha = \beta$. Therefore, $S = G$. Let $p \neq 2$ so $t(2,G) = 2$. Hence $t(2,S) = 2$. Now we consider the following two cases:

• Let $\rho(2,G) = \{2, r_{n-1}\}$, hence $q \equiv 3 \pmod 4$. Also let $\rho(2,S) = \{2, u_{n-1}\}$, hence $q' \equiv 3 \pmod 4$, and we have $R_{n-1}(q) = U_{n-1}(q')$. We know that $r_1 \sim r_{n-1} \nsim r_2$, and $u_1 \sim u_{n-1} \nsim u_2$. Consequently, $R_1(q) = U_1(q')$. Similarly to the above we get $R_{2(n-1)}(q) = U_{2(n-1)}(q')$ and so $\{r_{n-1}, r_{2(n-1)}\} \leftrightarrow \{u_{n-1}, u_{2(n-1)}\}$. Similarly to the above, $\{p\} \cup R_4(q) = \{p'\} \cup U_4(q')$. If $p = p'$, then similarly we get $S = G$. Otherwise, $p \in U_4(q')$ and $p' \in R_4(q)$. Therefore, $4 \mid (p-1)$ and $4 \mid (p'-1)$, which is a contradiction, since $q \equiv 3 \pmod 4$ and $q' \equiv 3 \pmod 4$.

If $\rho(2,S) = \{2, u_{2(n-1)}\}$, then, we get a contradiction.

• Let $\rho(2,G) = \{2, r_{2(n-1)}\}$, hence $q \equiv 1 \pmod 4$. Also let $\rho(2,S) = \{2, u_{n-1}\}$, hence $q' \equiv 3 \pmod 4$. Therefore, we get a contradiction. If $\rho(2,S) = \{2, u_{2(n-1)}\}$, then $q' \equiv 1 \pmod 4$. Also we have $R_{2(n-1)}(q) = U_{2(n-1)}(q')$. We know that $r_1 \nsim r_{2(n-1)} \sim r_2$, and $u_1 \nsim u_{2(n-1)} \sim u_2$. Consequently,

$R_1(q) = U_1(q')$ and $R_2(q) = U_2(q')$. Similarly we get $R_{n-1}(q) = U_{n-1}(q')$. Therefore, $\{r_{n-1}, r_{2(n-1)}\} \leftrightarrow \{u_{n-1}, u_{2(n-1)}\}$ and so $\{p\} \cup R_4(q) = \{p'\} \cup U_4(q')$. If $p = p'$, then we get $S = G$. Otherwise, $p \in U_4(q')$ and $p' \in R_4(q)$, it follows that $p \equiv 1 \pmod 4$ and $p' \equiv 1 \pmod 4$. Therefore, we have $S = D_n(r_4^{\beta})$. □

Remark 4.2. Let $S = D_{n'}(q')$ such that $\Gamma(S) = \Gamma(G)$ and $S \neq G$. By Lemma 4.1, we get that $n' = n$ and n is even, $p' \neq p$ and $p \equiv 1 \pmod 4$ and $p' \equiv 1 \pmod 4$. Also we have $R_1(q) = U_1(q')$, $R_2(q) = U_2(q')$, $\{p\} \cup R_4(q) = \{p'\} \cup U_4(q')$, $R_{2(n-1)}(q) = U_{2(n-1)}(q')$ and $R_{n-1}(q) = U_{n-1}(q')$. Moreover, we know that for every $r \in \pi(G)$, we have $t(r, G) = t(r, S)$.

• If $n \equiv 1 \pmod 3$, then r_3 and r_6 are the only vertices in $\Gamma(G)$ such that their independence number is 4, and also u_3 and u_6 are the only vertices in $\Gamma(S)$ such that their independence numbers are 4, by Remark 3.1. It implies that $R_3(q) \cup R_6(q) = U_3(q') \cup U_6(q')$. In addition, $\rho(r_3, G) = \{r_3, r_{2(n-1)}, r_{2(n-2)}, r_{2(n-3)}\}$, $\rho(u_3, G) = \{u_3, u_{2(n-1)}, u_{2(n-2)}, u_{2(n-3)}\}$, $\rho(r_6, G) = \{r_6, r_{2(n-2)}, r_{n-1}, r_{n-3}\}$ and $\rho(u_6, G) = \{u_6, u_{2(n-2)}, u_{n-1}, u_{n-3}\}$. We know that $R_{2(n-1)}(q) = U_{2(n-1)}(q')$. Consequently, $R_{2(n-2)}(q) \cup R_{2(n-3)}(q) \cup R_{n-3}(q) = U_{2(n-2)}(q') \cup U_{2(n-3)}(q') \cup U_{n-3}(q')$.

• Otherwise, $t(r_3, G) = t(r_6, G) = t(u_3, S) = t(u_6, S) = 5$. If $(n-2)/4$ is odd number, then r_3 and r_6 are the only vertices in $\Gamma(G)$ such that their independence numbers are 5, and also u_3 and u_6 are the only vertices in $\Gamma(S)$ such that their independence numbers are 5. It implies that $R_3(q) \cup R_6(q) = U_3(q') \cup U_6(q')$. Otherwise, r_3, r_6 and r_8 are the only vertices in $\Gamma(G)$ such that their independence numbers are 5, and also u_3, u_6 and u_8 are the only vertices in $\Gamma(S)$ such that their independence numbers are 5. Therefore, $R_3(q) \cup R_6(q) \cup R_8(q) = U_3(q') \cup U_6(q') \cup U_8(q')$. Also for other vertices in $\Gamma(G)$ and $\Gamma(S)$ we can find some relation similar to the above.

Furthermore, we know that $\{p\} \cup R_4(q) = \{p'\} \cup U_4(q')$. We claim that $R_4(q)$ and also $U_4(q')$ have more than one element. Otherwise, there exist natural numbers m and m' such that $q^2 + 1 = 2p'^{m'}$ and $q'^2 + 1 = 2p^m$. By Lemma 2.3, we consider the following cases:

(1) Let $m' = 1$. Therefore, $(q^2 - 1)/2 = p' - 1$, hence $\pi(q^2 - 1) \subseteq \pi(q' - 1)$. Consequently, there exists a natural number s such that $q' + 1 = 2^s$, which is a contradiction, since $p' \equiv 1 \pmod 4$.

(2) Let $\alpha = 1$ and $m' = 2$, hence $p^2 + 1 = 2p'^2$. On the other hand, since $q'^2 + 1 = 2p^m$, so by Lemma 2.3 we can consider the three following cases. If $m = 1$, then similarly to the above we get a contradiction. If $m = 2$ and $\beta = 1$, then $p'^2 + 1 = 2p^2$, which is a contradiction. So $p' = 239$, $\beta = 1$ and $p = 13$. Therefore, $13^2 + 1 = 2(239)^2$, which is a contradiction.

(3) Let $p = 239$, $\alpha = 1$ and $p' = 13$. Since $q'^2 + 1 = 2p^m$, so by Lemma 2.3 we consider the three following cases. If $m = 1$, then $13^{2\beta} + 1 = 2(239)$, which is a contradiction. If $m = 2$ and $\beta = 1$, then $13^2 + 1 = 2(239^2)$, which is a

contradiction. So $p' = 239$, $\beta = 1$ and $p = 13$, which is a contradiction. Consequently, $R_4(q)$ and $U_4(q')$ have more than one element.

In this situation, we conjecture that there is no q and q' satisfying all above conditions.

Lemma 4.3. *If $S = {}^2D_{n'}(q')$, where $q' = p'^{\beta}$ and $n' \geq 4$, then $\Gamma(S) \neq \Gamma(G)$.*

Proof. On the contrary let $\Gamma(S) = \Gamma(G)$. Therefore, $t(2, S) = t(2, G)$ and for every $r \in \pi(G)$, we have $t(r, G) = t(r, S)$. Also we know that $t(p, S) \geq 3$ and $t(u_1, S) = t(u_2, S) = 2$ and for every $u_i \in \pi(S)$, where $i > 2$, we have $t(u_i, S) > 2$, by Remark 3.2. Therefore, $R_1(q) \cup R_2(q) = U_1(q') \cup U_2(q')$. Also we have $t(S) = t(G)$, where $t(S) = [(3n'+4)/4]$ and $t(G)$ is equal to $[(3n+1)/4]$ or $(3n+3)/4$, so $[(3n'+4)/4] = [(3n+1)/4]$ or $[(3n'+4)/4] = (3n+3)/4$. Therefore, $n = n'$, $n'+2 = n$ or $n'+1 = n$.

Case (1). Let n be odd.

(1-1) Let $n = n'$. We know that $t(2, S) = 2$ or 3. We consider the following two cases:

► Let $t(2, S) = 3$, hence $\rho(2, S) = \{2, u_{2(n-1)}, u_{2n}\}$, by [11, Tables 4, 6]. So $t(2, G) = 3$ and hence $\rho(2, G) = \{2, r_n, r_{2(n-1)}\}$, by [11, Tables 4, 6]. Therefore, $R_n(q) \cup R_{2(n-1)}(q) = U_{2(n-1)}(q') \cup U_{2n}(q')$. Consequently, $\{r_n, r_{2(n-1)}\} \leftrightarrow \{u_{2(n-1)}, u_{2n}\}$. By Remarks 3.1 and 3.2, p and 2 are the only vertices in $\Gamma(G)$, which are adjacent to all vertices except $r_{2(n-1)}$ and r_n and p' and 2 are the only vertices in $\Gamma(S)$, which are adjacent to all vertices except $u_{2(n-1)}$ and u_{2n}. It follows that $\{p, 2\} = \{p', 2\}$ and so $p = p'$. Since $\pi(S) = \pi(G)$, so $2(n-1)\alpha = 2n\beta$, by Lemma 2.4, so $(\alpha)_2 < (\beta)_2$. Suppose $r_n = u_{2n}$, by Lemma 2.9, we get that $n\alpha = 2n\beta$, which is a contradiction. Therefore, $r_n = u_{2(n-1)}$, so by Lemma 2.9, $n\alpha = 2(n-1)\beta$, which is a contradiction, since n is odd and $(\alpha)_2 < (\beta)_2$.

► Let $t(2, S) = 2$ and so $t(2, G) = 2$. Let $\rho(2, G) = \{2, r_n\}$. Now we consider the following two cases:

- ∗ Let $\rho(2, S) = \{2, u_{2n}\}$, so $R_n(q) = U_{2n}(q')$. We know that $r_1 \sim r_n \nsim r_2$ and $u_1 \nsim u_{2n} \sim u_2$. Therefore, $R_1(q) = U_2(q')$ and so $R_{2(n-1)}(q) = U_{2(n-1)}(q')$.
- ∗ Let $\rho(2, S) = \{2, u_{2(n-1)}\}$, so $R_n(q) = U_{2(n-1)}(q')$. We know that $r_1 \sim r_n \nsim r_2$ and $u_1 \sim u_{2(n-1)} \nsim u_2$. Therefore, $R_1(q) = U_1(q')$ and so $R_{2(n-1)}(q) = U_{2n}(q')$.

Consequently, $R_n(q) \cup R_{2(n-1)}(q) = U_{2(n-1)}(q') \cup U_{2n}(q')$ and similarly $\{r_n, r_{2(n-1)}\} \leftrightarrow \{u_{2(n-1)}, u_{2n}\}$. So by Remarks 3.1 and 3.2, $p = p'$. Similarly we have $2(n-1)\alpha = 2n\beta$ so $(\alpha)_2 < (\beta)_2$. On the other hand, either $r_n = u_{2(n-1)}$ or $r_n = u_{2n}$, and we get a contradiction.

Let $\rho(2, G) = \{2, r_{2(n-1)}\}$. Then we get a contradiction.

(1-2) Let $n = n'+1$. If $p' = 2$, then $t(2, S) = 4$, which is a contradiction, since $t(2, G) \leq 3$, by [11, Tables 4, 6]. Therefore, $p' \neq 2$ and so $p \neq 2$, since

$\pi(q^2-1)=\pi(q'^2-1)$. It follows that $t(2,S)=t(2,G)=2$ and $\rho(2,S)=\{2,u_{2n'}\}$, by [11, Table 6]. Now we consider the following two cases:

• Let $\rho(2,G)=\{2,r_n\}$. Therefore, $R_n(q)=U_{2n'}(q')$. We know that $r_1\sim r_n\nsim r_2$ and $u_1\nsim u_{2n'}\nsim u_2$, which is a contradiction.

• Let $\rho(2,G)=\{2,r_{2(n-1)}\}$. Therefore, $R_{2(n-1)}(q)=U_{2n'}(q')$. We know that $r_1\nsim r_{2(n-1)}\sim r_2$ and $u_1\nsim u_{2n'}\nsim u_2$, which is a contradiction.

(1-3) Let $n'=n+2$, so n' is odd. It is clear that, either $n\equiv 1 \pmod 4$ or $n'\equiv 1 \pmod 4$. Now we consider the following two cases:

• Let $n\equiv 1 \pmod 4$. If $t(2,S)=t(2,G)=3$, then p and 2 are the only vertices in $\Gamma(G)$, which their independence numbers are 3. Also independence number of p', 2 and u_4 are 3 in $\Gamma(S)$. Therefore, $\{p,2\}=\{p',2\}\cup U_4(q')$. We know that $p'=2$ if and only if $p=2$. If $p'=2$, then $\{2\}=\{2\}\cup U_4(q')$, which is a contradiction. Hence, $p'\neq 2$. Therefore, $p=p'$ and so $\{2\}=\{2\}\cup U_4(q')$, which is a contradiction.
So $t(2,S)=t(2,G)=2$ and similarly we get $\{p\}=\{p'\}\cup U_4(q')$, which is a contradiction.

• Let $n'\equiv 1 \pmod 4$. If $t(2,S)=t(2,G)=3$, then p, 2 and r_4 are the only vertices in $\Gamma(G)$, which their independence numbers are 3. Also independence number of p' and 2 are 3 in $\Gamma(S)$. Therefore, $\{p,2\}\cup R_4(q)=\{p',2\}$ and we get a contradiction. So $t(2,S)=t(2,G)=2$ and we get $\{p\}\cup R_4(q)=\{p'\}$, which is a contradiction.

Case (2). Let n be even.

(2-1) Let $n=n'$. We know that $p'=2$ if and only if $p=2$. Let $p'=p=2$. Therefore, $t(2,S)=4$ and $t(2,G)=3$, which is a contradiction. Consequently, $p\neq 2$ and $p'\neq 2$ and so $\rho(2,S)=\{2,u_{2n}\}$. Let $\rho(2,G)=\{2,r_{n-1}\}$, so $R_{n-1}(q)=U_{2n}(q')$. We know that $r_1\sim r_{n-1}\nsim r_2$ and $u_1\nsim u_{2n}\nsim u_2$, which is a contradiction. Let $\rho(2,G)=\{2,r_{2(n-1)}\}$, so $R_{2(n-1)}(q)=U_{2n}(q')$. We know that $r_1\nsim r_{n-1}\sim r_2$ and $u_1\nsim u_{2n}\nsim u_2$, which is a contradiction.

If $n=n'+2$, then similarly we get a contradiction.

(2-2) Let $n=n'+1$, so n' is odd. We know that $p'=2$ if and only if $p=2$. Let $p'=p=2$, so $\rho(2,S)=\{2,u_{2(n'-1)},u_{2n'}\}$ and $\rho(2,G)=\{2,r_{n-1},r_{2(n-1)}\}$, by [11, Table 4]. Therefore, $\{r_{n-1},r_{2(n-1)}\}\leftrightarrow\{u_{2(n'-1)},u_{2n'}\}$. By Remarks 3.1 and 3.2, r_4 and 2 are the only vertices in $\Gamma(G)$, which are adjacent to all vertices except $r_{2(n-1)}$ and r_{n-1} and 2 is the only vertex in $\Gamma(S)$, which is adjacent to all vertices except $u_{2(n'-1)}$ and $u_{2n'}$. It follows that $\{2\}\cup R_4(q)=\{2\}$, which is a contradiction. Therefore, $p\neq 2$ and $p'\neq 2$. Since n is even, hence $t(2,G)=2$ and so $t(2,S)=2$. Now we consider the following two cases:

• Let $\rho(2,G)=\{2,r_{n-1}\}$. Also let $\rho(2,S)=\{2,u_{2n'}\}$ it follows that $R_{n-1}(q)=U_{2n'}(q')$. We know that $r_1\sim r_{n-1}\nsim r_2$, and $u_1\nsim u_{2n'}\sim u_2$. Consequently, $R_1(q)=U_2(q')$. Similarly we get $R_{2(n-1)}(q)=U_{2(n'-1)}(q')$ and so we get that $\{r_{n-1},r_{2(n-1)}\}\leftrightarrow\{u_{2(n'-1)},u_{2n'}\}$. Similarly to the above,

$\{p\} \cup R_4(q) = \{p'\}$, which is a contradiction. If $\rho(2, S) = \{2, u_{2(n'-1)}\}$, then we get a contradiction similarly to the above.

- Let $\rho(2, G) = \{2, r_{2(n-1)}\}$. Similarly we get a contradiction. □

Lemma 4.4. *If $S = C_{n'}(q')$ or $S = B_{n'}(q')$, where $q' = p'^{\beta}$ such that $\Gamma(S) = \Gamma(G)$, then $S = C_{n-1}(q')$ or $S = B_{n-1}(q')$, where $n \equiv 0 \pmod 4$, $p \neq p'$, $p \equiv 1 \pmod 4$ and $p' \equiv 1 \pmod 4$. Also we have $R_1(q) = U_1(q')$, $R_2(q) = U_2(q')$, $R_{2(n-1)}(q) = U_{2n'}(q')$, $R_{n-1}(q) = U_{n'}(q')$ and $\{p\} \cup R_4(q) = \{p'\} \cup U_4(q')$.*

Proof. Since $n > 4$, we get that $n' > 3$. We have $t(S) = t(G)$ so $[(3n'+5)/4] = [(3n+1)/4]$ or $[(3n'+5)/4] = (3n+3)/4$. Therefore, $n = n'+2$, $n = n'+1$ or $n = n'$.

Case (1). Let n be odd.

(1-1) Let $n = n'$. We know that $t(p, S) = 3$ and $t(u_1, S) = t(u_2, S) = 2$ and for every $u_i \in \pi(S)$, where $i > 2$, we have $t(u_i, S) > 2$, by Remark 3.3. Therefore, $R_1(q) \cup R_2(q) = U_1(q') \cup U_2(q')$.

- If $p = 2$, then $p' = 2$, since $\pi(q^2 - 1) = \pi(q'^2 - 1)$. Since $\pi(G) = \pi(S)$, so $2(n-1)\alpha = 2n\beta$, by Lemma 2.4. On the other hand, by 2-independent sets of S and G, we have $R_n(q) \cup R_{2(n-1)}(q) = U_n(q') \cup U_{2n}(q')$ and $\{r_n, r_{2(n-1)}\} \leftrightarrow \{u_n, u_{2n}\}$. Therefore, either $r_n = u_{2n}$ or $r_n = u_n$. If $r_n = u_{2n}$, then $n\alpha = 2n\beta$, by Lemma 2.9, which is a contradiction. Otherwise, $r_n = u_n$ and by Lemma 2.9, $n\alpha = n\beta$, which is a contradiction.
- Therefore, $p \neq 2$ and $p' \neq 2$, and so $t(2, S) = t(2, G) = 2$. Let $\rho(2, S) = \{2, u_n\}$.
  - ∗ If $\rho(2, G) = \{2, r_n\}$, then $R_n(q) = U_n(q')$. We know that $r_1 \sim r_n \nsim r_2$ and $u_1 \sim u_n \nsim u_2$. Consequently, $R_1(q) = U_1(q')$ and so $R_{2(n-1)}(q) = U_{2n}(q')$. Similarly $\{r_n, r_{2(n-1)}\} \leftrightarrow \{u_n, u_{2n}\}$. By Remarks 3.1 and 3.3, if $n \equiv 1 \pmod 4$, then $p = p'$. Since $\pi(S) = \pi(G)$, so $2(n-1)\alpha = 2n\beta$, by Lemma 2.4. On the other hand, since $R_n(q) = U_n(q')$, by Lemma 2.9, $n\alpha = n\beta$, which is a contradiction. Otherwise, $\{p\} = \{p'\} \cup U_4(q')$, which is a contradiction.
  - ∗ If $\rho(2, G) = \{2, r_{2(n-1)}\}$, then $R_{2(n-1)}(q) = U_n(q')$. Similarly we get a contradiction.

If $\rho(2, S) = \{2, u_{2n}\}$, then we get a contradiction.

(1-2) If $n = n' + 2$, then similarly we get a contradiction.

(1-3) Let $n = n'+1$, so n' is even. Hence $\rho(2, S) = \{2, u_{2n'}\}$ and $t(p, S) = 2$, by [11, Tables 4, 6]. By Remarks 3.1 and 3.3, $R_1(q) \cup R_2(q) = U_1(q') \cup U_2(q') \cup \{p'\}$ and $u_{2n'}$ is not adjacent to u_1, u_2 and p'. If $\rho(2, G) = \{2, r_n\}$, then $R_n(q) = U_{2n'}(q')$. On the other hand, we know that $r_n \sim r_1$, which is a contradiction. If $\rho(2, G) = \{2, r_{2(n-1)}\}$, then $R_{2(n-1)}(q) = U_{2n'}(q')$. We know that $r_{2(n-1)} \sim r_2$, which is a contradiction.

Case (2). Let n be even.

(2-1) Let $n = n'$, so $\rho(2, S) = \{2, u_{2n}\}$. By Remarks 3.1 and 3.3, $R_1(q) \cup R_2(q) = U_1(q') \cup U_2(q') \cup \{p'\}$. We know that u_{2n} is not adjacent to u_1, u_2 and p'. If $\rho(2, G) = \{2, r_{n-1}\}$, then $R_{n-1}(q) = U_{2n}(q')$. We know that $r_1 \sim r_{n-1}$, which is a contradiction. If $\rho(2, G) = \{2, r_{2(n-1)}\}$, then $R_{2(n-1)}(q) = U_{2n}(q')$. We know that $r_2 \sim r_{2(n-1)}$, which is a contradiction.

(2-2) If $n = n' + 2$, then similarly we get a contradiction.

(2-3) Let $n = n'+1$ and so n' is odd. By Remarks 3.1 and 3.3, $R_1(q) \cup R_2(q) = U_1(q') \cup U_2(q')$. If $p = 2$, then $p' = 2$. Since $\pi(G) = \pi(S)$, so $2(n-1)\alpha = 2n'\beta$, by Lemma 2.4. Therefore, $\alpha = \beta$ and so $S = C_{n-1}(q)$ or $S = B_{n-1}(q)$. We know that $r_6 \sim r_{n-3}$ in $\Gamma(S)$, while $r_6 \nsim r_{n-3}$ in $\Gamma(G)$, which is a contradiction. Therefore, $p \neq 2$ and $p' \neq 2$, so $t(2, S) = t(2, G) = 2$. Now we consider the following two cases:

- Let $\rho(2, G) = \{2, r_{n-1}\}$, and hence $q \equiv 3 \pmod 4$. Also let $\rho(2, S) = \{2, u_{n'}\}$, which implies that $R_{n-1}(q) = U_{n'}(q')$. We know that $r_1 \sim r_{n-1} \nsim r_2$, and $u_1 \sim u_{n'} \nsim u_2$. Consequently, $R_1(q) = U_1(q')$. Similarly we get $R_{2(n-1)}(q) = U_{2n'}(q')$ and so $\{r_{n-1}, r_{2(n-1)}\} \leftrightarrow \{u_{n'}, u_{2n'}\}$. Let $n' \equiv 1 \pmod 4$ so $\{p\} \cup R_4(q) = \{p'\}$, which is a contradiction. Therefore, $n' \not\equiv 1 \pmod 4$ and so $\{p\} \cup R_4(q) = \{p'\} \cup U_4(q')$. If $p = p'$, then we get a contradiction otherwise, $p \in U_4(q')$ and $p' \in R_4(q)$. It follows that $p \equiv 1 \pmod 4$, which is a contradiction. If $\rho(2, S) = \{2, u_{2n'}\}$, then similarly to the above we get a contradiction.
- Let $\rho(2, G) = \{2, r_{2(n-1)}\}$, and hence $q \equiv 1 \pmod 4$. Let $\rho(2, S) = \{2, u_{n'}\}$, hence $(q' - 1)_2 = 2$ so $q' = 2h + 1$, where h is odd, by [11, Table 6]. Similarly we get that $p \in U_4(q')$ and $p' \in R_4(q)$. This implies that $p' \equiv 1 \pmod 4$, which is a contradiction. So $\rho(2, S) = \{2, u_{2n'}\}$. Consequently, $R_{2(n-1)}(q) = U_{2n'}(q')$. We know that $r_1 \nsim r_{2(n-1)} \sim r_2$, and $u_1 \nsim u_{2n'} \sim u_2$. So $R_1(q) = U_1(q')$ and $R_2(q) = U_2(q')$. Similarly we get that $R_{n-1}(q) = U_{n'}(q')$ and so $\{r_{n-1}, r_{2(n-1)}\} \leftrightarrow \{u_{n'}, u_{2n'}\}$. If $n' \equiv 1 \pmod 4$ so $\{p\} \cup R_4(q) = \{p'\}$, which is a contradiction. Therefore, $n' \not\equiv 1 \pmod 4$ and so $\{p\} \cup R_4(q) = \{p'\} \cup U_4(q')$. If $p = p'$, then we get a contradiction otherwise, $p \in U_4(q')$ and $p' \in R_4(q)$, which implies that $p \equiv 1 \pmod 4$ and $p' \equiv 1 \pmod 4$. Therefore, $S = C_{n-1}(r_4^\beta)$ or $S = B_{n-1}(r_4^\beta)$. □

Remark 4.5. Let $S = C_{n'}(q')$ or $S = B_{n'}(q')$ such that $\Gamma(S) = \Gamma(G)$. By Lemma 4.4, we get that $n' + 1 = n$ and $n \equiv 0 \pmod 4$, $p' \neq p$ and $p \equiv 1 \pmod 4$ and $p' \equiv 1 \pmod 4$. Also we have $R_1(q) = U_1(q')$, $R_2(q) = U_2(q')$, $R_{2(n-1)}(q) = U_{2n'}(q')$ and $R_{n-1}(q) = U_{n'}(q')$. Moreover, we know that for every $r \in \pi(G)$, we have $t(r, G) = t(r, S)$.

By Remark 3.1, if $n \equiv 1 \pmod 3$, then r_3 and r_6 are vertices in $\Gamma(G)$ such that their independence numbers are 4, while there is no member in $\Gamma(S)$ such that its independence number is equal 4, by Remark 3.3, so we get a contradiction. Consequently, $n \not\equiv 1 \pmod 3$ and so $t(r_3, G) = t(r_6, G) = 5$. Let $n \equiv 2 \pmod 3$, so $n' \equiv 1 \pmod 3$ and by Remark 3.3, there is no member in $\Gamma(S)$ such that its independence number is equal 5, which is a contradiction. Therefore, $n \equiv 0 \pmod 3$ and u_3 and u_6 are only vertices in $\Gamma(S)$ such that their independence numbers are equal to 5. Now, as in Remark 4.2, if $(n-2)/4$ is odd, then $R_3(q) \cup R_6(q) = U_3(q') \cup U_6(q')$ otherwise, $R_3(q) \cup R_6(q) \cup R_8(q) = U_3(q') \cup U_6(q')$. Also for other vertices in $\Gamma(G)$ and $\Gamma(S)$ we can find some relation similar to the above.

Also as in Remark 4.2, we can show $R_4(q)$ and $U_4(q')$ have more than one member.

In this situation, we conjecture that there is no q and q' satisfying all the above conditions.

Lemma 4.6. *If $S = {}^2A_{n'-1}(q')$, where $q' = p'^{\beta}$, then $\Gamma(S) \neq \Gamma(G)$.*

Proof. On the contrary let $\Gamma(S) = \Gamma(G)$. Therefore, $t(S) = t(G)$, where $t(S) = [(n'+1)/2]$ and $t(G) = [(3n+1)/4]$ or $t(G) = (3n+3)/4$. So $[(n'+1)/2] = [(3n+1)/4]$ or $[(n'+1)/2] = (3n+3)/4$. Therefore, $n' \geq 7$, since $n > 4$.

Case (1). Let n be odd.

By [11, Proposition 4.2], we know that $t(u_2, S) = 2$ or 3. Therefore, we consider the following two cases:

(1-1) Let $t(u_2, S) = 3$, so by Remarks 3.1 and 3.5, $R_1(q) \cup R_2(q) = U_1(q')$. In the sequel we consider each possibility for $\rho(2, S)$. Let $\rho(2, S) = \{2, u_{2n'}\}$. Hence $t(2, G) = 2$. If $\rho(2, G) = \{2, r_n\}$, then $R_n(q) = U_{2n'}(q')$. We know that $r_1 \sim r_n \nsim r_2$, which is a contradiction, since $R_1(q) \cup R_2(q) = U_1(q')$. Consequently, $\rho(2, G) = \{2, r_{2(n-1)}\}$, and so $R_{2(n-1)}(q) = U_{2n'}(q')$. We have $r_1 \nsim r_{2(n-1)} \sim r_2$, which is a contradiction. If $\rho(2, S) = \{2, u_{2(n'-1)}\}$, $\rho(2, S) = \{2, u_{n'}\}$ or $\rho(2, S) = \{2, u_{n'/2}\}$, then we get a contradiction. Now let $\rho(2, S) = \{2, u_{n'}, u_{2(n'-1)}\}$, so $t(2, G) = 3$. By [11, Tables 4, 6], $\rho(2, G) = \{2, r_n, r_{2(n-1)}\}$. Similarly we get that $\{r_n, r_{2(n-1)}\} \leftrightarrow \{u_{n'}, u_{2(n'-1)}\}$, i.e. $r_n = u_{n'}$ or $r_n = u_{2(n'-1)}$ and we get a contradiction. If $\rho(2, S) = \{2, u_{n'-1}, u_{2n'}\}$, $\rho(2, S) = \{2, u_{n'/2}, u_{2(n'-1)}\}$ or $\rho(2, S) = \{2, u_{(n'-1)/2}, u_{2n'}\}$, then we get a contradiction.

(1-2) Let $t(u_2, S) = 2$, hence $R_1(q) \cup R_2(q) = U_1(q') \cup U_2(q')$. We know that $t(2, S) = 2$ or 3, so we consider the following two cases:

(1-2-a) Let $t(2, S) = 2$, so $t(2, G) = 2$. By Remark 3.5, we know that p' and u_6 are the only vertices in $\Gamma(S)$ such that their independence numbers are 3. We consider the following two cases:

$*$ Let $n \equiv 1 \pmod 4$. In this case, p is the only vertex in $\Gamma(G)$ such that its independence number is 3 so $\{p\} = \{p'\} \cup U_6(q')$, which is a contradiction.

$*$ Let $n \not\equiv 1 \pmod 4$. Therefore, p and r_4 are the only vertices in $\Gamma(G)$ such that their independence numbers are 3. So $\{p\} \cup R_4(q) = \{p'\} \cup U_6(q')$. Let $p = p'$. Now we consider two possibilities for n', separately.

▶ If n' is even, then $2(n-1)\alpha = 2(n'-1)\beta$, and so $(\alpha)_2 < (\beta)_2$. On the other hand, we know that $\rho(2, S) = \rho(2, G)$. Now we consider the following two cases:

• Let $\rho(2,G)=\{2,r_n\}$. If $\rho(2,S)=\{2,u_{2(n'-1)}\}$, then $R_n(q)=U_{2(n'-1)}(q')$. So by Lemma 2.9, we get that $n\alpha=2(n'-1)\beta$, which is a contradiction. If $\rho(2,S)=\{2,u_{n'}\}$, then $R_n(q)=U_{n'}(q')$, and so by Lemma 2.9, we get that $n\alpha=n'\beta$, since n is odd and n' is even. Hence $(\alpha)_2>(\beta)_2$, which is a contradiction. Finally, let $\rho(2,S)=\{2,u_{n'/2}\}$. By Lemma 2.9, we get that $n\alpha=(n'/2)\beta$ and so $\alpha=(1-n'/2)\beta$, which is a contradiction.

• Let $\rho(2,G)=\{2,r_{2(n-1)}\}$. If $\rho(2,S)=\{2,u_{2(n'-1)}\}$, then $R_{2(n-1)}(q)=U_{2(n'-1)}(q')$. We know that $r_1\nsim r_{2(n-1)}\sim r_2$ in $\Gamma(G)$ and $u_1\nsim u_{2(n'-1)}$ in $\Gamma(S)$. Since $R_1(q)\cup R_2(q)=U_1(q')\cup U_2(q')$ it follows that $R_2(q)=U_2(q')$. By Remark 3.1, r_n is the only vertex in $\Gamma(G)$ which is not adjacent to r_2. If $n'\equiv 0 \pmod 4$, then $u_{n'}$ is the only vertex in $\Gamma(S)$ which is not adjacent to u_2, so $R_n(q)=U_{n'}(q')$. Consequently, $n\alpha=n'\beta$ and hence $(\alpha)_2>(\beta)_2$, which is a contradiction. If $n'\equiv 2 \pmod 4$, then $u_{n'/2}$ is the only vertex in $\Gamma(S)$ which is not adjacent to u_2, so $R_n(q)=U_{n'/2}(q')$. Consequently, $n\alpha=(n'/2)\beta$ hence $(\alpha)_2=(\beta)_2$, which is a contradiction. If $\rho(2,S)=\{2,u_{n'}\}$, then $R_{2(n-1)}(q)=U_{n'}(q')$, by Lemma 2.9, we get that $2(n-1)\alpha=n'\beta$, which is a contradiction. Finally, let $\rho(2,S)=\{2,u_{n'/2}\}$, by Lemma 2.9, we get that $n\alpha=(n'/2)\beta$, which is a contradiction.

► Therefore, n' is odd and so $2(n-1)\alpha=2n'\beta$, hence $(\alpha)_2<(\beta)_2$. Since n' is odd, so $\rho(2,S)=\{2,u_{2n'}\}$. Let $\rho(2,G)=\{2,r_n\}$. It follows that $R_n(q)=U_{2n'}(q')$. By Lemma 2.9, we have $n\alpha=2n'\beta$, which is a contradiction. Let $\rho(2,G)=\{2,r_{2(n-1)}\}$. It follows that $R_{2(n-1)}(q)=U_{2n'}(q')$. We know that $r_1\nsim r_{2(n-1)}\sim r_2$ in $\Gamma(G)$ and $u_1\nsim u_{2n'}\sim u_2$ in $\Gamma(S)$. Therefore, $R_2(q)=U_2(q')$. If $n'\not\equiv 1 \pmod 4$, then $R_2(q)=U_2(q')$, which implies that $R_n(q)=U_{(n'-1)/2}(q')$, and so by Lemma 2.9, $n\alpha=((n'-1)/2)\beta$. Since n and $(n'-1)/2$ are odd, so $(\alpha)_2=(\beta)_2$, which is a contradiction. Otherwise, $n'\equiv 1 \pmod 4$, and the above relation implies that $R_n(q)=U_{n'-1}(q')$, by Lemma 2.9, we have $n\alpha=(n'-1)\beta$. Since n is odd, so $(\alpha)_2>(\beta)_2$, which is a contradiction.

Consequently, $p\neq p'$.

Therefore, $p'\in R_4(q)$ and $p\in U_6(q')$. Now we claim that $|R_4(q)|=|U_6(q')|=1$. Otherwise, let $p_0=r'_4=u'_6\in R_4(q)\cap U_6(q')$.
We know that $\{p,r_n,r_{2(n-1)}\}$ and $\{r_4,r_{n-2},r_n\}$ are unique maximal independent sets which contain p and r_4 in $\Gamma(G)$, respectively. We consider the following cases:

(1) Let $n'\equiv 0 \pmod 4$. So $\{p',u_{n'},u_{2(n'-1)}\}$ is the unique maximal independent set which contains p'. Since $r_4=p'$, so $\{r_{n-2},r_n\}\leftrightarrow\{u_{n'},u_{2(n'-1)}\}$. In other words, $R_{n-2}(q)\cup R_n(q)=U_{n'}(q')\cup U_{2(n'-1)}(q')$. On the other hand, $r'_4=u'_6$, and so u'_6 is not adjacent to $U_{n'}(q')$ and $U_{2(n'-1)}(q')$. We know that $p=u_6$ and it follows that $R_n(q)\cup R_{2(n-1)}(q)=U_{n'}(q')\cup U_{2(n'-1)}(q')$. Therefore, $R_{2(n-1)}(q)=R_{n-2}(q)$, which is a contradiction.

(2) Let $n'\equiv 1 \pmod 4$, so $\{p',u_{n'-1},u_{2n'}\}$ is the unique maximal independent set which contains p'. Since $r_4=p'$, $R_{n-2}(q)\cup R_n(q)=U_{n'-1}(q')\cup$

$U_{2n'}(q')$. On the other hand, $r'_4 = u'_6$, so u'_6 is not adjacent to $U_{n'-1}(q')$ and $U_{2n'}(q')$. We know that $p = u_6$ and it follows that $R_n(q) \cup R_{2(n-1)}(q) = U_{n'-1}(q') \cup U_{2n'}(q')$. Therefore, $R_{2(n-1)}(q) = R_{n-2}(q)$, which is a contradiction.

(3) Let $n' \equiv 2 \pmod 4$, so $\{p', u_{n'/2}, u_{2(n'-1)}\}$ is the unique maximal independent set which contains p'. Now since $r_4 = p'$, $r'_4 = u'_6$ and $p = u_6$, it follows that $R_{n-2}(q) \cup R_n(q) = U_{n'/2}(q') \cup U_{2(n'-1)}(q') = R_n(q) \cup R_{2(n-1)}(q)$, which is a contradiction.

(4) Let $n' \equiv 3 \pmod 4$, so $\{p', u_{(n'-1)/2}, u_{2n'}\}$. Since $r_4 = p'$, $r'_4 = u'_6$ and $p = u_6$, it follows that $R_{n-2}(q) \cup R_n(q) = U_{(n'-1)/2}(q') \cup U_{2n'}(q') = R_n(q) \cup R_{2(n-1)}(q)$, which is a contradiction.

Consequently, $|R_4(q)| = |U_6(q')| = 1$. Since $t(2, S) = t(2, G) = 2$, so $p \neq 2$ and $p' \neq 2$. So there exists a natural number m such that $q^2 + 1 = 2p'^m$, and so $p^{2\alpha} - 2p'^m = -1$. Now by Lemma 2.3, we consider the following three cases:

(1) If $m = 1$, then $p' - 1 = (p^{2\alpha} - 1)/2 = (q^2 - 1)/2$. Therefore, $\pi(q^2 - 1) \subseteq \pi(q' - 1)$, and also there exists natural number h such that $q' + 1 = 2^h$, since $\pi(q^2 - 1) = \pi(q'^2 - 1)$. By Lemma 2.2, we have $\beta = 1$ and so $q' = p'$. On the other hand, we know that p is the only member of $U_6(q')$. Moreover, we know that $p' + 1 = 2^h$, so $(p' + 1, p'^2 - p' + 1) = 1$. It follows that $p'^2 - p' + 1 = p^{m'}$, for some natural number m'. Also we know that $(q^2 + 1)/2 = p'$. Hence, $q^4 + 3 = 4p^{m'}$, and so $p = 3$. Consequently, $3^{4\alpha} + 3 = 4 \cdot 3^{m'}$, which is a contradiction.

(2) Let $m = 2$ and $\alpha = 1$, so $p = q$ and $p^2 + 1 = 2p'^2$. We consider the following two cases:

(2-1) Let $3 \nmid (q' + 1)$, we have $q'^2 - q' + 1 = p^{m'}$ and so $q'(q' - 1) = p^{m'} - 1$. It follows that either $r_{m'} = p'$ or $r_{m'} = u_1$. If $r_{m'} = u_1$, then $m' = 1$ or 2, since $\pi(q^2 - 1) = \pi(q'^2 - 1)$. If $m' = 1$, then $q'^2 - q' + 1 = p$, and so $p^2 + 1 = q'^2(q'^2 - 2q' + 1) + 2p$. On the other hand, we know that $p^2 + 1 = 2p'^2$, which is a contradiction, since $p \neq p'$. If $m' = 2$, then $q'^2 - q' + 1 = p^2$. So $2p'^2 = p^2 + 1 = q'^2 - q' + 2$, which is a contradiction, since $p' \neq 2$. Therefore, $r_{m'} = p'$ and so by assumption $m' = 4$. Consequently, $q'^2 - q' + 1 = p^4$ and so $p^2 + 1 = q'(q' - 1)/(p^2 - 1)$. Hence $q'(q' - 1) = 2p'(p^2 - 1)$. It follows that $\pi(q' - 1) = \pi(q^2 - 1)$, and so $q' + 1 = 2^h$ for some natural number h. By Lemma 2.2, $\beta = 1$ hence $q' = p'$ and so $p'^2 - p' + 1 = p^4$. On the other hand, we know that $p^2 = 2p'^2 - 1$, which is a contradiction.

(2-2) Let $3 \mid (q' + 1)$, so there exists a natural number t such that $q'^2 - q' + 1 = 3^t \cdot p^{m'}$. Also $q' = 3s - 1$, for some natural number s. Therefore, $9(s^2 - s) + 3 = 3^t \cdot p^{m'}$, it follows that $t = 1$. Moreover, $3 \mid (p^{m'} - 1)$. We know that $e(3, p) = 1$ or 2. If $e(3, p) = 2$, then m' is even, hence $m' = 2l$, for some natural number l. Since $p^2 + 1 = 2p'^2$, so we have $q'(q' - 1) + 1 = 3(2p'^2 - 1)^l$. Consequently, $p' \mid (3(-1)^l - 1)$ and so $p' = 2$, which is a contradiction. Therefore, $e(3, p) = 1$, and so $3 \nsim r_{2(n-1)}$ in $\Gamma(G)$. We know that $p' \in R_4(q)$ and $p \in U_6(q')$, also

$\{p, r_n, r_{2(n-1)}\}$ and $\{r_4, r_{n-2}, r_n\}$ are unique maximal independent sets which contain p and r_4 in $\Gamma(G)$, respectively. We consider the following cases:

(2-2-1) Let $n' \equiv 0 \pmod 4$, so $\{p', u_{n'}, u_{2(n'-1)}\}$ is the unique maximal independent set which contains p'. Since $r'_p = 4$, so $R_{n-2}(q) \cup R_n(q) = U_{n'}(q') \cup U_{2(n'-1)}(q')$. Also we know that u_6 is not adjacent to two members of $\{u_{(n'-2)/2}, u_{n'}, u_{2(n'-1)}\}$. Since $p = u_6$, so $R_{2(n-1)}(q) = U_{(n'-2)/2}(q')$. On the other hand, by [11, Proposition 4.2], we have $3 \sim u_{(n'-2)/2}$, which is a contradiction, since $3 \nsim r_{2(n-1)}$.

(2-2-2) Let $n' \equiv 1 \pmod 4$, so $\{p', u_{n'-1}, u_{2n'}\}$ is the unique maximal independent set which contains p'. Since $p' = r_4$, so $R_{n-2}(q) \cup R_n(q) = U_{n'-1}(q') \cup U_{2n'}(q')$. Also we know that u_6 is not adjacent to two members of $\{u_{n'-1}, u_{2(n'-2)}, u_{2n'}\}$. Similarly we get that $R_{2(n-1)}(q) = U_{2(n'-2)}(q')$, by [11, Proposition 4.2], we have $3 \sim u_{2(n'-2)}$, which is a contradiction.

(2-2-3) Let $n' \equiv 2 \pmod 4$, so $\{p', u_{n'/2}, u_{2(n'-1)}\}$ is the unique maximal independent set which contains p'. Now since $r'_p = 4$, so $R_{n-2}(q) \cup R_n(q) = U_{n'/2}(q') \cup U_{2(n'-1)}(q')$. Also we know that u_6 is not adjacent to two members of $\{u_{n'/2}, u_{n'-2}, u_{2(n'-1)}\}$. Similarly we get that $R_{2(n-1)}(q) = U_{n'-2}(q')$, by [11, Proposition 4.2], we have $3 \sim u_{n'-2}$, which is a contradiction.

(2-2-4) Let $n' \equiv 3 \pmod 4$, so $\{p', u_{(n'-1)/2}, u_{2n'}\}$ is the unique maximal independent set which contains p'. Now since $p' = r_4$, so $R_{n-2}(q) \cup R_n(q) = U_{(n'-1)/2}(q') \cup U_{2n'}(q')$. Also we know that u_6 is not adjacent to two members of $\{u_{(n'-1)/2}, u_{2(n'-2)}, u_{2n'}\}$. Similarly we get that $R_{2(n-1)}(q) = U_{2(n'-2)}(q')$, by [11, Proposition 4.2], we have $3 \sim u_{2(n'-2)}$, which is a contradiction.

(3) Let $p = 239$, $\alpha = 1$ and $p' = 13$. We know that $\{2, 3, 5, 7, 17\} = \pi(239^2 - 1) = \pi(13^{2\beta} - 1)$. Therefore, $\beta = 2$. Since $p = u_6$, so $239 \mid (13^{12} - 1)$, which is a contradiction.

(1-2-b) Therefore, $t(2, S) = t(2, G) = 3$, and we can consider the following two cases:

$*$ Let $n \equiv 1 \pmod 4$, so $\{2, p\} = \{2, p'\} \cup U_6(q')$. If $p = 2$, then $p' = 2$. Hence $\{2\} = \{2\} \cup U_6(q')$. Consequently, $U_6(q') = \emptyset$ and $\beta = 1$, by Lemma 2.4. Since $\pi(q^2 - 1) = \pi(p'^2 - 1)$, so $\alpha = 1$. On the other hand, we know $\pi(G) = \pi(S)$. Therefore, if n' is even, then $2(n-1) = 2(n'-1)$, by Lemma 2.4. Hence $n = n'$, which is a contradiction, since n is odd and n' is even. Otherwise, n' is odd, then $2(n-1) = 2n'$, by Lemma 2.4, which is a contradiction. Consequently, $p \neq 2$ and $p' \neq 2$. Therefore $p = p'$ and $\{2\} = \{2\} \cup U_6(q')$, which is a contradiction.

$*$ Let $n \not\equiv 1 \pmod 4$, so $\{2, p\} \cup R_4(q) = \{2, p'\} \cup U_6(q')$. If $p = 2$, then $p' = 2$, so $p = p'$. We know that $\rho(2, G) = \{2, r_n, r_{2(n-1)}\}$. Now we consider two possibilities for n'.

▶ If n' is even, then $2(n-1)\alpha = 2(n'-1)\beta$, and so $(\alpha)_2 < (\beta)_2$. On the other hand, we know that $\rho(2, S) = \rho(2, G)$. Now we consider the following two cases:

• Let $n' \equiv 0 \pmod 4$, so $\rho(2,S) = \{2, u_{n'}, u_{2(n'-1)}\}$. Therefore, $R_n(q) \cup R_{2(n-1)}(q) = U_{n'}(q') \cup U_{2(n'-1)}(q')$. So similarly to the above, $\{r_n, r_{2(n-1)}\} \leftrightarrow \{u_{n'}, u_{2(n'-1)}\}$. Let $r_n = u_{n'}$, then $n\alpha = n'\beta$, by Lemma 2.9. Since n is odd and n' is even, so $(\beta)_2 < (\alpha)_2$, which is a contradiction. Let $r_n = u_{2(n'-1)}$, by Lemma 2.9, we get that $n\alpha = 2(n'-1)\beta$, which is a contradiction.

• Let $n' \equiv 2 \pmod 4$, so $\rho(2,S) = \{2, u_{n'/2}, u_{2(n'-1)}\}$. Therefore, $R_n(q) \cup R_{2(n-1)}(q) = U_{n'/2}(q') \cup U_{2(n'-1)}(q')$. So, $\{r_n, r_{2(n-1)}\} \leftrightarrow \{u_{n'/2}, u_{2(n'-1)}\}$. Let $r_n = u_{n'/2}$, by Lemma 2.9, we get that $n\alpha = (n'/2)\beta$. Since n and $n'/2$ are odd, so $(\beta)_2 = (\alpha)_2$, which is a contradiction. Let $r_n = u_{2(n'-1)}$, so we have $n\alpha = 2(n'-1)\beta$, by Lemma 2.9, which is a contradiction.

▶ If n' is odd, then $2(n-1)\alpha = 2n'\beta$, and so $(\alpha)_2 < (\beta)_2$. On the other hand, we know that $\rho(2,S) = \rho(2,G)$. Now we consider the following two cases:

• Let $n' \equiv 1 \pmod 4$, so $\rho(2,S) = \{2, u_{n'-1}, u_{2n'}\}$. Therefore, $\{r_n, r_{2(n-1)}\} \leftrightarrow \{u_{n'-1}, u_{2n'}\}$. Let $r_n = u_{n'-1}$, by Lemma 2.9, we have $n\alpha = (n'-1)\beta$. Since n and n' are odd, so $(\beta)_2 < (\alpha)_2$, which is a contradiction. Let $r_n = u_{2n'}$, by Lemma 2.9, we get that $n\alpha = 2n'\beta$, and so $(\beta)_2 < (\alpha)_2$, which is a contradiction.

• Let $n' \equiv 3 \pmod 4$, so $\rho(2,S)=\{2, u_{(n'-1)/2}, u_{2n'}\}$. Therefore, $\{r_n, r_{2(n-1)}\} \leftrightarrow \{u_{(n'-1)/2}, u_{2n'}\}$. Let $r_n = u_{(n'-1)/2}$, we have $n\alpha = ((n'-1)/2)\beta$, by Lemma 2.9. Since n and $(n'-1)/2$ are odd, so $(\beta)_2 = (\alpha)_2$, which is a contradiction. Let $r_n = u_{2n'}$, so we have $n\alpha = 2n'\beta$, hence $(\beta)_2 < (\alpha)_2$, which is a contradiction.

Consequently, $p \neq 2$ and $p' \neq 2$. It is clear that $r_4 \neq 2$ and $u_6 \neq 2$ so $\{p\} \cup R_4(q) = \{p'\} \cup U_6(q')$ and we get a contradiction.

Case (2). Let n be even.

Now we consider the following two cases:

(2-1) Let $t(u_2, S) = 3$, completely similar to the above case we get a contradiction.

(2-2) Let $t(u_2, S) = 2$, hence $R_1(q) \cup R_2(q) = U_1(q') \cup U_2(q')$. We know that $t(2,S) = 2$ or 3, so we consider the following two cases:

(2-1-a) Let $t(2,S) = t(2,G) = 2$. Now we will prove that $p \neq p'$. By Remark 3.5, we know that p' and u_6 are the only vertices in $\Gamma(S)$ such that their independence numbers are 3. Also p and r_4 are the only vertices in $\Gamma(G)$ such that their independence numbers are 3. So $\{p\} \cup R_4(q) = \{p'\} \cup U_6(q')$. Let $p = p'$. Now we consider two possibilities for n':

▶ If n' is even, then $2(n-1)\alpha = 2(n'-1)\beta$. On the other hand, we know that $\rho(2,S) = \rho(2,G)$. Now we consider the following two cases:

• Let $\rho(2,G) = \{2, r_{n-1}\}$. If $\rho(2,S) = \{2, u_{2(n'-1)}\}$, then $R_{n-1}(q) = U_{2(n'-1)}(q')$ and so by Lemma 2.9, $(n-1)\alpha = 2(n'-1)\beta$, which is a contradiction. If $\rho(2,S) = \{2, u_{n'}\}$, then $R_{n-1}(q) = U_{n'}(q')$, by Lemma 2.9, we get that $(n-1)\alpha = n'\beta$, which is a contradiction. Finally, let $\rho(2,S) = \{2, u_{n'/2}\}$, similarly to the above, we get $(n-1)\alpha = (n'/2)\beta$, which is a contradiction.

• Let $\rho(2,G) = \{2, r_{2(n-1)}\}$. If $\rho(2,S) = \{2, u_{2(n'-1)}\}$, then $R_{2(n-1)}(q) = U_{2(n'-1)}(q')$. We know that $r_1 \nsim r_{2(n-1)} \sim r_2$. in $\Gamma(G)$ and $u_{2(n'-1)} \nsim u_1$ in $\Gamma(S)$. Since $R_1(q) \cup R_2(q) = U_1(q') \cup U_2(q')$ it follows that $R_2(q) = U_2(q')$. We know that the only vertex in $\Gamma(G)$, which is not adjacent to r_2 is r_{n-1}. If $n' \equiv 0 \pmod 4$, then $u_{n'}$ is the only vertex in $\Gamma(S)$ which is not adjacent to u_2, by [11, Proposition 4.2]. Therefore, $R_{n-1}(q) = U_{n'}(q')$, by Lemma 2.9, we get that $(n-1)\alpha = n'\beta$, which is a contradiction. If $n' \equiv 2 \pmod 4$, then $u_{n'/2}$ is the only vertex in $\Gamma(S)$ which is not adjacent to u_2, by [11, Proposition 4.2]. Similarly $(n-1)\alpha = (n'/2)\beta$, which is a contradiction. If $\rho(2,S) = \{2, u_{n'}\}$, then $R_{2(n-1)}(q) = U_{n'}(q')$. Similarly, we get $2(n-1)\alpha = n'\beta$, which is a contradiction. Finally, let $\rho(2,S) = \{2, u_{n'/2}\}$, we get $2(n-1)\alpha = (n'/2)\beta$, which is a contradiction.

► Therefore, n' is odd and so $2(n-1)\alpha = 2n'\beta$. Since n' is odd, so $\rho(2,S) = \{2, u_{2n'}\}$. If $\rho(2,G) = \{2, r_{n-1}\}$, then $R_{n-1}(q) = U_{2n'}(q')$ and so $(n-1)\alpha = 2n'\beta$, which is a contradiction. Therefore, $\rho(2,G) = \{2, r_{2(n-1)}\}$, and so $R_{2(n-1)}(q) = U_{2n'}(q')$. We know that $r_1 \nsim r_{2(n-1)} \sim r_2$ in $\Gamma(G)$ and $u_1 \nsim u_{2n'} \sim u_2$ in $\Gamma(S)$. Similarly $R_2(q) = U_2(q')$, which implies that $\rho(r_2, G) = \rho(u_2, S)$. Therefore, if $n' \equiv 1 \pmod 4$, then $R_{n-1}(q) = U_{n'-1}(q')$, by [11, Proposition 4.2]. Hence, by Lemma 2.9, $(n-1)\alpha = (n'-1)\beta$, which is a contradiction. Consequently, $n' \not\equiv 1 \pmod 4$, and so $R_{n-1}(q) = U_{(n'-1)/2}(q')$. Therefore, $(n-1)\alpha = ((n'-1)/2)\beta$, which is a contradiction. Consequently, $p \neq p'$.

By Remark 3.1, we know that if $n \not\equiv 1 \pmod 3$, then for every $r_i \in \pi(G)$, we have $t(r_i, G) \neq 4$ and otherwise, r_3 and r_6 are the only vertices in $\Gamma(G)$ such that their independence numbers are 4. On the other hand, we know that $t(u_4, S) = 4$, by Remark 3.5. Therefore, $n \equiv 1 \pmod 3$ and $R_3(q) \cup R_6(q) = U_4(q')$.

If $\rho(2,S) = \{2, u_{2(n'-1)}\}$, then we have either $R_{n-1}(q) = U_{2(n'-1)}(q')$ or $R_{2(n-1)}(q) = U_{2(n'-1)}(q')$, by [11, Table 6]. Also we know that $r_3 \sim r_{n-1} \nsim r_6$ and $r_3 \nsim r_{2(n-1)} \sim r_6$, which is a contradiction, since $R_3(q) \cup R_6(q) = U_4(q')$. If $\rho(2,S) = \{2, u_{2n'}\}$, $\rho(2,S) = \{2, u_{n'}\}$ or $\rho(2,S) = \{2, u_{n'/2}\}$, then we get a contradiction.

(2-1-b) If $t(2,S) = t(2,G) = 3$, then we have $\{2,p\} \cup R_4(q) = \{2,p'\} \cup U_6(q')$. Now we get $p \neq p'$. While since $t(2,G) = 3$, then $p = 2$. Consequently, $p' = 2$, so $p = p'$, which is a contradiction.

□

Lemma 4.7. *If $S = A_{n'-1}(q')$, where $q' = p'^{\beta}$, then $\Gamma(S) \neq \Gamma(G)$.*

Proof. On the contrary let $\Gamma(S) = \Gamma(G)$. Therefore, $t(S) = t(G)$, so $[(n'+1)/2] = [(3n+1)/4]$ or $[(n'+1)/2] = (3n+3)/4$.

Case (1). Let n be odd.

By [11, Proposition 4.2], we know that $t(u_1, S) = 2$ or 3. Therefore, we consider the following two cases:

(1-1) Let $t(u_1, S) = 3$, so by Remarks 3.1 and 3.4, $R_1(q) \cup R_2(q) = U_2(q')$. Now we consider the following cases:

▶ Let $\rho(2, S) = \{2, u_{n'}\}$. Hence $t(2, G) = 2$. Let $\rho(2, G) = \{2, r_n\}$, so $R_n(q) = U_{n'}(q')$. We know that $r_1 \sim r_n \nsim r_2$, which is a contradiction, since $R_1(q) \cup R_2(q) = U_2(q')$. Consequently, $\rho(2, G) = \{2, r_{2(n-1)}\}$, and so $R_{2(n-1)}(q) = U_{n'}(q')$. Now we have $r_1 \nsim r_{2(n-1)} \sim r_2$, this is a contradiction.

▶ Let $\rho(2, S) = \{2, u_{n'-1}\}$. If $\rho(2, G) = \{2, r_n\}$, then $R_n(q) = U_{n'-1}(q')$. Since $r_1 \sim r_n \nsim r_2$, and $R_1(q) \cup R_2(q) = U_2(q')$, we get a contradiction. Similarly, if $\rho(2, G) = \{2, r_{2(n-1)}\}$, then $R_{2(n-1)}(q) = U_{n'-1}(q')$, while we know that $r_1 \nsim r_{2(n-1)} \sim r_2$, and this is a contradiction.

▶ Let $\rho(2, S) = \{2, u_{n'-1}, u_{n'}\}$. Therefore, $t(2, G) = 3$, and we get that $\{r_n, r_{2(n-1)}\} \leftrightarrow \{u_{n'-1}, u_{n'}\}$. It follows that $r_n = u_{n'}$ or $r_n = u_{n'-1}$ and we get a contradiction.

(1-2) Let $t(u_1, S) = 2$. Hence $R_1(q) \cup R_2(q) = U_1(q') \cup U_2(q')$. We consider two possibilities for n:

$*$ Let $n \equiv 1 \pmod 4$. Let $t(2, S) = t(2, G) = 2$. By Remark 3.4, we know that p' and u_3 are the only vertices in $\Gamma(S)$ such that their independence numbers are 3. On the other hand, p is the only vertex in $\Gamma(G)$ such that its independence number is 3 so $\{p\} = \{p'\} \cup U_3(q')$, which is a contradiction. So $t(2, S) = t(2, G) = 3$. Similarly we have, $\{2, p\} = \{2, p'\} \cup U_3(q')$. If $p' = 2$, then $p = 2$, since $\pi(q^2 - 1) = \pi(q'^2 - 1)$. It follows that $u_3 = 2$, which is a contradiction. Therefore, $p = p'$ and so $\{2\} = \{2\} \cup U_3(q')$, which is impossible.

$*$ Therefore, $n \not\equiv 1 \pmod 4$. We claim that $t(2, S) = t(2, G) = 3$. Otherwise, $t(2, S) = t(2, G) = 2$ and $\{p\} \cup R_4(q) = \{p'\} \cup U_3(q')$. If $p = p'$, then by p-independent sets of S and G we have $\{r_n, r_{2(n-1)}\} \leftrightarrow \{u_{n'-1}, u_{n'}\}$. On the other hand, $2(n-1)\alpha = n'\beta$, by Lemma 2.4. If $r_n = u_{n'}$, then by Lemma 2.9, $n'\beta = n\alpha$, which is a contradiction. Otherwise, $r_n = u_{n'-1}$ and so by Lemma 2.9, we get that $n\alpha = (n'-1)\beta$. Also $2(n-1)\alpha = n'\beta$, which implies that $\beta \geq 3\alpha$. On the other hand, $t(S) = t(G)$ so $n' \in \{(3n-3)/2, (3n-1)/2, (3n+1)/2, (3n+3)/2\}$. Since $\beta \geq 3\alpha$, in each case we can see that $\pi(S) \neq \pi(G)$, and this is a contradiction. Therefore, $p \neq p'$. Hence $p' \in R_4(q)$ and $p \in U_3(q')$. Now we claim that $|R_4(q)| = |U_3(q')| = 1$. Otherwise, let $p_0 = r'_4 = u'_3 \in R_4(q) \cap U_3(q')$.

We know that $\{p, r_n, r_{2(n-1)}\}$ and $\{r_4, r_{n-2}, r_n\}$ are the unique maximal independent sets which contain p and r_4 in $\Gamma(G)$, respectively. Also$\{p', u_{n'-1}, u_{n'}\}$ is the unique maximal independent set which contains p' in $\Gamma(S)$. Since $p' = r_4$, then similarly to the above we get that $\{r_{n-2}, r_n\} \leftrightarrow \{u_{n'-1}, u_{n'}\}$. In other words, $R_{n-2}(q) \cup R_n(q) = U_{n'-1}(q') \cup U_{n'}(q')$. On the other hand, $r'_4 = u'_3$, so u'_3 is not adjacent to $U_{n'}(q')$ and $U_{n'-1}(q')$. We know that $p = u_3$, similarly it follows that $R_{2(n-1)}(q) \cup R_n(q) = U_{n'-1}(q') \cup U_{n'}(q')$.

Therefore, $R_{2(n-1)}(q) = R_{n-2}(q)$, which is a contradiction. Consequently, $|R_4(q)| = |U_3(q')| = 1$.

If $p = 2$, then $u_3 = 2$, which is a contradiction. Consequently, $p \neq 2$. Also we know that $4 \nmid (q^2+1)$. It follows that there exists a natural number m such that $q^2 + 1 = 2p'^m$, hence $p^{2\alpha} - 2p'^m = -1$. Now by Lemma 2.3, we consider the three following cases:

(I) Let $m = 2$ and $\alpha = 1$, so $p = q$ and $p^2 + 1 = 2p'^2$. We consider the following two cases:

(I-1) Let $3 \nmid (q'-1)$, we have $q'^2 + q' + 1 = p^{m'}$ and so $q'(q'+1) = p^{m'} - 1$. It follows that either $r_{m'} = p'$ or $r_{m'} = u_2$. If $r_{m'} = u_2$, then $m' = 1$ or 2, since $\pi(q^2-1) = \pi(q'^2-1)$. If $m' = 1$, then $q'^2 + q' + 1 = p$, and so $p^2+1 = q'^2(q'^2+2q'+1)+2p$. On the other hand, we know that $p^2+1 = 2p'^2$, which is a contradiction, since $p \neq p'$. If $m' = 2$, then $q'^2 + q' + 1 = p^2$. So $2p'^2 = p^2 + 1 = q'^2 + q' + 2$, which is a contradiction, since $p' \neq 2$.
Therefore, $r_{m'} = p'$ and so by assumption $m' = 4$. Consequently, $q'^2 + q' + 1 = p^4$ and so $p^2 + 1 = q'(q'+1)/(p^2-1)$. Hence $q'(q'+1) = 2p'^2(p^2-1)$. It follows that $\pi(q'+1) = \pi(q^2-1)$, and so $q' - 1 = 2^h$, for some natural number h. By Lemma 2.2, $\beta = 1$, since $q' = 9$ is impossible. Hence $q' = p'$ and so $p'^2 + p' + 1 = p^4$. On the other hand, we know that $p^2 = 2p'^2 - 1$, which is a contradiction.

(I-2) Let $3 \mid (q'-1)$. Then there exists a natural number t such that $q'^2 + q' + 1 = 3^t \cdot p^{m'}$. Also $q' = 3s+1$, for some natural number s. Therefore, $9(s^2+s) + 3 = 3^t \cdot p^{m'}$, it follows that $t = 1$. Moreover, $3 \mid (p^{m'} - 1)$. We know that $e(3,p) = 1$ or 2. If $e(3,p) = 2$, then m' is even, hence $m' = 2l$, for some natural number l. Since $p^2 + 1 = 2p'^2$, so we have $q'(q'+1) + 1 = 3(2p'^2-1)^l$. Consequently, $p' \mid (3(-1)^l - 1)$ and so $p' = 2$, which is a contradiction. Therefore, $e(3,p) = 1$, and so $3 \nsim r_{2(n-1)}$ in $\Gamma(G)$. We know that $p' \in R_4(q)$ and $p \in U_3(q')$, also $\{p, r_n, r_{2(n-1)}\}$ and $\{r_4, r_{n-2}, r_n\}$ are the unique maximal independent sets which contain p and r_4 in $\Gamma(G)$, respectively. Also $\{p', u_{n'}, u_{2(n'-1)}\}$ is the unique maximal independent set which contains p'. Since $r_4 = p'$, so $R_{n-2}(q) \cup R_n(q) = U_{n'}(q') \cup U_{n'-1}(q')$. Also we know that u_3 is not adjacent to two members of $\{u_{n'-2}, u_{n'-1}, u_{n'}\}$. Since $p = u_3$, so $R_{2(n-1)}(q) = U_{n'-2}(q')$. On the other hand, by [11, Proposition 4.1], we have $3 \sim u_{n'-2}$, which is a contradiction, since $3 \nsim r_{2(n-1)}$.

(II) If $m = 1$, then $p' - 1 = (p^{2\alpha} - 1)/2 = (q^2-1)/2$. Therefore, $\pi(q^2-1) \subseteq \pi(q'-1)$, and also there exists natural number h such that $q' + 1 = 2^h$, since $\pi(q^2-1) = \pi(q'^2-1)$. By Lemma 2.2, we have $\beta = 1$ and so $q' = p'$. On the other hand, we know that p is the only member of $U_3(q')$. Moreover, $d = (p'-1, p'^2+p'+1) = 1$ or 3 and $9 \nmid (p'^2+p'+1)$, for each p'. Therefore, $p'^2+p'+1 = d \cdot p^{m'}$, for some natural number m'. Therefore, $p'^2+p'+1 = p^{m'}$. Also we know that $(q^2+1)/2 = p'$. Hence, $q^4 + 4q^2 + 7 = 4d \cdot p^{m'}$, and so $p = 7$ and $m' = 1$, which is a contradiction.

(III) Let $p = 239$, $\alpha = 1$ and $p' = 13$. We know that $\{2, 3, 5, 7, 17\} = \pi(239^2 - 1) = \pi(13^{2\beta} - 1)$. Therefore, $\beta = 2$. Since $p = u_3$, so $239 \mid (13^6 - 1)$, which is a contradiction.

Therefore, $t(2, S) = t(2, G) = 3$, so $\{2, p\} \cup R_4(q) = \{2, p'\} \cup U_3(q')$. If $p = 2$, then $p' = 2$, since $\pi(q^2 - 1) = \pi(q'^2 - 1)$. Therefore, $p = p'$ and similarly we get a contradiction. Consequently, since $r_4 \neq 2$ and $u_3 \neq 2$ so $\{p\} \cup R_4(q) = \{p'\} \cup U_3(q')$, we get a contradiction.

Case (2). Let n be even.

If $t(u_1, S) = 3$, then completely similar to (1-1) we get a contradiction. Therefore, $t(u_1, S) = 2$. Hence $R_1(q) \cup R_2(q) = U_1(q') \cup U_2(q')$. Let $t(2, S) = t(2, G) = 3$, by [11, Table 6], $p = 2$ and so $p' = 2$. Therefore, $2(n-1)\alpha = n'\beta$, by Lemma 2.4. Also by 2-independent sets of S and G we have $R_{n-1}(q) \cup R_{2(n-1)}(q) = U_{n'-1}(q') \cup U_{n'}(q')$. If $r_{n-1} = u_{n'-1}$, then by Lemma 2.9, $(n-1)\alpha = (n'-1)\beta$, which is a contradiction. Let $r_{n-1} = u_{n'}$. Then $(n-1)\alpha = n'\beta$, by Lemma 2.9, which is a contradiction. Therefore, $t(2, S) = t(2, G) = 2$, so we get $\{p\} \cup R_4(q) = \{p'\} \cup U_3(q')$. If $p = p'$, then we get a contradiction similarly to the above. Therefore, $p \neq p'$.

By Remark 3.1, we know that if $n \not\equiv 1 \pmod 3$, then for every $r_i \in \pi(G)$, we have $t(r_i, G) \neq 4$. Otherwise, r_3 and r_6 are the only vertices in $\Gamma(G)$ such that their independence number is 4. On the other hand, we know that $t(u_4, S) = 4$, by Remark 3.5. Therefore, $n \equiv 1 \pmod 3$ and $R_3(q) \cup R_6(q) = U_4(q')$.

Let $\rho(2, S) = \{2, u_{n'}\}$, so we have either $R_{n-1}(q) = U_{n'}(q')$ or $R_{2(n-1)}(q) = U_{n'}(q')$, by [11, Table 6]. Also we know that $r_3 \sim r_{n-1} \nsim r_6$ and $r_3 \nsim r_{2(n-1)} \sim r_6$, which is a contradiction. If $\rho(2, S) = \{2, u_{n'-1}\}$, then we get a contradiction. □

Lemma 4.8. *Let $G = D_n(q)$, where $q = p^\alpha$ and $n > 4$, and also S be an exceptional group of Lie type. Then $\Gamma(S)$ and $\Gamma(G)$ are not the same.*

Proof. We consider the following cases:
(1) Let $S = {}^3D_4(q')$. Since $t({}^3D_4(q')) = 3$ or 2, so $t(G) = 3$ or 2. Therefore, $n = 3$ or 4, which is a contradiction. Similarly $S \neq {}^2F_4(2')$ and $G_2(q')$.
(2) Let $S = E_8(q')$. We know that $s(S) \geq 4$, while $s(G) \leq 2$, which is a contradiction. Similarly $S \neq {}^2F_4(q')$, where $q' = 2^{2m+1}$, ${}^2B_2(q')$, where $q' = 2^{2m+1}$ and ${}^2G_2(q')$, where $q' = 3^{2m+1}$.
(3) Let $S = E_6(q')$. We know that $t(G) = t(S)$ and $t(S) = 5$. So either $[(3n+1)/4] = 5$ or $(3n+3)/4 = 5$ hence $n = 7$. We know that $\Gamma(S)$ has two components so $s(G) = 2$. Therefore, $G = D_7(q)$, where $q \in \{2, 3, 5\}$ or $G = D_8(q)$, where $q \in \{2, 3\}$, by [13, Tables 1a-1c]. Let $G = D_7(2)$, so $\pi(G) = \{2, 3, 5, 7, 11, 13, 17, 31, 127\}$. Hence, $|\pi(S)| = 9$. Therefore, $\beta = 1$, by order of S. Also we know that $\pi(S) = \pi(G)$, so $p' \in \pi(G)$. If $p' = 2$, then $73 \in \pi(S) \setminus \pi(G)$, which is a contradiction. Otherwise, $\pi(p'^{12} - 1) \subseteq \pi(S)$, while $\pi(p'^{12} - 1) \nsubseteq \pi(G)$, which is a contradiction. Similarly, if $G = D_7(q)$, where $q \in \{3, 5\}$ or $G = D_8(q)$, where $q \in \{2, 3\}$, then we get a contradiction.

Similarly S is not isomorphic to ${}^2E_6(q')$, $F_4(q')$ and $E_7(q')$.

□

Lemma 4.9. *Let $G = D_n(q)$, where $q = p^\alpha$ and $n > 4$, and also S be an alternating or sporadic group. Then $\Gamma(S)$ and $\Gamma(G)$ are not the same.*

Proof. We consider the following cases:
(1) Let $S = M_{22}$. Since $s(M_{22}) = 4$ and $s(G) \leq 2$, by [13, Tables 1a-1c], so we get a contradiction. Similarly $S \neq M_{11}$, M_{23}, M_{24}, J_1, J_3, J_4, Suz, Co_2, ON, HS, Ly, F_{23}, F'_{24}, $F_1 = M$, $F_2 = B$ and $F_3 = Th$.
(2) Let $S = M_{12}$. Since $t(M_{12}) = 3$, so $t(G) = 3$. Therefore, $n = 4$, which is a contradiction. Similarly $S \neq J_2$, He, McL and HN.
(3) Let $S = Ru$. Since $t(S) = t(G)$, so $n = 5$ or 6. On the other hand, $s(Ru) = 2$, so $G = D_5(q)$, where $q \in \{2, 3, 5\}$ or $G = D_6(q)$, where $q \in \{2, 3\}$, by [13, Tables 1a-1c]. Moreover, we know that $|\pi(G)| \geq 8$, while $|\pi(S)| \leq 6$, which is a contradiction. Similarly $S \neq Co_1$, Co_3 and Fi_{22}.
By [16], it is clear that S cannot be equal to an alternating group. □

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References

[1] A. Babai, B. Khosravi and N. Hasani, Quasirecognition by prime graph of ${}^2D_p(3)$ where $p = 2^n + 1 \geq 5$ is a prime, *Bull. Malays. Math. Sci. Soc. (2)* **32** (2009), no. 3, 343–350.
[2] A. Babai and B. Khosravi, Recognition by prime graph of ${}^2D_{2^m+1}(3)$, *Sib. Math. J.* **52** (2011), no. 5, 788–795.
[3] A. Babai and B. Khosravi, Quasirecognition by prime graph of ${}^2D_n(3^\alpha)$, where $n = 4m + 1 \geq 21$ and α is odd, *Math. Notes,* to appear.
[4] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, Atlas of Finite Groups, Oxford Univ. Press, Oxford, 1985.
[5] P. Crescenzo, A diophantine equation which arises in the theory of finite groups, *Adv. Math.* **17** (1975), no. 1, 25–29.
[6] M. Hagie, The prime graph of a sporadic simple group, *Comm. Algebra* **31** (2003), no. 9, 4405–4424.
[7] A. Khosravi and B. Khosravi, A new characterization of some alternating and symmetric groups (II), *Houston J. Math.* **30** (2004) 465–478.
[8] A. Khosravi and B. Khosravi, Quasirecognition by prime graph of the simple group ${}^2G_2(q)$, *Sib. Math. J.* **48** (2007), no. 3, 570–577.
[9] V. D. Mazurov and E. I. Khukhro (eds.), The Kourovka Notebook: Unsolved Problems in Group Theory, Sobolev Inst. Math. Novosibirsk, 16th edition, 2006.
[10] W. Sierpiński, Elementary Theory of Numbers, Monografie Matematyczne, 42, Panstwowe Wydawnictwo Naukowe, Warsaw, 1964.
[11] A. V. Vasil'ev and E. P. Vdovin, An adjacency criterion in the prime graph of a finite simple group, *Algebra Logic* **44** (2005), no. 6, 381–405.
[12] A. V. Vasil'ev and E. P. Vdovin, Cocliques of maximal size in the prime graph of a finite simple group, *Algebra Logic* **50** (2011), no. 4, 291–322.

[13] A. V. Vasil'ev and M. A. Grechkoseeva, On the recognition of the finite simple orthogonal groups of dimension 2^m, $2^m + 1$ and $2^m + 2$ over a field of characteristic 2, *Sib. Math. J.* **45** (2004), no. 3, 420–431.
[14] A. V. Zavarnitsin, On the recognition of finite groups by the prime graph, *Algebra Logic* **43** (2006), no. 4, 220–231.
[15] K. Zsigmondy, Zur theorie der potenzreste, *Monatsh. Math. Phys.* **3** (1892), no. 1, 265–284.
[16] M. A. Zvezdina, On nonabelian simple groups having the same prime graph as an alternating group, *Sib. Math. J.* **54** (2013), no. 1, 47–55.

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