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## POSITIVE SOLUTIONS OF $n$ TH-ORDER $m$ -POINT BOUNDARY VALUE PROBLEMS

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**ABSTRACT.** In this paper, by using four functionals fixed point theorem, we obtain sufficient conditions for the existence of at least one positive solution of an  $n$ th-order  $m$ -point boundary value problem. As an application, we give an example to demonstrate our main result.

**Keywords:** Fixed-point theorem, positive solution,  $m$ -point boundary value problem.

**MSC(2010):** Primary: 34B10; Secondary: 34B18.

### 1. Introduction

Boundary value problems (BVPs) for differential equations arise in a variety of different areas of applied mathematics and physics such as the deflection of a curved beam having a constant or varying cross section, electromagnetic waves or gravity driven flow and so on. The study of multi-point BVPs for second-order ordinary differential equations was initiated by Il'in and Moiseev [6, 7] and since then, such second order problems have been studied by several authors [1, 5, 8, 10, 13, 14, 16, 17]. Recently, there is an increasing interest in the literature on multi-point BVPs for higher-order differential equations, see for example [3, 4, 9, 11, 12, 15]. In particular, we would like to mention some results of Graef and Yang [3], Guo et al. [4], and Su and Wang [15].

In [4], Guo et al. studied the existence of at least three positive solutions for the nonlinear  $n$ th-order  $m$ -point BVP

$$\begin{cases} u^{(n)}(t) + f(t, u) = 0, & t \in (0, 1), \\ u(0) = 0, \quad u'(0) = \dots = u^{(n-2)}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} k_i u(\xi_i). \end{cases}$$

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Using the Leggett-Williams fixed point theorem and the Green's function, they get the existence of at least three positive solutions.

On the other hand, Graef and Yang [3] considered a higher-order multi-point BVP

$$\begin{cases} u^{(n)}(t) + \lambda g(t)f(u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \dots = u^{(n-3)}(0) = u^{(n-2)}(0) \\ = \sum_{i=1}^m a_i u^{(n-2)}(\xi_i) - u^{(n-2)}(1) = 0 \end{cases}$$

By using the Krasnosel'skii's fixed point theorem, the authors obtained criteria for the existence and nonexistence results for positive solutions for the problem.

In [15], Su and Wang studied the existence of positive solutions by means of fixed-point index theorem to the following singular semipositone  $m$ -point  $n$ th-order BVP

$$\begin{cases} (-1)^{(n-k)} x^{(n)}(t) = \lambda f(t, x(t)), & 0 < t < 1, \\ x(1) = \sum_{i=1}^{m-2} a_i x(\eta_i), & x^{(i)}(0) = 0, \quad 0 \leq i \leq k-1, \\ x^{(j)}(1) = 0, & 1 \leq j \leq n-k-1, \end{cases}$$

where  $m \geq 3$ ,  $\lambda > 0$ ,  $a_i \in [0, \infty)$ ,  $(i = 1, 2, \dots, m-2)$ .

Motivated by the results above, in this study, we consider the following  $n$ th-order  $m$ -point BVP

$$(1.1) \quad \begin{cases} u^{(n)}(t) + q(t)f(t, u(t)) = 0, & t \in [0, 1], \\ au^{(n-2)}(0) - bu^{(n-1)}(0) = \sum_{i=1}^{m-2} \alpha_i u^{(n-2)}(\xi_i), \\ cu^{(n-2)}(1) + du^{(n-1)}(1) = \sum_{i=1}^{m-2} \beta_i u^{(n-2)}(\xi_i), \\ u^{(j)}(0) = 0, & 0 \leq j \leq n-3. \end{cases}$$

where  $n \geq 3$ . By using the four functionals fixed point theorem [2], we get the existence of at least one positive solution for the BVP (1.1).

This paper is organized as follows. In Section 2, we provide some definitions and preliminary lemmas which are key tools for our main result. We give and prove our main result in Section 3. Finally, in Section 4, we give an example to demonstrate our main result.

## 2. Preliminaries

Throughout the paper, we assume that the following conditions hold:

- (C1)  $a, b, c, d \in [0, +\infty)$  with  $ac + ad + bc > 0$ ;  $\alpha_i, \beta_i \in [0, +\infty)$ ,  $\xi_i \in (0, 1)$  for  $i \in \{1, 2, \dots, m-2\}$ ,
- (C2)  $f \in \mathcal{C}([0, 1] \times \mathbb{R}_+, \mathbb{R}_+)$ , where  $\mathbb{R}_+ = [0, +\infty)$ ,  $q \in \mathcal{C}([0, 1], \mathbb{R}_+)$ .

We shall reduce problem (1.1) to an integral equation in  $\mathcal{C}([0, 1])$ . To this goal, firstly using the transformation

$$(2.1) \quad u^{(n-2)}(t) = y(t),$$

we convert BVP (1.1) into

$$(2.2) \quad \begin{cases} u^{(n-2)}(t) = y(t), & t \in [0, 1], \\ u^{(j)}(0) = 0, & j = 1, 2, \dots, n-3, \end{cases}$$

and

$$(2.3) \quad \begin{cases} -y''(t) + q(t)f(t, u(t)) = 0, & t \in [0, 1] \\ ay(0) - by'(0) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i), \\ cy(1) + dy'(1) = \sum_{i=1}^{m-2} \beta_i y(\xi_i). \end{cases}$$

**Lemma 2.1.** *If  $y \in \mathcal{C}([0, 1])$ , then BVP (2.2) has a unique solution  $u$  and  $u$  can be expressed in the form*

$$(2.4) \quad u(t) = \int_0^t \frac{(t-s)^{n-3}}{(n-3)!} y(s) ds.$$

*Proof.* The proof follows by routine calculations. □

Set

$$(2.5) \quad \Delta := \begin{vmatrix} -\sum_{i=1}^{m-2} \alpha_i (b + a\xi_i) & \rho - \sum_{i=1}^{m-2} \alpha_i (d + c(1 - \xi_i)) \\ \rho - \sum_{i=1}^{m-2} \beta_i (b + a\xi_i) & -\sum_{i=1}^{m-2} \beta_i (d + c(1 - \xi_i)) \end{vmatrix},$$

and

$$(2.6) \quad \rho := ad + ac + bc.$$

**Lemma 2.2.** *Let (C1) and (C2) hold. Assume that  $\Delta \neq 0$ . If  $y \in \mathcal{C}[0, 1]$  is a solution of the equation*

$$(2.7) \quad y(t) = \int_0^1 G(t, s) q(s) f(s, u(s)) ds + A(f)(b + at) + B(f)(d + c(1 - t)),$$

where

$$(2.8) \quad G(t, s) = \frac{1}{\rho} \begin{cases} (b + as)(d + c(1 - t)), & s \leq t, \\ (b + at)(d + c(1 - s)), & t \leq s, \end{cases}$$

$$(2.9) \quad A(f) = \frac{1}{\Delta} \left[ \begin{array}{cc} \sum_{i=1}^{m-2} \alpha_i \left( \int_0^1 G(\xi_i, s) q(s) f(s, u(s)) ds \right) & \rho - \sum_{i=1}^{m-2} \alpha_i (d + c(1 - \xi_i)) \\ \sum_{i=1}^{m-2} \beta_i \left( \int_0^1 G(\xi_i, s) q(s) f(s, u(s)) ds \right) & - \sum_{i=1}^{m-2} \beta_i (d + c(1 - \xi_i)) \end{array} \right],$$

and

$$(2.10) \quad B(f) = \frac{1}{\Delta} \left[ \begin{array}{cc} - \sum_{i=1}^{m-2} \alpha_i (b + a\xi_i) & \sum_{i=1}^{m-2} \alpha_i \left( \int_0^1 G(\xi_i, s) q(s) f(s, u(s)) ds \right) \\ \rho - \sum_{i=1}^{m-2} \beta_i (b + a\xi_i) & \sum_{i=1}^{m-2} \beta_i \left( \int_0^1 G(\xi_i, s) q(s) f(s, u(s)) ds \right) \end{array} \right],$$

then  $y$  is a solution of the BVP (2.3).

*Proof.* If  $y$  satisfies the integral equation (2.7), then we have

$$\begin{aligned} y(t) &= \int_0^t \frac{1}{\rho} (b + as)(d + c(1 - t))q(s)f(s, u(s)) ds \\ &\quad + \int_t^1 \frac{1}{\rho} (b + at)(d + c(1 - s))q(s)f(s, u(s)) ds + A(f)(b + at) \\ &\quad + B(f)(d + c(1 - t)), \\ y'(t) &= - \int_0^t \frac{c}{\rho} (b + as)q(s)f(s, u(s)) ds \\ &\quad + \int_t^1 \frac{a}{\rho} (d + c(1 - s))q(s)f(s, u(s)) ds + A(f)a - B(f)c. \end{aligned}$$

Thus

$$\begin{aligned} y''(t) &= \frac{1}{\rho} (-ad + ac + bc)q(t)f(t, u(t)) = -q(t)f(t, u(t)), \\ y''(t) + q(t)f(t, u(t)) &= 0. \end{aligned}$$

Since

$$\begin{aligned} y(0) &= \int_0^1 \frac{b}{\rho} (d + c(1 - s))q(s)f(s, u(s)) ds + A(f)b + B(f)(d + c), \\ y'(0) &= \int_0^1 \frac{a}{\rho} (d + c(1 - s))q(s)f(s, u(s)) ds + A(f)a - B(f)c, \end{aligned}$$

we get

$$\begin{aligned} ay(0) - by'(0) &= B(f)(ad + ac + bc) \\ &= \sum_{i=1}^{m-2} \alpha_i \left[ \int_0^1 G(\xi_i, s) q(s) f(s, u(s)) ds \right. \\ (2.11) \quad &\quad \left. + A(f)(b + a\xi_i) + B(f)(d + c(1 - \xi_i)) \right]. \end{aligned}$$

Since

$$\begin{aligned} y(1) &= \int_0^1 \frac{d}{\rho} (b + a(s)q(s)f(s, u(s))) ds + A(f)(b + a) + B(f)d, \\ y'(1) &= - \int_0^1 \frac{c}{\rho} (b + a(s)q(s)f(s, u(s))) + A(f)a - B(f)c, \end{aligned}$$

we have

$$\begin{aligned} cy(1) + dy'(1) &= A(f)(ad + ac + bc) \\ &= \sum_{i=1}^{m-2} \beta_i \left[ \int_0^1 G(\xi_i, s) q(s) f(s, u(s)) ds \right. \\ (2.12) \quad &\left. + A(f)(b + a\xi_i) + B(f)(d + c(1 - \xi_i)) \right]. \end{aligned}$$

Using the equations (2.6), (2.11) and (2.12), we get that

$$\left\{ \begin{aligned} &\left[ - \sum_{i=1}^{m-2} \alpha_i (b + a\xi_i) \right] A(f) + \left[ \rho - \sum_{i=1}^{m-2} \alpha_i (d + c(1 - \xi_i)) \right] B(f) \\ &= \sum_{i=1}^{m-2} \alpha_i \left( \int_0^1 G(\xi_i, s) q(s) f(s, u(s)) ds \right) \\ &\left[ \rho - \sum_{i=1}^{m-2} \beta_i (b + a\xi_i) \right] A(f) + \left[ - \sum_{i=1}^{m-2} \beta_i (d + c(1 - \xi_i)) \right] B(f) \\ &= \sum_{i=1}^{m-2} \beta_i \left( \int_0^1 G(\xi_i, s) q(s) f(s, u(s)) ds \right) \end{aligned} \right.$$

So, this implies that  $A(f)$  and  $B(f)$  satisfy (2.9) and (2.10), respectively.  $\square$

**Lemma 2.3.** Assume (C1) and (C2) hold. Assume

$$(C3) \quad \Delta < 0, \quad \rho - \sum_{i=1}^{m-2} \beta_i (b + a\xi_i) > 0, \quad a - \sum_{i=1}^{m-2} \alpha_i > 0.$$

Then for  $y \in C[0, 1]$  with  $f, q \geq 0$ , the solution  $y$  of the problem (2.3) satisfies

$$y(t) \geq 0 \text{ for } t \in [0, 1].$$

*Proof.* It is an immediate subsequence of the facts that  $G \geq 0$  on  $[0, 1] \times [0, 1]$  and  $A(f) \geq 0$ ,  $B(f) \geq 0$ .  $\square$

**Lemma 2.4.** Suppose (C1) – (C3) hold. Assume that

$$(C4) \quad c - \sum_{i=1}^{m-2} \beta_i < 0.$$

Then the solution  $y \in C[0, 1]$  of the problem (2.3) satisfies  $y'(t) \geq 0$  for  $t \in [0, 1]$ .

*Proof.* Assume that the inequality  $y'(t) < 0$  holds. Since  $y'(t)$  is nonincreasing on  $[0, 1]$ , one can verify that

$$y'(1) \leq y'(t), \quad t \in [0, 1].$$

From the boundary conditions of the problem (2.3), we have

$$-\frac{c}{d}y(1) + \frac{1}{d} \sum_{i=1}^{m-2} \beta_i y(\xi_i) \leq y'(t) < 0.$$

Therefore, we conclude that

$$\sum_{i=1}^{m-2} \beta_i y(1) < \sum_{i=1}^{m-2} \beta_i y(\xi_i) < cy(1),$$

i.e.,

$$\left( c - \sum_{i=1}^{m-2} \beta_i \right) y(1) > 0.$$

Using Lemma 2.3, we have  $y(1) \geq 0$ . So,  $c - \sum_{i=1}^{m-2} \beta_i > 0$ . However, this contradicts to condition (C4). Consequently,  $y'(t) \geq 0$  for  $t \in [0, 1]$ . □

Let the Banach space  $\mathbb{B} = \mathcal{C}([0, 1])$  be equipped with the norm  $\|y\| = \max_{t \in [0, 1]} |y(t)|$ , and we define a cone  $\mathcal{P}$  in  $\mathbb{B}$  by

$$(2.13) \quad \mathcal{P} = \{y \in \mathbb{B} : y(t) \text{ is nonnegative, nondecreasing and concave on } [0, 1]\}.$$

**Lemma 2.5.** *Let  $y \in \mathcal{P}$ . Then,*

$$\min_{t \in [\xi_1, \xi_{m-2}]} y(t) \geq \xi_1 \|y\|.$$

*Proof.* Since  $y \in \mathcal{P}$  we know that  $y(t)$  is nondecreasing on  $[0, 1]$ . So,  $\min_{t \in [\xi_1, \xi_{m-2}]} y(t) = y(\xi_1)$  and  $\|y\| = \max_{t \in [0, 1]} y(t) = y(1)$ . Since the graph of  $y$  is concave down on  $[0, 1]$ , we have

$$y(\xi_1) \geq \xi_1 y(1) + (1 - \xi_1)y(0).$$

So,  $y(\xi_1) \geq \xi_1 y(1)$ . The proof is complete. □

We define the operator  $T : \mathbb{B} \rightarrow \mathbb{B}$  by

$$(2.14) \quad (Ty)(t) = \int_0^1 G(t, s) F(s, y(s))q(s)ds + A(f)(b + at) + B(f)(d + c(1 - t)),$$

where  $F(t, y(t)) = f\left(t, \int_0^t \frac{(t-r)^{n-3}}{(n-3)!} y(r) dr\right)$ ,  $G$ ,  $A(f)$  and  $B(f)$  are defined as in (2.8), (2.9) and (2.10), respectively.

Solving BVP (1.1) is equivalent to finding fixed points of the operator  $T$  defined by (2.14).

### 3. Main result

Let  $\alpha$  and  $\Psi$  be nonnegative continuous concave functionals on  $\mathcal{P}$ , and let  $\beta$  and  $\Phi$  be nonnegative continuous convex functionals on  $\mathcal{P}$ , then for positive numbers  $r$ ,  $j$ ,  $l$  and  $R$ , we define the sets:

$$(3.1) \quad \begin{aligned} Q(\alpha, \beta, r, R) &= \{y \in \mathcal{P} : r \leq \alpha(y), \beta(y) \leq R\}, \\ U(\Psi, \tau) &= \{y \in Q(\alpha, \beta, r, R) : \tau \leq \Psi(y)\}, \\ V(\Phi, \nu) &= \{y \in Q(\alpha, \beta, r, R) : \Phi(y) \leq \nu\}. \end{aligned}$$

**Lemma 3.1.** [2] *If  $\mathcal{P}$  is a cone in a real Banach space  $\mathbb{B}$ ,  $\alpha$  and  $\Psi$  are nonnegative continuous concave functionals on  $\mathcal{P}$ ,  $\beta$  and  $\Phi$  are nonnegative continuous convex functionals on  $\mathcal{P}$  and there exist positive numbers  $r$ ,  $\tau$ ,  $\nu$  and  $R$ , such that*

$$T : Q(\alpha, \beta, r, R) \rightarrow \mathcal{P}$$

is a completely continuous operator, and  $Q(\alpha, \beta, r, R)$  is a bounded set. If

- (i)  $\{y \in U(\Psi, \tau) : \beta(y) < R\} \cap \{y \in V(\Phi, \nu) : r < \alpha(y)\} \neq \emptyset$ ;
- (ii)  $\alpha(Ty) \geq r$ , for all  $y \in Q(\alpha, \beta, r, R)$ , with  $\alpha(y) = r$  and  $\nu < \Phi(Tu)$ ;
- (iii)  $\alpha(Ty) \geq r$ , for all  $y \in V(\Phi, \nu)$ , with  $\alpha(y) = r$ ;
- (iv)  $\beta(Ty) \leq R$ , for all  $u \in Q(\alpha, \beta, r, R)$ , with  $\beta(y) = R$  and  $\Psi(Ty) < \tau$ ;
- (v)  $\beta(Ty) \leq R$ , for all  $y \in U(\Psi, \tau)$ , with  $\beta(y) = R$ .

Then  $T$  has a fixed point  $y$  in  $Q(\alpha, \beta, r, R)$ .

For the convenience, we take the notations

$$A = \frac{1}{\Delta} \begin{vmatrix} \sum_{i=1}^{m-2} \alpha_i \int_0^1 G(\xi_i, s) q(s) ds & \rho - \sum_{i=1}^{m-2} \alpha_i (d + c(1 - \xi_i)) \\ \sum_{i=1}^{m-2} \beta_i \int_0^1 G(\xi_i, s) q(s) ds & - \sum_{i=1}^{m-2} \beta_i (d + c(1 - \xi_i)) \end{vmatrix},$$

$$B = \frac{1}{\Delta} \begin{vmatrix} - \sum_{i=1}^{m-2} \alpha_i (b + a\xi_i) & \sum_{i=1}^{m-2} \alpha_i \int_0^1 G(\xi_i, s) q(s) ds \\ \rho - \sum_{i=1}^{m-2} \beta_i (b + a\xi_i) & \sum_{i=1}^{m-2} \beta_i \int_0^1 G(\xi_i, s) q(s) ds \end{vmatrix},$$



$$M = \int_{\xi_1}^{\xi_{m-2}} G(\xi_1, s) q(s) ds,$$

$$N = \int_0^1 G(1, s) q(s) ds + A(b+a) + Bd.$$

Let  $Q(\alpha, \beta, r, R)$ ,  $U(\Psi, \tau)$  and  $V(\Phi, \nu)$  be defined by (3.1).

**Theorem 3.2.** Assume (C1) – (C4) hold. Suppose that there exist constants  $r, \tau, \nu, R$  with  $0 < r < \tau \leq \nu < R$ ,  $\max\left\{\frac{r}{\nu}, \frac{\tau}{R}\right\} \leq \xi_1$ . If the function  $f$  satisfies the following conditions:

$$(C5) \quad f(t, u) \geq \frac{r}{M} \text{ for } (t, u) \in [\xi_1, \xi_{m-2}] \times \left[0, \frac{r}{\xi_1}\right];$$

$$(C6) \quad f(t, u) \leq \frac{R}{N} \text{ for } (t, u) \in [0, 1] \times [0, R].$$

Then the BVP (1.1) has at least one positive solution.

*Proof.* Define the maps

$$\alpha(y) = \Psi(y) = \min_{t \in [\xi_1, \xi_{m-2}]} y(t),$$

$$\beta(y) = \max_{t \in [0, 1]} y(t),$$

$$\Phi(y) = \max_{t \in [\xi_1, \xi_{m-2}]} y(t).$$

Then  $\alpha$  and  $\Psi$  are nonnegative continuous concave functionals on  $\mathcal{P}$ , and  $\beta$  and  $\Phi$  are nonnegative continuous convex functionals on  $\mathcal{P}$ . Since

$$\|y\| = \max_{t \in [0, 1]} |y(t)| = \beta(y) \leq R$$

for all  $y \in Q(\alpha, \beta, r, R)$ ,  $Q(\alpha, \beta, r, R)$  is a bounded set. Note that the operator  $T : Q(\alpha, \beta, r, R) \rightarrow \mathcal{P}$  is completely continuous by a standard application of the Arzela-Ascoli theorem.

Now, we verify that the remaining conditions of Lemma 3.1. Let

$$y_0 = \nu.$$

Clearly,  $y_0 \in \mathcal{P}$ . By direct calculation,

$$\alpha(y_0) = \min_{t \in [\xi_1, \xi_{m-2}]} y_0(t) = \nu > r,$$

$$\beta(y_0) = \max_{t \in [0, 1]} y_0^{(n-2)}(t) = \nu < R,$$

$$\Psi(y_0) = \min_{t \in [\xi_1, \xi_{m-2}]} y_0(t) = \nu \geq \tau,$$

$$\Phi(y_0) = \max_{t \in [\xi_1, \xi_{m-2}]} y_0(t) = \nu.$$

So,  $y_0 \in \{y \in U(\Psi, \tau) : \beta(y) < R\} \cap \{y \in V(\Phi, \nu) : r < \alpha(y)\}$ , which means that (i) in Lemma 3.1 is satisfied.

Now, we shall show that conditions (ii) and (iv) of Lemma 3.1 hold. By Lemma 2.5, we have

$$\alpha(Ty) = Ty(\xi_1) \geq \xi_1 \|Ty\| \geq \xi_1 \Phi(Ty) > \xi_1 \nu \geq r,$$

and for all  $y \in U(\Psi, \tau)$ , with  $\beta(y) = R$ ,

$$\beta(Ty) = \max_{t \in [0,1]} (Ty)(t) = (Ty)(1) \leq \frac{1}{\xi_1} (Ty)(\xi_1) = \frac{1}{\xi_1} \Psi(Ty) < \frac{1}{\xi_1} \tau \leq R.$$

So,  $\alpha(Ty) > r$  and  $\beta(Ty) < R$ . Hence (ii) and (iv) in Lemma 3.1 is fulfilled.

Now, using (C5), we shall verify that condition (iii) of Lemma 3.1 is satisfied. For any  $y \in V(\Phi, \nu)$ , with  $\alpha(y) = r$ , we have that  $\min_{t \in [\xi_1, \xi_{m-2}]} y(t) = r$  and

$$0 \leq \|y\| \leq \frac{1}{\xi_1} \min_{t \in [\xi_1, \xi_{m-2}]} y(t) = \frac{r}{\xi_1}. \text{ Since the following inequality holds}$$

$$(3.2) \quad \int_0^t \frac{(t-r)^{n-3}}{(n-3)!} y(r) dr \leq y(t) \leq \|y\|, \quad t \in [0, 1],$$

we have

$$\left( t, \int_0^t \frac{(t-r)^{n-3}}{(n-3)!} y(r) dr \right) \in [\xi_1, \xi_{m-2}] \times \left[ 0, \frac{r}{\xi_1} \right].$$

Then one has

$$\begin{aligned} \alpha(Ty) &= \min_{t \in [\xi_1, \xi_{m-2}]} (Ty)(t) = (Ty)(\xi_1) \\ &\geq \int_0^1 G(\xi_1, s) q(s) F(s, y(s)) ds \\ &\geq \frac{r}{M} \int_{\xi_1}^{\xi_{m-2}} G(\xi_1, s) q(s) ds = r. \end{aligned}$$

Finally, using (C6), we shall show that condition (v) of Lemma 3.1 is satisfied. For all  $y \in U(\Psi, \tau)$ , with  $\beta(y) = R$ , we have that  $0 \leq y(t) \leq \|y\| = R$  for  $t \in [0, 1]$ . From (3.2), we have

$$\left( t, \int_0^t \frac{(t-r)^{n-3}}{(n-3)!} y(r) dr \right) \in [0, 1] \times [0, R].$$

Thus,

$$\begin{aligned} \beta(Ty) &= \max_{t \in [0,1]} (Ty)(t) = (Ty)(1) \\ &= \int_0^1 G(1, s) F(s, y(s)) q(s) ds + A(f)(b+a) + B(f)d \\ &\leq \frac{R}{N} \left( \int_0^1 G(1, s) q(s) ds + A(b+a) + Bd \right) \\ &= R. \end{aligned}$$

Hence, by Lemma 3.1, the BVP (2.3) has at least one positive solution  $y$  such that  $r \leq \alpha(y)$ ,  $\beta(y) \leq R$  for  $t \in [0, 1]$ . Then the  $n$ th-order BVP (1.1) has at least one positive solution

$$u(t) = \int_0^t \frac{(t-r)^{n-3}}{(n-3)!} y(r) dr.$$

This completes the proof.  $\square$

#### 4. Numerical example

**Example 4.1.** Consider the following problem

$$(4.1) \quad \begin{cases} u'''(t) + \frac{7100}{32281}u + 100 = 0, & t \in [0, 1], \\ 4u'(0) - 8u''(0) = \frac{1}{4}u'\left(\frac{1}{6}\right) + \frac{1}{2}u'\left(\frac{1}{3}\right), \\ \frac{1}{12}u'(1) + 3u''(1) = \frac{1}{3}u'\left(\frac{1}{6}\right) + \frac{1}{12}u'\left(\frac{1}{3}\right), \\ u(0) = 0. \end{cases}$$

Then  $n = 3$ ,  $m = 4$ ,  $a = 4$ ,  $b = 8$ ,  $c = \beta_2 = \frac{1}{12}$ ,  $d = 3$ ,  $\alpha_1 = \frac{1}{4}$ ,  $\alpha_2 = \frac{1}{2}$ ,  $\beta_1 = \frac{1}{3}$ ,  $\xi_1 = \frac{1}{6}$ ,  $\xi_2 = \frac{1}{3}$ ,  $q(t) = 1$  and  $f(t, u) = \frac{7100}{32281}u + 100$ .

By simple calculation, we get  $\rho = 13$ ,  $\Delta = -\frac{6565}{72}$ ,  $A = \frac{8966633}{73738080}$ ,  $B = \frac{344201}{1536210}$ ,  $M = \frac{49}{96}$ ,  $N = \frac{32281}{7272}$  and

$$G(t, s) = \frac{1}{156} \begin{cases} (8 + 4s)(37 - t), & s \leq t, \\ (8 + 4t)(37 - s), & t \leq s. \end{cases}$$

Taking  $r = 49$ ,  $\tau = 404$ ,  $\nu = 1000$ ,  $R = 32281$ , we can obtain that  $0 < r < \tau \leq \nu < R$ ,  $\max\left\{\frac{r}{\nu}, \frac{\tau}{R}\right\} \leq \xi_1$ . It is clear that (C1) – (C4) are satisfied. Next, we show that (C5) and (C6) are also satisfied.

For  $(t, u) \in \left[\frac{1}{6}, \frac{1}{3}\right] \times [0, 294]$ , since  $f(t, u) \geq 100 \geq \frac{r}{M} = 96$ . So (C5) is satisfied.

For  $(t, u) \in [0, 1] \times [0, 32281]$ , since  $f(t, u) \leq 7200 \leq \frac{R}{N} = 7272$ . Hence (C6) holds. Then all the conditions in Theorem 3.2 are satisfied. Thus, BVP (4.1) has at least one positive solution.

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## REFERENCES

- [1] D. R. Anderson and R. Ma, Second-order  $n$ -point eigenvalue problems on time scales, *Adv. Difference Equ.* (2006), Article ID 59572, 17 pages.
- [2] R. Avery, J. Henderson and D. O'Regan, Four functionals fixed point theorem, *Math. Comput. Model.* **48** (2008), no. 7-8, 1081–1089.
- [3] J. R. Graef and B. Yang, Positive solutions to a multi-point higher order boundary value problem, *J. Math. Anal. Appl.* **316** (2006), no. 2, 409–421.
- [4] Y. Guo, Y. Ji and J. Zhang, Three positive solutions for a nonlinear  $n$ th-order  $m$ -point boundary value problem, *Nonlinear Anal.* **68** (2008) 3485–3492.
- [5] J. Henderson and R. Luca, Positive solutions for systems of nonlinear second-order multipoint boundary value problems, *Math. Methods Appl. Sci.* **37** (2014), no. 16, 2502–2516.
- [6] V. A. Il'in and E. I. Moiseev, A nonlocal boundary value problem of the second kind for the Sturm-Liouville operator, *Differ. Uravn.* **23** (1987), no. 8, 1422–1431.
- [7] V. A. Il'in and E. I. Moiseev, A nonlocal boundary value problem of the first kind for the Sturm-Liouville operator in differential and difference interpretations, *Differ. Uravn.* **23** (1987), no. 7, 1198–1207.
- [8] I. Y. Karaca and F. Tokmak, Existence of three positive solutions for  $m$ -point time scale boundary value problems on infinite intervals, *Dynam. Systems Appl.* **20** (2011), no. 2-3, 355–367.
- [9] I. Y. Karaca, Positive solutions of an  $n$ th order three-point boundary value problem, *Rocky Mountain J. Math.* **43** (2013) 205–224.
- [10] I. Y. Karaca and F. Tokmak, Eigenvalues for iterative systems of nonlinear  $m$ -point boundary value problems on time scales, *Bound. Value Probl.* **2014** (2014), no. 63, 17 pages.
- [11] L. Liu, B. Liu and Y. Wu, Nontrivial solutions for higher-order  $m$ -point boundary value problem with a sign-changing nonlinear term, *Appl. Math. Comput.* **217** (2010), no. 8, 3792–3800.
- [12] R. Luca, Positive solutions for a higher-order  $m$ -point boundary value problem, *Mediterr. J. Math.* **9** (2012), no. 2, 379–392.
- [13] R. Ma, Multiple positive solutions for nonlinear  $m$ -point boundary value problems, *Appl. Math. Comput.* **148** (2004), no. 1, 249–262.
- [14] Y. B. Sang, Z. Wei and W. Dong, Existence and uniqueness of positive solutions for second-order Sturm-Liouville and multi-point problems on time scales, *Bull. Korean Math. Soc.* **48** (2011), no. 5, 1047–1061.
- [15] H. Su and X. Wang, Positive solutions to singular semipositone  $m$ -point  $n$ -order boundary value problems, *J. Appl. Math. Comput.* **36** (2011), no. 1–2, 187–200.
- [16] F. Tokmak and I. Y. Karaca, Existence of symmetric positive solutions for a multipoint boundary value problem with sign-changing nonlinearity on time scales, *Bound. Value Probl.* **2013** (2013), no. 52, 12 pages.
- [17] Y. L. Zhao, H. B. Chen and X. G. Liu, Triple positive solutions of second-order  $m$ -point singular boundary value problems, *Math. Pract. Theory* **41** (2011), no. 1, 171–177.

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