ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 42 (2016), No. 6, pp. 1441–1450

Title:

On generalizations of semiperfect and perfect rings

Author(s):

Y. Mehmet Demirci

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 6, pp. 1441–1450
Online ISSN: 1735-8515

ON GENERALIZATIONS OF SEMIPERFECT AND PERFECT RINGS

Y. MEHMET DEMİRCİ

(Communicated by Omid Ali S. Karamzadeh)

Abstract. We call a ring R right generalized semiperfect if every simple right R-module is an epimorphic image of a flat right R-module with small kernel, that is, every simple right R-module has a flat B-cover. We give some properties of such rings along with examples. We introduce flat strong covers as flat covers which are also flat B-covers and give characterizations of A-perfect, B-perfect and perfect rings in terms of flat strong covers.
Keywords: Flat cover, flat B-cover, flat strong cover, G-semiperfect ring, perfect ring.
MSC(2010): Primary: 16D40; Secondary: 16L30.

1. Introduction

Throughout the article R denotes an associative ring with identity element and all modules considered are right R-modules unless stated otherwise. $J = J(R)$ denotes the Jacobson radical of the ring R and by a regular ring we mean a von Neumann regular ring. For a module M, $\mathrm{Rad}(M)$ denotes the Jacobson radical of M and we use the notation $N \ll M$ to indicate that N is a small submodule of M.

Amini et al. introduced in [1] flat covers of modules by weakening the condition of projectivity in the definition of a projective cover due to H. Bass (see [5]). For a module M, a flat cover of M is an epimorphism $f : F \longrightarrow M$ with F flat and $\mathrm{Ker}\, f \ll F$.

The term flat cover is introduced by E. Enochs (see [8]). A flat cover of a module M is a homomorphism $\alpha : F \longrightarrow M$ with the properties,

(i) F is a flat module;
(ii) for any homomorphism $\beta : F' \longrightarrow M$ with F' a flat module, there is a homomorphism $\gamma : F' \longrightarrow F$ satisfying $\alpha \circ \gamma = \beta$;

Article electronically published on December 18, 2016.
Received: 2 July 2015, Accepted: 15 September 2015.

(iii) if θ is an endomorphism of F with $\alpha \circ \theta = \alpha$, then θ is an automorphism.

To distinguish between these two concepts, we refer to the definition in [1] as a flat B-cover to indicate that it is in the sense of H. Bass, while the definition due to E. Enochs remains unchanged because of its common use.

We call a ring R right generalized semiperfect (right G-semiperfect for short) if every simple right R-module has a flat B-cover. We investigate some properties of G-semiperfect rings and give some examples of such rings.

We introduce flat strong covers of modules as flat covers which are also flat B-covers. They constitute a special case of uniqueness (up to isomorphism) of flat B-covers when the ring considered is right duo. In [2] and [6] A-perfect (B-perfect resp.) rings are introduced and characterized as the rings over which flat covers of cyclic (simple resp.) modules are projective. It turns out that A-perfect (B-perfect resp.) rings are exactly semilocal rings over which every cyclic (simple resp.) module has a flat strong cover. We also show that a ring R is perfect if and only if flat covers of semisimple modules are projective if and only if every semisimple module has a flat B-cover and flat covers of simple modules are projective.

2. **Generalized Semiperfect Rings**

Definition 2.1. For an R-module M, by a flat B-cover of M we mean an epimorphism $\psi : F \longrightarrow M$ where F is a flat R-module and $\operatorname{Ker}\psi \ll F$. In this case, we also say that F is a flat B-cover of M.

In [1], flat B-covers are mentioned as flat covers and a right G-perfect ring is defined to be the ring over which every right module has a flat B-cover. By restriction of this condition to simple right modules we give the following definition.

Definition 2.2. We call a ring R right generalized semiperfect (right G-semiperfect for short) if every simple right R-module has a flat B-cover. Left G-semiperfect rings are defined similarly. If R is both right and left G-semiperfect, we call R a G-semiperfect ring.

Before giving examples let us examine flat B-covers when the module at hand is simple. Let S be a simple right R-module which has a flat B-cover $\psi : F \longrightarrow S$. Since S is simple, there is an epimorphism $\phi : R \longrightarrow S$. Projectivity of R guaranties the existence of the homomorphism $f : R \longrightarrow F$ satisfying $\psi f = \phi$. Since $\operatorname{Ker}\psi \ll F$, f is an epimorphism and $F \cong R/I$ for some right ideal I of R. Then $S \cong R/M$ for some maximal right ideal M of R that contains I. Moreover, $F \cong R/I$ is local. Note that using similar arguments flat B-cover of a cyclic right module is necessarily cyclic.

Example 2.3. (a) Clearly, every G-perfect ring is G-semiperfect.

(b) Every flat module is a flat B-cover of itself, therefore every regular ring is G-perfect.

(c) Every perfect ring is G-perfect and every semiperfect ring is G-semiperfect.
(d) By the previous observation the ring of integers is not a G-semiperfect ring.

It is not true in general that for a flat B-cover F of R/M for a maximal right ideal M of R, $F \cong R/I$ for some right ideal I of R contained in M. The following lemma gives a condition under which this is true. The proof is straightforward, therefore it is omitted.

Lemma 2.4. *Let R be a ring and M be a maximal right ideal of R. If R is right G-semiperfect, then there is a flat B-cover of R/M isomorphic to R/I for some right ideal I contained in M.*

Proposition 2.5. *Let R be a right duo ring and M be a maximal right ideal of R. If R/I and R/K are flat B-covers of R/M for right ideals I and K contained in M, then $R/(I \cap K)$ is also a flat B-cover of R/M.*

Proof. Intersection of finitely many pure right ideals of a right duo ring is pure, therefore $R/(I \cap K)$ is a flat right R-module. Let $M/(I \cap K) + N/(I \cap K) = R/(I \cap K)$ for some right ideal N. We have $M + N = R$ and $M/I + (N+I)/I = R/I$. Then $N + I = R$, since $M/I \ll R/I$ and similarly we have $N + K = R$. Since R is right duo and I, K pure in R, $K = RK = (N + I)K = NK + IK \subseteq (N \cap K) + (I \cap K) \subseteq N$ and $R = N + K \subseteq N$. Hence $M/(I \cap K) \ll R/(I \cap K)$ and $R/(I \cap K)$ is a flat B-cover of R/M. □

The following example shows that the class of G-semiperfect rings differs from classes of G-perfect and semiperfect rings.

Example 2.6. (a) Let R be a semiperfect ring which is not right perfect. Then by [1, Theorem 2.3], R is a G-semiperfect ring which is not G-perfect.
(b) Let K be a regular ring and $R = \prod_{i=1}^{\infty} K_i$ with $K_i = K$ for $i = 1, 2, 3, \ldots$. Then R is a regular ring which is not semisimple (see [9, §10.4]), therefore R is a regular ring which is not semiperfect. Hence R is a G-semiperfect ring which is not semiperfect.

Theorem 2.7. *Let R and S be Morita equivalent rings. If R is right G-semiperfect, then so is S.*

Proof. It is a consequence of [3, Proposition 21.6] and [3, Exercise 22.12]. □

Proposition 2.8. *Let R and S be right G-semiperfect rings. Then,*
(a) *every factor ring of R is G-semiperfect,*
(b) *$R \times S$ is a G-semiperfect ring.*

Proof. (a) Let I be an ideal of R and $T = R/I$. For the simple T-module R/M, let $f : F \longrightarrow R/M$ be a flat B-cover of R/M as an R-module. Then $f(FI) = 0$ and there is a homomorphism $\bar{f} : F/FI \longrightarrow R/M$ induced by f.

$F/FI \cong F \otimes_R R/I \cong F \otimes_R T$ is a flat T-module and $\operatorname{Ker} \bar{f} = \operatorname{Ker} f/FI \ll F/FI$. Hence $\bar{f} : F/FI \longrightarrow R/M$ is a flat B-cover of R/M as a T-module.

(b) Let $T = R \times S$ and M be a simple T-module. Then $e = (1_R, 0) \in T$ is a central idempotent and $M = U \oplus V$ where $U = Me$ and $V = M(1-e)$. Since M is simple $U = 0$ or $U = M$. Without loss of generality we may assume that $U = M$, then $M = Me$ has an R-module structure since $M(0 \times S) = 0$. Every R-module is naturally a T-module, therefore M_R is simple. By assumption M_R has a flat B-cover $f : F_R \longrightarrow M$. Using [10, Theorem 4.24] we can easily show that F_T is flat. Since $\operatorname{Ker} f \ll F_R$, we have $\operatorname{Ker} f \ll F_T$. Therefore $f : F_T \longrightarrow M_T$ is a flat B-cover of M_T. □

Over a noetherian ring, finitely generated modules are finitely presented and so a flat B-cover of a simple module is also a projective cover. Therefore noetherian G-semiperfect rings are semiperfect. Moreover, we have the following result due to C. Lomp about G-semiperfect and semiperfect rings, we include it for completeness.

Theorem 2.9. *[11, Theorem 3.8] For a ring R the following statements are equivalent.*

(a) *R is semiperfect.*
(b) *R is semilocal and every simple R-module has a flat B-cover.*
(c) *R is semilocal and every finitely generated R-module has a flat B-cover.*

Proposition 2.10. *Let R be a commutative domain. Then R is G-semiperfect if and only if it is local.*

Proof. Let F be a flat B-cover of a simple module. Then F is local and torsion-free. Therefore $R \cong F$ is local. The converse is clear. □

Proposition 2.11. *Let R be a commutative ring and S be a multiplicatively closed subset of R such that every maximal ideal of the ring $S^{-1}R$ is of the form $S^{-1}M$ for some maximal ideal M of R. If R is G-semiperfect then so is $S^{-1}R$.*

Proof. Let U be maximal ideal of $S^{-1}R$, then $U = S^{-1}M$ for some maximal ideal M of R. Let R/I be a flat B-cover of R/M. Since R/I is a flat R-module, $S^{-1}R/S^{-1}I \cong S^{-1}(R/I)$ is a flat $S^{-1}R$-module. Let $(S^{-1}M/S^{-1}I) + (S^{-1}K/S^{-1}I) = S^{-1}R/S^{-1}I$ for some ideal K of R containing I. Then $S^{-1}(M+K) = S^{-1}M + S^{-1}K = S^{-1}R$ and so $(M+K) \cap S \neq \emptyset$. Since $M \cap S = \emptyset$, we have $M \subsetneq M + K$ and $M + K = R$. Since $M/I \ll R/I$ and $I \subseteq K$, we have $S^{-1}K = S^{-1}R$ and therefore $S^{-1}R/S^{-1}I$ is a flat B-cover of $S^{-1}R/S^{-1}M$. □

The following result is a consequence of Theorem 2.9 and Proposition 2.11.

Corollary 2.12. *Let R be a commutative G-semiperfect ring. Then $S^{-1}R$ is semiperfect for every finite number of maximal ideals $M_1, M_2, \cdots, M_n$ and $S = R\backslash \bigcup_{i=1}^{n} M_i$.*

Example 2.13. (Remark in [11]) Let $R = S^{-1}\mathbb{Z}$ with $S = \mathbb{Z}\backslash(p\mathbb{Z} \cup q\mathbb{Z})$ for prime numbers $p \neq q$. Then R is semilocal but not G-semiperfect.

Using the ideas in the proof of [1, Theorem 2.7] one can obtain the following result.

Proposition 2.14. *Let R be a right duo ring and $J(R/K)$ be nil for every pure right ideal K of R. If every cyclic right R-module has a flat B-cover, then R/J is regular.*

The question if $R[x]$ is G-semiperfect for a G-semiperfect ring R is answered as a corollary to the following proposition. Its proof is similar to the proof of [1, Proposition 2.4], therefore it is omitted.

Proposition 2.15. *Let R be a right G-semiperfect ring and $J(R)$ be nil. Then R is right noetherian if and only if R is right artinian.*

Corollary 2.16. *$R[x]$ is not a G-semiperfect ring for every commutative noetherian ring R.*

Proof. Let $R[x]$ be G-semiperfect for a commutative noetherian ring R. By [4, Exercise 1.4] J is nil. Then by the previous result it is artinian which is not true, since $xR[x]$ is not artinian. □

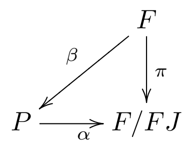
Proposition 2.17. *Let R be a ring. Then R is right G-semiperfect if and only if $R[[x]]$ is right G-semiperfect.*

Proof. Let R be a G-semiperfect ring, $T = R[[x]]$ and S be a simple T-module. Considering S as an R-module there is a flat B-cover $f : F \longrightarrow S$ of S_R. By [2, Lemma 3.13], $F \cong G/GI$ for some flat T-module G, where $I = xT$. Then G_R is local, since $I \leq \operatorname{Rad} T$ and G/GI is local and so G_T is local. For the isomorphism $g : G/GI \longrightarrow F$ and the canonical epimorphism $\pi : G \longrightarrow G/GI$, $f \circ g \circ \pi : G \longrightarrow S$ is a flat B-cover of S_T. The converse follows from Proposition 2.8 since $R \cong T/xT$. □

Using [1, Theorem 2.3] it is easy to see that a semilocal ring R is right perfect if and only if it is right G-perfect. Indeed, the condition in this result about flat B-covers can be weakened. The following result can be found in [7], we include it for future references.

Theorem 2.18. *(see [7, Theorem 4]) Let R be a semilocal ring. Then R is right perfect if and only if every semisimple R-module has a flat B-cover.*

Proof. Necessity part is clear. For sufficiency let F be a free right R-module. Since R/J is semisimple, F/FJ is a semisimple right R-module. By assumption F/FJ has a flat B-cover $\alpha : P \longrightarrow F/FJ$ for some flat right R-module P. Since F is projective, we have the commutative diagram



where $\pi : F \longrightarrow F/FJ$ is the canonical epimorphism. Since π is an epimorphism and $\operatorname{Ker}\alpha \ll P$, β is an epimorphism. Then we have $P \cong F/\operatorname{Ker}\beta$ and $\operatorname{Ker}\beta \leq \operatorname{Ker}\pi = FJ = \operatorname{Rad} F$. By [10, Exercise 4.20], $\operatorname{Ker}\beta = 0$, so β is an isomorphism. Then $FJ = \beta^{-1}(\operatorname{Ker}\alpha) \ll F$, since $\operatorname{Ker}\alpha \ll P$. By [3, Lemma 28.3], J is right T-nilpotent. Hence R is right perfect. □

If F is a flat B-cover of a module M, then any module F' isomorphic to F is also a flat B-cover of M. However, flat B-cover of a module need not be unique (up to isomorphism) in general (see [1, Example 3.1]). The following result is a consequence of this example. Let us recall that a ring R is called a right V-ring if every simple right R-module is injective and an unpublished result, which is not true in general, due to I. Kaplansky states that being regular and V-ring are equivalent conditions for a commutative ring.

Proposition 2.19. *Let R be a regular ring and flat B-covers of modules be unique (up to isomorphism), then R is a right V-ring.*

A special case of uniqueness of flat B-covers will be given in the next section under the hypothesis of R being right duo.

3. **Flat Strong Covers**

It is not true in general that every module has a flat B-cover. Even if a module has a flat B-cover, that cover need not be a flat cover. In this section, we investigate the case when flat covers are also flat B-covers.

Definition 3.1. A right R-module M is said to have a flat strong cover if a flat cover $f : F \longrightarrow M$ of M is also a flat B-cover. In this case, we also say that F is a flat strong cover of M.

A submodule of a module with flat strong cover need not have a flat strong cover in general. We have the following result in a special case. Recall that for a module M a homomorphism $\alpha : F \longrightarrow M$ satisfying the first two of conditions of a flat cover is called a flat precover of M.

Proposition 3.2. *Let R be a ring such that $\operatorname{Rad} M = MJ \ll M$ for any module M. Let $K \leq L$ and L/K be flat. If L has a flat strong cover, then so does K.*

Proof. Let $\phi : F \longrightarrow L$ be a flat strong cover of L. Then for $P = \phi^{-1}(K)$, $\phi' : P \longrightarrow K$ induced by ϕ is a flat precover of K by the proof of [13, Lemma 3.1.3]. By [13, Theorem 1.2.7] $P = X \oplus Y$ for submodules X and Y such that $\phi'_{|X} : X \longrightarrow K$ is a flat cover of K and $Y \leq \operatorname{Ker}\phi' = \operatorname{Ker}\phi$. Let $W + \operatorname{Ker}\phi'_{|X} = W + (\operatorname{Ker}\phi \cap X) = X$ for some submodule W of X. Then $\operatorname{Ker}\phi = \operatorname{Ker}\phi \cap P = \operatorname{Ker}\phi \cap (X+Y) = (\operatorname{Ker}\phi \cap X) + Y$ and $P = X + Y = W + (\operatorname{Ker}\phi \cap X) + Y = W + \operatorname{Ker}\phi$. Since $\operatorname{Ker}\phi \ll P$ by the proof of [1, Proposition 3.11], $P = W = X$ and hence $\phi' : P \longrightarrow K$ is a flat strong cover of K. □

Over a regular ring R which is not a V-ring, every module has a flat strong cover, but flat B-covers of modules need not be unique by Proposition 2.19. Example of such a ring can be found in [12, §2]. However, right duo rings with simple (cyclic resp.) modules having flat strong covers constitute a special case for uniqueness (up to isomorphism) of flat B-covers of simple (cyclic resp.) modules. Before proceeding we need the following lemma whose proof is straightforward and hence is omitted.

Lemma 3.3. *Let R be a right duo ring, B be a cyclic R-module and A be a small submodule of B. If B and B/A are flat, then $A = 0$.*

Proposition 3.4. *Let R be a right duo ring. If every simple module has a flat strong cover, then flat B-covers of simple modules are flat strong covers.*

Proof. Let $f : F \longrightarrow S$ be a flat strong cover and $g : P \longrightarrow S$ be a flat B-cover of simple module S for flat modules F and P. Since F is a flat cover, there is a homomorphism $h : P \longrightarrow F$ such that $g = fh$. h is an epimorphism, since $\operatorname{Ker} f \ll F$. Moreover, F and P are cyclic and $\operatorname{Ker} h \leq \operatorname{Ker} g \ll P$. Then $\operatorname{Ker} h = 0$ by Lemma 3.3 and so $P \cong F$. □

Flat B-covers of cyclic modules are cyclic, therefore the next result can be proved just by taking S to be cyclic in the previous proof.

Proposition 3.5. *Let R be right duo ring. If every cyclic module has a flat strong cover, then flat B-covers of cyclic modules are flat strong covers.*

Flat cover of a module need not be projective even if the module admits a projective cover. The following lemma gives a condition under which such a case occurs. Its proof is straightforward, we give it for completeness.

Lemma 3.6. *Let M be an R-module. Then flat cover of M is projective if and only if M has a projective cover and a flat strong cover.*

Proof. Necessity part is clear by [13, Theorem 1.2.12]. For sufficiency let $f : F \longrightarrow M$ be a flat strong cover of a right R-module M and $g : P \longrightarrow M$ be a projective cover of M. Then the homomorphism $h : P \longrightarrow F$ satisfying $g = fh$ is an epimorphism. Since P is projective, F is flat and $\operatorname{Ker} h \leq \operatorname{Ker} g \ll P$, F is also projective by [10, Exercise 4.20]. □

In [2] a ring R is called right A-perfect if every flat right R-module is projective relative to R. In [6] a ring R is called right B-perfect if for every flat module F and simple right R-module S, and homomorphisms $f : R \longrightarrow S$, $h : F \longrightarrow S$ there exists a homomorphism $g : F \longrightarrow R$ such that $h = fg$. Several characterizations of A-perfect and B-perfect rings are given in [2] and [6]. Both classes of rings are strictly contained in the class of semiperfect rings as shown in [2] and [6]. It turns out that they can be characterized as semilocal rings with certain type of modules having strong flat covers.

Theorem 3.7. *For a ring R the following statements are equivalent:*

(i) *Flat covers of cyclic modules are projective.*
(ii) *R is semilocal and every cyclic module has a flat strong cover.*

Proof. $(i)\Rightarrow(ii)$: R is semilocal by [2, Theorem 3.7]. If C is a cyclic R-module and $f : F \longrightarrow C$ is flat cover of C with F projective, then $\operatorname{Ker} f \ll F$ by [13, Theorem 1.2.12]. Hence $f : F \longrightarrow C$ is a flat strong cover of C.

$(ii)\Rightarrow(i)$: Let C be a cyclic module. R is semiperfect by [11, Theorem 3.8]. Therefore C has a projective cover. Then C has a projective cover and a flat strong cover. By Lemma 3.6, flat cover of C is projective. □

The following result can be proved by taking C to be simple in the previous proof.

Theorem 3.8. *For a ring R the following statements are equivalent:*

(i) *Flat covers of simple modules are projective.*
(ii) *R is semilocal and every simple module has a flat strong cover.*

The implications

$$A\text{-perfect} \Rightarrow B\text{-perfect} \Rightarrow \text{semiperfect} \Rightarrow \text{semilocal}$$

for rings are all strict. Examples for showing the first two implications are not invertible can be found in [2] and [6] and Example 2.13 can be given to show the converse of the last implication is not true.

Let R be a semiperfect ring which is not B-perfect. Then every simple module has a projective cover and therefore flat B-covers of simple modules are unique (up to isomorphism). However, there is a simple module that does not have a flat strong cover by Theorem 3.8. This shows that uniqueness of flat B-covers of simple modules does not imply that every simple module has a flat strong cover while the converse implication holds for right duo rings by Proposition 3.4.

Adding a condition about flat strong covers to semilocal rings we obtain classes that strictly contain class of semiperfect rings as the above results show. However, this is not the case for perfect rings. Next result explains this situation and gives characterizations of perfect rings in terms of flat covers and flat B-covers.

Theorem 3.9. *For a ring R the following statements are equivalent.*

(i) *R is right perfect.*
(ii) *Flat covers of semisimple modules are projective.*
(iii) *R is semilocal and every semisimple module has a flat strong cover.*
(iv) *Every semisimple module has a flat B-cover and flat covers of simple modules are projective.*
(v) *Every semisimple module has a flat B-cover and flat B-covers of simple modules are projective.*

Proof. $(i)\Rightarrow(ii)$ follows from the fact that flat modules are projective over a perfect ring.

$(ii)\Rightarrow(iii)$: R is semilocal by [6, Theorem 2.4]. If $f : F \longrightarrow X$ is a flat cover of a semisimple module X with F projective, then it is also a flat B-cover by [13, Theorem 1.2.12]. Hence every semisimple module has a flat strong cover.

$(iii)\Rightarrow(iv)$ is a consequence of Theorem 3.8.

$(iv)\Rightarrow(v)$: Let S be a simple module, $f : F \longrightarrow S$ be a flat B-cover and $g : P \longrightarrow S$ be a flat cover of S with P projective. Then there is a homomorphism $h : P \longrightarrow F$ such that $g = fh$. h is an epimorphism since $\operatorname{Ker} f \ll F$. By Lemma 3.6, $\operatorname{Ker} g \ll P$. Then we have $F \cong P/\operatorname{Ker} h$ with P projective, F flat and $\operatorname{Ker} h \leq \operatorname{Ker} g \ll P$. By [10, Exercise 4.20] $\operatorname{Ker} h = 0$ and so $F \cong P$ is projective.

$(v)\Rightarrow(i)$: Every simple module has a projective cover by [13, Theorem 1.2.12] in this case and so R is semiperfect. Theorem 2.18 completes the proof. □

Acknowledgements

The author would like to thank the referee for his/her careful reading of the manuscript, suggestions and comments which have improved the presentation of the paper.

References

[1] A. Amini, B. Amini, M. Ershad and H. Sharif, On generalized perfect rings, *Comm. Algebra* **35** (2007), no. 3, 953–963.

[2] A. Amini, M. Ershad and H. Sharif, Rings over which flat covers of finitely generated modules are projective, *Comm. Algebra* **36** (2008), no. 8, 2862–2871.

[3] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, Grad. Texts in Math., 13, Springer-Verlag, New York, 1992.

[4] M. F. Atiyah, and I. G. Macdonald, Introduction to Commutative Algebra, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969.

[5] H. Bass, Finitistic dimension and a homological generalization of semi-primary rings, *Trans. Amer. Math. Soc.* **95** (1960) 466–488.

[6] E. Büyükaşık, Rings over which flat covers of simple modules are projective, *J. Algebra Appl.* **11** (2012), no. 3, 1250046, 7 pages.

[7] N. Ding and J. Chen, On a characterization of perfect rings, *Comm. Algebra* **27** (1999), no. 2, 785–791.

[8] E. E. Enochs, Injective and flat covers, envelopes and resolvents, *Israel J. Math.* **39** (1981), no. 3, 189–209.
[9] F. Kasch, Modules and Rigs, London Math. Soc. Monogr. Ser. 17, Translated from German and edited by D. A. R. Wallace, Academic Press, London- New York, 1982.
[10] T. Y. Lam, Lectures on Modules and Rings, Grad. Texts in Math., 189, Springer-Verlag, New York, 1999.
[11] C. Lomp, On semilocal modules and rings, *Comm. Algebra* **27** (1999), no. 4, 1921–1935.
[12] A. K. Srivastava, On Σ-V rings, *Comm. Algebra* **39** (2011), no. 7, 2430–2436.
[13] J. Xu, Flat Covers of Modules, Lecture Notes in Math., 1634, Springer-Verlag, Berlin, 1996.

(Yılmaz Mehmet Demirci) Department of Mathematics, Sinop University, 57000, Sinop, Turkey.
E-mail address: ymdemirci@sinop.edu.tr