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EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTION FOR SIXTH-ORDER BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we formulate the sixth-order boundary value problem as Fredholm integral equation by finding Green's function and obtain the sufficient conditions for existence and multiplicity of positive solution for this problem. Also nonexistence results are obtained. An example is given to illustrate the results of paper.
Keywords: Sixth order boundary value problem, positive solution, Green's function.
MSC(2010): Primary: 34B15; Secondary: 34B18, 34B27.

1. Introduction

Boundary value problems arise in many applications of engineering and sciences, see [1,2,7] and references there in for more details. Recently some papers studied the existence, nonexistence and multiplicity of positive solution for ordinary and fractional boundary value problems [6, 8, 10, 11]. In this paper, we investigate existence, nonexistence and multiplicity results of positive solutions for nonlinear sixth-order boundary value problem (SBVP) of the form

$$-u^{(6)}(t) = \lambda f(t, u(t)), \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = u'(0) = u''(0) = 0, \quad u(1) = u'(1) = u''(1) = 0. \tag{1.2}$$

If λ is a number for which the equation (1.1) with boundary condition (1.2) has a nontrivial solution, then λ is called an eigenvalue and nontrivial solution for that λ is called an eigenfunction. Spectral problems for differential equations arise in many different physical applications. SBVP arise in astrophysics, i.e., the narrow convecting layers bounded by stable layers which are believed to surround A-type stars may be modeled by sixth-order boundary

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value problems, also this problem arise in hydrodynamic and magnetohydrodynamic stability theory [3,4,7]. Wuest [9] derived a model for beams and pipes that the resulting differential equation after separation of variables leads to a sixth-order differential equation. Also, sixth order boundary value problems appear in vibrations of circular arches [1].

2. **Preliminaries**

We find the integral representation of the SBVP. Indeed we show that the SBVP equivalent to a nonlinear homogeneous fredholm integral equation.

Lemma 2.1. *Suppose that $f(t,x) \in C([0,1] \times [0,\infty))$, then SBVP equivalent to integral equation*

$$u(t) = \lambda \int_0^1 G(t,s) f(s,u(s))ds, \tag{2.1}$$

where

$$G(t,s) = \begin{cases} G_1(t,s), & 0 \le s \le t \le 1 \\ G_2(t,s), & 0 \le t \le s \le 1 \end{cases}, \tag{2.2}$$

with

$$G_1(t,s) = \frac{1}{120} s^3 (1-t)^3 \{(6s^2 - 15s + 10)t^2 + (3s^2 - 5s)t + s^2\},$$

$$G_2(t,s) = \frac{1}{120} t^3 (1-s)^3 \{(6t^2 - 15t + 10)s^2 + (3t^2 - 5t)s + t^2\}.$$

The function $G(t,s)$ is Green's function of problem (1.1) and (1.2).

Proof. By multiple integration of equation (1.1), we obtain

$$u(t) = -\frac{1}{120} \lambda \int_0^t (t-s)^5 f(s,u(s))ds + \sum_{k=0}^{5} c_k t^k, \tag{2.3}$$

the boundary conditions $u(0) = u'(0) = u''(0) = 0$ implies that $c_0 = c_1 = c_2 = 0$. By applying the boundary conditions $u(1) = u'(1) = u''(1) = 0$, we obtain

$$\begin{cases} c_3 + c_4 + c_5 = \frac{\lambda}{120} \int_0^1 (1-s)^5 f(s,u(s))ds \\ 3c_3 + 4c_4 + 5c_5 = \frac{\lambda}{24} \int_0^1 (1-s)^4 f(s,u(s))ds \\ 6c_3 + 12c_4 + 20c_5 = \frac{\lambda}{6} \int_0^1 (1-s)^3 f(s,u(s))ds \end{cases}. \tag{2.4}$$

By solving the above system we obtain

$$\begin{aligned} c_3 &= \tfrac{\lambda}{120} \textstyle\int_0^1 \{-10(1-s)^5 + 20(1-s)^4 - 10(1-s)^3\} f(s,u(s))ds \\ c_4 &= \tfrac{\lambda}{120} \textstyle\int_0^1 \{15(1-s)^5 - 35(1-s)^4 + 20(1-s)^3\} f(s,u(s))ds \\ c_5 &= \tfrac{\lambda}{120} \textstyle\int_0^1 \{-6(1-s)^5 + 15(1-s)^4 - 10(1-s)^3\} f(s,u(s))ds \end{aligned}. \tag{2.5}$$

Thus

$$\begin{aligned} u(t) &= -\lambda \int_0^t \frac{(t-s)^5}{120} f(s,u(s))ds \\ &\quad + \lambda \int_0^t \frac{(1-s)^5}{120}[-6t^2+15t^4-10t^3] f(s,u(s))ds, \\ &\quad + \lambda \int_0^t \frac{(1-s)^4}{120}[15t^5-35t^4+20t^3] f(s,u(s))ds \\ &\quad + \lambda \int_0^t \frac{(1-s)^3}{120}[-10t^5+20t^4-10t^3] f(s,u(s))ds \\ &= \lambda \int_0^1 G(t,s) f(s,u(s))ds \end{aligned}$$

where $G(t,s)$ is defined by (2.2). Thus the SBVP is equivalent to integral equation (2.1). □

Lemma 2.2. *The Green's function $G(t,s)$ in (2.2) satisfies the following conditions*

(1) $G(t,s) > 0, \quad t,s \in (0,1)$

(2) $\max_{0\le t\le 1} G(t,s) = G(J(s),s)$, *where*

$$J(s) = \begin{cases} \frac{2-s+(1-s)\sqrt{4-6s}}{6s^2-15s+10}, & s \in (0,\frac{1}{2}) \\ \frac{2s+6s^2-s\sqrt{6s-2}}{6s^2+3s+1}, & s \in (\frac{1}{2},1) \end{cases},$$

(3) There exist a function $\gamma(s)$ such that

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} G(t,s) \ge \gamma(s)G(J(s),s) \ge \frac{398}{1565}G(J(s),s).$$

Proof. From (2.2), it is obvious that $G(t,s) > 0$ for $t,s \in (0,1)$.

$$\frac{\partial G_1}{\partial t} = -5s^3(1-t)^2\{(6s^2-15s+10)t^2+(2s-4)t+s\},\ t\in[s,1]$$

For $s \in (\frac{1}{2},1)$, $\frac{\partial G_1}{\partial t} < 0$ thus $G_1(t,s) \le G_1(s,s)$.
For $s \in (0,1/2), t \in (s,1)$ we have $J(s) \ge s$ and, $G_1(t,s) \le G(J(s),s)$. Thus

$$G_1(t,s) \le \begin{cases} G_1(J(s),s), & 0\le s\le \frac{1}{2} \\ G_1(s,s), & \frac{1}{2}\le s\le 1 \end{cases} \tag{2.6}$$

$$\frac{\partial G_2}{\partial t} = 5t^2(1-s)^3\{(6s^2+3s+1)t^2+(-12s^2-4s)t+6s^2\},\ t\in[0,s]$$

For $s \in (0,\frac{1}{2})$, $\frac{\partial G_2}{\partial t} > 0$ thus $G_2(t,s) \le G_2(s,s)$.
For $s \in (\frac{1}{2},1)$, $t \in [0,s]$ we have $J(s) \le s$ and $G_2(t,s) \le G_2(J(s),s)$. Thus

$$G_2(t,s) \le \begin{cases} G_2(s,s), & 0\le s\le \frac{1}{2} \\ G_2(J(s),s), & \frac{1}{2}\le s\le 1 \end{cases} \tag{2.7}$$

It is not difficult to verify that

$$\forall s \in [0, \frac{1}{2}],\ G_2(s,s) \leq G_1(J(s),s),$$

and

$$\forall s \in [\frac{1}{2}, 1],\ G_1(s,s) \leq G_2(J(s),s).$$

Also for $s \in [0, \frac{1}{2}]$, $J(s) \geq s$ and for $s \in [\frac{1}{2}, 1]$, $J(s) \leq s$, Therefore

$$\max_{0\leq t\leq 1} G(t,s) = G(J(s),s). \tag{2.8}$$

Now we prove case (3):

$$\begin{aligned}\min_{\frac{1}{4}\leq t\leq \frac{3}{4}} G(t,s) &= \begin{cases} \min G_1(t,s), & s \in (0, \frac{1}{4}) \\ \min\{G_1(t,s), G_2(t,s)\}, & s \in (\frac{1}{4}, \frac{3}{4}) \\ \min G_2(t,s), & s \in (\frac{3}{4}, 1) \end{cases}, \\ &= \begin{cases} \min G_1(\frac{3}{4},s), & s \in (0, \frac{1}{4}) \\ \min\{G_1(\frac{3}{4},s), G_2(\frac{1}{4},s)\}, & s \in (\frac{1}{4}, \frac{3}{4}) \\ \min G_2(\frac{1}{4},s), & s \in (\frac{3}{4}, 1) \end{cases},\end{aligned}$$

by continuity of G we obtain that

$$\min_{\frac{1}{4}\leq t\leq \frac{3}{4}} G(t,s) = \begin{cases} G_1(\frac{3}{4},s), & s \in (0, \frac{1}{2}) \\ G_2(\frac{1}{4},s), & s \in (\frac{1}{2}, 1) \end{cases}.$$

If we define

$$\gamma(s) = \begin{cases} \frac{G_1(\frac{3}{4},s)}{G(J(s),s)}, & s \in (0, \frac{1}{2}) \\ \frac{G_2(\frac{1}{4},s)}{G(J(s),s)}, & s \in (\frac{1}{2}, 1) \end{cases}, \tag{2.9}$$

then $\min_{\frac{1}{4}\leq t\leq \frac{3}{4}} G(t,s) \geq \gamma(s)G(J(s),s)$. It is easy to verify that $\gamma(s) \geq \frac{398}{1565}$, therefore

$$\min_{\frac{1}{4}\leq t\leq \frac{3}{4}} G(t,s) \geq \gamma(s)G(J(s),s) \geq \frac{398}{1565} G(J(s),s). \tag{2.10}$$

□

For $u \in C[0,1]$, define $\| u \| = \max_{0\leq t\leq 1} | u(t) |$, and let

$$P = \{u \in C[0,1], u(t) \geq 0, \quad \min_{\frac{1}{4}\leq t\leq \frac{3}{4}} u(t) \geq \tau\|u\|\}, \quad \tau = \frac{398}{1565}.$$

Lemma 2.3. *Suppose that $f : [0,1] \times [0,\infty) \longrightarrow [0,\infty)$ is continuous. For $u \in C[0,1]$, define the operator T by*

$$Tu(t) = \lambda \int_0^1 G(t,s) f(s,u(s)) ds, \tag{2.11}$$

then $TP \subset P$, and $T : P \longrightarrow P$ is completely continuous.

Proof. Let $u \in P$, since $G(t,s) \geq 0,\ f(t,x) \geq 0$, thus

$$Tu(t) = \lambda \int_0^1 G(t,s)f(s,u(s))ds \geq 0.$$

$$\begin{aligned}\min_{\frac{1}{4}\leq t\leq \frac{3}{4}} Tu(t) &= \lambda \min_{\frac{1}{4}\leq t\leq \frac{3}{4}} \int_0^1 G(t,s)f(s,u(s))ds \\ &= \lambda \int_0^1 \min_{\frac{1}{4}\leq t\leq \frac{3}{4}} G(t,s)f(s,u(s))ds \\ &\geq \lambda\tau \int_0^1 \max_{0\leq t\leq 1} G(t,s)f(s,u(s))ds \\ &= \tau \max_{0\leq t\leq 1} \lambda \int_0^1 G(t,s)f(s,u(s))ds = \tau\|Tu\|.\end{aligned}$$

Thus $Tu \in P$. By continuity of $G(t,s)$ and $f(t,x)$, the operator T is continuous. We prove that T is bounded. Let $\Omega \subset P$ is bounded, i.e, there exist a positive number $M > 0$, such that for all $u \in \Omega$, $\quad \| u \| \leq M$. The function $f(t,x)$ is continuous, thus $f(t,x)$ bounded in $[0,1]\times[0,M]$. We suppose that $\mid f(t,x) \mid \leq L$ and we have

$$\begin{aligned}\|Tu\| &= \lambda \max_{0\leq t\leq 1} \int_0^1 G(t,s)f(s,u(s))ds \\ &\leq \lambda L \max_{0\leq t\leq 1} \int_0^1 G(t,s)ds = \lambda L \int_0^1 G(J(s),s)ds < \infty.\end{aligned}$$

Thus $T(\Omega)$ is bounded. Also $T\Omega$ is equicontinuous, thus by Arzela-Ascoli Theorem, $T : P \longrightarrow P$ is completely continuous. $\square$

Theorem 2.4. *[5] Let X be a real Banach space and $P \subset X$ be a cone in X. Assume Ω_1, Ω_2 are two open bounded subsets of X with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$ and $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \longrightarrow P$ be a completely continuous operator such that*

(1) $\forall u \in P \cap \partial\Omega_1$, $\|Tu\| \leq \|u\|$ and $\forall u \in P \cap \partial\Omega_2$, $\|Tu\| \geq \|u\|$, or

(2) $\forall u \in P \cap \partial\Omega_2$, $\|Tu\| \leq \|u\|$ and $\forall u \in P \cap \partial\Omega_1$, $\|Tu\| \geq \|u\|$,

Then T has a fixed point in $P \cap (\overline{\Omega_0}\ \Omega_0)$.

3. **Main results**

We introduce the following notations:

$$f_0 = \liminf_{u\longrightarrow 0^+} \min_{0\le t\le 1} \frac{f(t,u)}{u}, \qquad f_\infty = \liminf_{u\longrightarrow \infty} \min_{0\le t\le 1} \frac{f(t,u)}{u}$$

$$f^0 = \limsup_{u\longrightarrow 0^+} \max_{0\le t\le 1} \frac{f(t,u)}{u}, \qquad f^\infty = \limsup_{u\longrightarrow \infty} \max_{0\le t\le 1} \frac{f(t,u)}{u}$$

$$A = (\int_0^1 G(J(s),s)ds)^{-1}, \qquad B = (\tau \int_{\frac{1}{4}}^{\frac{3}{4}} G(J(s),s)ds)^{-1}$$

Theorem 3.1. *Suppose that $f : [0,1]\times[0,\infty) \longrightarrow [0,\infty)$ is continuous, Then SBVP has at least one positive solution on P in two cases:*

(1) For every $\lambda \in (\frac{B}{f_0}, \frac{A}{f^\infty})$, if $f_0, f^\infty \in (0,\infty)$ and $Af_0 > Bf^\infty$, or
(2) For every $\lambda \in (\frac{B}{f_\infty}, \frac{A}{f^0})$, if $f^0, f_\infty \in (0,\infty)$ and $Af_\infty > Bf^0$.

Proof. Let $Af_0 > Bf^\infty$ and $\lambda \in (\frac{B}{f_0}, \frac{A}{f^\infty})$, thus there exist $\epsilon > 0$, such that

$$\frac{B}{f_0 - \epsilon} < \lambda < \frac{A}{f^\infty + \epsilon}.$$

Since $f_0 \in (0,\infty)$, there exist $R_1 > 0$, such that for every $t \in [0,1]$ and $u \in [0,R_1]$

$$\frac{f(t,u)}{u} \ge f_0 - \epsilon \Longrightarrow f(t,u) \ge u(f_0 - \epsilon).$$

Define $\Omega_1 = \{u : \ \|u\| \le R_1\}$. Thus for every $u \in \partial\Omega_1 \cap P$, we have

$$Tu(t) = \lambda \int_0^1 G(t,s)F(s,u(s))ds \ge \lambda(f_0 - \epsilon)\int_0^1 G(t,s)u(s)ds,$$

$$\begin{aligned}\|Tu\| &\ge \lambda(f_0-\epsilon)\max_{0\le t\le 1}\int_0^1 G(t,s)u(s)ds\\ &= \lambda(f_0-\epsilon)\int_0^1 G(J(s),s)u(s)ds\\ &\ge \lambda(f_0-\epsilon)\int_{\frac{1}{4}}^{\frac{3}{4}} G(J(s),s)u(s)ds\\ &\ge \|u\|\lambda\tau(f_0-\epsilon)\int_{\frac{1}{4}}^{\frac{3}{4}} G(J(s),s)ds \ge \|u\|.\end{aligned}$$

Thus for $u \in P\cap\partial\Omega_1, \|Tu\| \ge \|u\|$. On the other hand, $f^\infty \in (0,\infty)$ thus

$$\exists R > 0, \forall t \in [0,1], \forall u \in [R,\infty), f(t,u) \le (f^\infty + \epsilon)u.$$

Let $R_2 = \max\{R_1 + 1, R\tau^{-1}\}$, then for all $u \in P \cap \partial\Omega_2$

$$\begin{aligned}\|Tu\| &= \lambda \max_{0\le t\le 1}\int_0^1 G(t,s)f(s,u(s))ds\\ &\le \lambda(f^\infty+\epsilon)\|u\|\max_{0\le t\le 1}\int_0^1 G(t,s)ds\\ &= \lambda(f^\infty+\epsilon)\|u\|\int_0^1 G(J(s),s)ds \le \|u\|.\end{aligned}$$

Thus using the Theorem 2.4, the SBVP has at least one positive solution in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$. The proof of case (2) is similar. □

In the following theorem we present the multiplicity and nonexistence of positive solution for SBVP.

Theorem 3.2. *Suppose that $f : [0,1] \times [0,\infty) \longrightarrow [0,\infty)$ is continuous, and $f(t,x)$ is nondecreasing with respect to the variable x. Also, there exist positive constants $R_2 > R_1$, such that*

$$\frac{BR_1}{\min_{\frac14\le t\le\frac34} f(t,\tau R_1)} < \lambda < \frac{AR_2}{\max_{0\le t\le 1} f(t,R_2)}, \tag{3.1}$$

Then the SBVP has at least two positive solutions u^*, w^* such that

$$R_1 \le \|u^*\| \le R_2,\quad \lim_{n\longrightarrow\infty} T^n u_0 = u^*,\quad u_0 = R_2,\quad \forall t \in [0,1],$$

and

$$R_1 \ge \|w^*\| \le R_2,\quad \lim_{n\longrightarrow\infty} T^n w_0 = w^*,\quad w_0 = R_1,\quad \forall t \in [0,1].$$

Theorem 3.3. *(i) If $f^0, f^\infty < \infty$, then for all $\lambda \in (0, \frac{A}{C})$,where $C = \sup_{0<u<\infty, t\in[0,1]} \frac{f(t,u)}{u}$, the SBVP has no positive solution,*

(ii) If $f_0, f_\infty < \infty$, then for all $\lambda > \frac{B}{M}$,where $M = \min_{0<u<\infty, t\in[0,1]} \frac{f(t,u)}{u}$, the SBVP has no positive solution,

Proof. Since f^0, f^∞, f_0 and f_∞ are finite, thus C and M exist finitely. The nonexistence results can be derive by contradiction. □

Example 3.4. Consider the following boundary value problem

$$-u^{(6)}(t) = \lambda(1+t^2)\frac{au^2+bu}{cu+d}, \tag{3.2}$$

with boundary conditions (1.2). The parameters a, b, c and d are real numbers such that the conditions (continuity, nonnegativity) of the function $f(t,u)$ in $[0,1] \times [0,\infty)$ hold. For continuity of the function $f(t,u) = (1+t^2)\frac{au^2+bu}{cu+d}$ with respect to variable u in $[0,\infty)$ both of the parameters c and d must be positive or negative (we suppose that c and d are positive). For nonnegativity of function $f(t,u)$ both of the parameters a and b must be positive. Of course some of these parameters can be zero, but for satisfying the conditions of theorem 3.1, we consider them nonzero. Thus, these parameters are real positive numbers.

We have

$$f_0 = \frac{b}{d}, \quad f_\infty = \frac{a}{c}, \quad f^0 = 2\frac{b}{d}, \quad f^\infty = 2\frac{a}{c}, \quad A = \frac{62693}{166}, \quad B = \frac{22902}{13}.$$

Also we suppose that $2adB < bcA$, then By Theorem 3.1 for $\lambda \in (\frac{Bd}{b}, \frac{Ac}{2a})$ the boundary value problem (3.2) has at least one positive solution.
The constants C and M in Theorem 3.3 are $2\frac{b}{d}$ and $2\frac{a}{c}$, respectively. Thus for all $\lambda \in (0, \frac{Ad}{2b}) \cup (\frac{Bc}{2a}, \infty)$ the boundary value problem has no positive solution. If we choose $a = 0.5$,$b = 10$, $c = 10$ and $d = 0.5$, then condition $2adB < bcA$ holds. Thus by Theorem 3.1 the boundary value problem (3.2) for $\lambda \in (\frac{11451}{130}, \frac{60427}{16})$ has at least one positive solution and by Theorem 3.3 for $\lambda \in (0, \frac{1539}{163}) \cup (\frac{458040}{13}, \infty)$ has no positive solution. By taking the limit of inequality (3.1) as $R_1 \to 0^+$ and $R_2 \to \infty$, we conclude that for $\lambda \in (\frac{39139}{113}, \frac{60427}{16})$ the boundary value problem (3.2) has at least two positive solutions.

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