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DIFFERENTIAL SUBORDINATION AND SUPERORDINATION RESULTS ASSOCIATED WITH THE WRIGHT FUNCTION

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Abstract. An operator associated with the Wright function is introduced in the open unit disk. Differential subordination and superordination results associated with this operator are obtained by investigating appropriate classes of admissible functions. In particular, some inequalities for modified Bessel functions are also obtained.
Keywords: Wright function, differential subordination and superordination, Hadamard product, Bessel and modified Bessel function of the first kind.
MSC(2010): 30C45, 33C10.

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the class of analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{S}(\mathbb{D})$ be the subclass of $\mathcal{H}(\mathbb{D})$ consisting of functions which are also univalent in $\mathbb{D}$. Moreover, for $a \in \mathbb{C}$ and $n \in \mathbb{N}$ consider

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(\mathbb{D}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots,\ z \in \mathbb{D}\},$$

with $\mathcal{H}_0 = \mathcal{H}[0,1]$ and $\mathcal{H}_1 = \mathcal{H}[1,1]$. We denote by $\mathcal{A}$ the class of the functions $\mathcal{H}[a,1]$ which are normalized by the condition $f(0) = 0 = f'(0) - 1$ and have representation of the form

$$f(z) = z + \sum_{n\geq 2} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

Given two functions $f \in \mathcal{H}(\mathbb{D})$ and $g \in \mathcal{H}(\mathbb{D})$, we say that f is subordinated to g in $\mathbb{D}$, and write $f(z) \prec g(z)$, if there exists a Schwarz function w analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < |z|$ for all $z \in \mathbb{D}$, such that $f(z) = g(w(z))$ for $z \in \mathbb{D}$. In particular, if g is univalent in $\mathbb{D}$, we have the following equivalence:

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$f(z) \prec g(z)$, $z \in \mathbb{D} \Longleftrightarrow f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. We denote by $\mathcal{Q}$ the class of functions q that are analytic and injective on $\overline{\mathbb{D}} \setminus \mathrm{E}(q)$, where

$$\mathrm{E}(q) = \left\{ \zeta \in \partial\mathbb{D} : \lim_{z \to \zeta} q(z) = \infty \right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} \setminus \mathrm{E}(q)$. Further let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$ with $\mathcal{Q}(0) \equiv \mathcal{Q}_0$ and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

The Wright function $W_{\lambda,\mu}(z)$ is defined by the series

$$W_{\lambda,\mu}(z) = \sum_{n \geq 0} \frac{z^n}{n!\,\Gamma(\lambda n + \mu)}, \quad \lambda > -1,\ \mu \in \mathbb{C}. \tag{1.2}$$

If $\lambda > -1$, the series (1.2) is absolutely convergent for all $z \in \mathbb{C}$, while for $\lambda = -1$ this series is absolutely convergent for $|z| < 1$. Moreover, for $\lambda > -1$, $W_{\lambda,\mu}$ is an entire function of z. The Wright functions was introduced by Wright in [22], and have been used widely in the asymptotic theory of partitions, in the Mikusinski operational calculus and in the theory of integral transforms of Hankel type. Recently these functions have appeared in the solution of partial differential equations of fractional order, and it was found that the corresponding Green functions can be represented in terms of the Wright function, see [16, 19].

If λ is a positive rational number, then the Wright function $W_{\lambda,\mu}$ can be represented in terms of more familiar generalized hypergeometric function, see [9, Sec. 2.1]. In particular, when $\lambda = 1$ and $\mu = \nu + \frac{b+1}{2}$ $(\nu, b \in \mathbb{C})$, the function $W_{1,\nu+\frac{b+1}{2}}\left(-cz^2/4\right)$ can be expressed in terms of the generalized Bessel function $w_{\nu,b,c}$, given by

$$w_{\nu,b,c}(z) = \left(\frac{z}{2}\right)^{\nu} W_{1,\nu+\frac{b+1}{2}}\left(-\frac{cz^2}{4}\right) = \sum_{n \geq 0} \frac{(-c)^n (z/2)^{2n+\nu}}{n!\,\Gamma\left(n + \nu + \frac{b+1}{2}\right)},$$

where $\nu, b, c, z \in \mathbb{C}$ and $c \neq 0$. Moreover, note that the generalized Bessel functions $w_{\nu,b,c}$ is the solution of differential equation

$$z^2 w''(z) + bzw'(z) + (cz^2 - \nu^2 + (1-b)\nu)w(z) = 0,$$

where z is a non-zero complex number. Further, observe that the function $w_{\nu,b,c}$ permits the study of Bessel, modified Bessel, spherical Bessel and modified spherical Bessel functions together. It is clear that for $c = 1$ and $b = 1$ the function $w_{\nu,b,c}$ reduces to J_ν, the Bessel function of the first kind of order ν; when $c = -1$ and $b = 1$ the function $w_{\nu,b,c}$ becomes I_ν, which is modified Bessel function of the first kind of order ν. Similarly, when $c = 1$ and $b = 2$ the function $w_{\nu,b,c}$ reduces to $2j_\nu/\sqrt{\pi}$, where j_ν is the spherical Bessel function of order ν; while if $c = -1$ and $b = 2$, then $w_{\nu,b,c}$ becomes $2i_\nu/\sqrt{\pi}$, where i_ν is the modified spherical Bessel function of order ν (see [5]). Also note that

$$W_{-\nu,1-\nu}(-z) = M_\nu(z) \ \ (0<\nu<1), W_{-\nu,0}(-z) = F_\nu(z) \ \ (0<\nu<1),$$
$$W_{-1/3,2/3}(z) = 3^{2/3}\mathrm{Ai}(-z/3^{1/3}),\ W_{-1/2,0}(z) = -\frac{z}{2\sqrt{\pi}}e^{-z^2/4},$$
$$W_{-1/2,1/2}(z) = -\frac{1}{\sqrt{\pi}}e^{-z^2/4},$$

where M_ν and F_ν are (Wright-type) entire auxiliary functions studied by [9,11] (see also [12]) and the function Ai is well known Airy function.

For $f\in\mathcal{A}$ given by (1.1) and g given by $g(z)=z+\sum_{n\geq 2} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by

$$(f*g)(z) = z+\sum_{n\geq 2} a_n b_n z^n = (g*f)(z), \quad z\in\mathbb{D}.$$

Note that $f*g\in\mathcal{A}$, also observe that the Wright function $W_{\lambda,\mu}\in\mathcal{H}[1/\Gamma(\mu),n]$, but does not belong to the class $\mathcal{A}$. Thus, to put the Wright function $W_{\lambda,\mu}$ in class $\mathcal{A}$, we consider here the following normalized form

$$\mathcal{W}_{\lambda,\mu}(z) = \Gamma(\mu)zW_{\lambda,\mu}\left(\frac{z}{4}\right) = \sum_{n\geq 0}\frac{\Gamma(\mu)\,z^{n+1}}{4^n\,n!\,\Gamma(\lambda\,n+\mu)},$$

where $\lambda>-1$, $\mu\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $z\in\mathbb{D}$. Note that the normalized Wright function $\mathcal{W}_{\lambda,\mu}$ was studied recently in [18].

Now, we define an operator $\mathbb{W}_{\lambda,\mu}$ as follows

$$\mathbb{W}_{\lambda,\mu}f(z) = \mathcal{W}_{\lambda,\mu}(z)*f(z) = z+\sum_{n\geq 2}\frac{\Gamma(\mu)a_n z^n}{4^{n-1}(n-1)!\Gamma(\lambda(n-1)+\mu)},$$

where $\lambda>-1$, $\mu\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $z\in\mathbb{D}$. Note that, if $f(z)=z/(1-z)$, then the operator $\mathbb{W}_{\lambda,\mu}f$ reduces to the functions

$$u_{\nu,b,c}(z) = \mathcal{W}_{1,\nu+\frac{b+1}{2}}(-cz)*\frac{z}{1-z} = (-c)2^\nu\,\Gamma(\nu+(b+1)/2)\,z^{1-\nu/2}\,w_{\nu,b,c}(\sqrt{z}),$$
$$g_\nu(z) = \mathcal{W}_{1,\nu+1}(-z)*\frac{z}{1-z} = (-1)2^\nu\,\Gamma(\nu+1)\,z^{1-\nu/2}J_\nu(\sqrt{z})$$

and

$$k_\nu(z) = \mathbb{W}_{1,\nu+1}\left(\frac{z}{1-z}\right) = \mathcal{W}_{1,\nu+1}(z)*\frac{z}{1-z} = 2^\nu\,\Gamma(\nu+1)\,z^{1-\nu/2}I_\nu(\sqrt{z}).$$

Note that the function $u_{\nu,b,c}$ was studied recently in [4,8,15] and g_ν was investigated in [7,17,21].

Let Ω and Δ be any set in $\mathbb{C}$, let p be an analytic function in $\mathbb{D}$ with $p(0)=1$ and let $\psi(r,s,t;z):\mathbb{C}^3\times\mathbb{D}\to\mathbb{C}$. Miller and Mocanu [13] studied implications of the form

$$\left\{\psi(p(z),zp'(z),z^2p''(z);\,z):\ z\in\mathbb{D}\right\}\subset\Omega \implies p(\mathbb{D})\subset\Delta. \tag{1.3}$$

If Δ is a simply connected domain containing the point a and $\Delta \neq \mathbb{C}$, then the Riemann mapping theorem ensures that there is a conformal mapping q of $\mathbb{D}$ onto Δ such that $q(0) = a$. In this case (1.3) can be rewritten as

$$\left\{\psi(p(z), zp'(z), z^2p''(z);\, z) : \, z \in \mathbb{D}\right\} \subset \Omega \implies p(z) \prec q(z). \tag{1.4}$$

Further, if Ω is a simply connected domain and $\Omega \neq \mathbb{C}$, then there is a conformal mapping h of $\mathbb{D}$ onto Ω such that $h(0) = \psi(a, 0, 0; 0)$. If in addition, the function $\psi(p(z), zp'(z), z^2p''(z);\, z)$ is analytic in $\mathbb{D}$, then (1.3) can be rewritten as

$$\psi(p(z), zp'(z), z^2p''(z);\, z) \prec h(z) \implies p(z) \prec q(z). \tag{1.5}$$

In this article, for suitably defined classes of admissible functions, involving the operator $\mathbb{W}_{\lambda,\mu}$, we study implications of the form (1.4) and (1.5). Through the simple algebraic check of admissible functions, we get various subordination, superordination and differential inequalities that would be difficult to obtain directly. Aghalary *et al.* [1], Ali *et al.* [2,3], Baricz *et al.* [6], Kim and Srivastava [10], Soni *et al.* [20] and Xiang *et al.* [23] have considered similar problem for various linear and multiplier operators. To prove our main results, we need the following definitions and lemmas.

Definition 1.1. [13, Definition 2.3a, p. 27] Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\Psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition

$$\psi(r, s, t; z) \notin \Omega,$$

whenever

$$r = q(\zeta),\ s = k\zeta q'(\zeta) \quad \text{and} \quad \Re\left(\frac{t}{s} + 1\right) \geq k\,\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right)$$

for $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}\backslash E(q)$ and $k \geq n$. In particular, $\Psi_1[\Omega, q] \equiv \Psi[\Omega, q]$.

Definition 1.2. [14, Definition 3, p. 817] Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those function $\psi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$ that satisfy the admissibility condition

$$\psi(r, s, t; \zeta) \in \Omega,$$

whenever

$$r = q(z), \quad s = \frac{zq'(z)}{m}, \quad \text{and} \quad \Re\left(\frac{t}{s} + 1\right) \leq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

for $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}\backslash E(q)$ and $m \geq n \geq 1$. In particular, $\Psi'_1[\Omega, q] \equiv \Psi'[\Omega, q]$.

Lemma 1.3. [13, Theorem 2.3b, p. 28] *Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If $p \in \mathcal{H}[a, n]$ satisfies*

$$\psi\left(p(z), zp'(z), z^2p''(z);\, z\right) \in \Omega,$$

then $p(z) \prec q(z)$.

Lemma 1.4. [14, Theorem 1, p. 887] *Let* $\psi \in \Psi'_n[\Omega, q]$ *with* $q(0) = a$. *If* $p \in \mathcal{Q}(a)$ *and we have that* $\psi\left(p(z), zp'(z), z^2p''(z); z\right) \in \mathcal{S}(\mathbb{D})$, *then*

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in \mathbb{D}\right\}$$

implies $q(z) \prec p(z)$.

2. Subordination results for the operator $\mathbb{W}_{\lambda,\mu}$

First we define the following class of admissible functions.

Definition 2.1. Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}_0 \cap \mathcal{H}_0$. The class of admissible functions $\Phi_H[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), \quad v = \frac{(\mu - \lambda - 1)q(\zeta) + k\lambda\zeta\, q'(\zeta)}{\mu - 1}$$

and

$$\Re\left(\frac{(\mu-1)(\mu-2)w - (\mu-\lambda-1)(\mu-\lambda-2)u}{\lambda((\mu-1)v - (\mu-\lambda-1)u)} - \frac{2\mu-3}{\lambda} + 2\right) \geq k\,\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right)$$

for $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}\backslash E(q)$, $k \geq 1$, $\lambda > -1$, $\mu - 2 \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Our first main result is the following theorem.

Theorem 2.2. *Let* $\lambda > -1$, $\mu - 2 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ *and* $\phi \in \Phi_H[\Omega, q]$. *If* $f \in \mathcal{A}$ *satisfies*

$$\left\{\phi\left(\mathbb{W}_{\lambda,\mu}f(z), \mathbb{W}_{\lambda,\mu-1}f(z), \mathbb{W}_{\lambda,\mu-2}f(z); z\right) : z \in \mathbb{D}\right\} \subset \Omega, \tag{2.1}$$

then

$$\mathbb{W}_{\lambda,\mu}f(z) \prec q(z), \quad z \in \mathbb{D}.$$

Proof. Let us consider the analytic function $p : \mathbb{D} \to \mathbb{C}$, defined by $p(z) = \mathbb{W}_{\lambda,\mu}f(z)$. By using the recurrence relation

$$\lambda z(\mathbb{W}_{\lambda,\mu}f(z))' = (\mu - 1)\mathbb{W}_{\lambda,\mu-1}f(z) - (\mu - \lambda - 1)\mathbb{W}_{\lambda,\mu}f(z)$$

we get

$$\mathbb{W}_{\lambda,\mu-1}f(z) = \frac{\lambda zp'(z) + (\mu - \lambda - 1)p(z)}{\mu - 1}$$

and

$$\mathbb{W}_{\lambda,\mu-2}f(z) = \frac{\lambda^2z^2p''(z) + \lambda(2\mu - \lambda - 3)zp'(z) + (\mu - \lambda - 1)(\mu - \lambda - 2)p(z)}{(\mu - 1)(\mu - 2)}.$$

Now, let us define the transformation from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \qquad v = \frac{\lambda s + (\mu - \lambda - 1)r}{\mu - 1}$$

and

$$w = \frac{\lambda^2 t + \lambda(2\mu - \lambda - 3)s + (\mu - \lambda - 1)(\mu - \lambda - 2)r}{(\mu - 1)(\mu - 2)}.$$

Let

$$\begin{aligned}
&\psi(r, s, t; z) = \phi(u, v, w; z)\\
&= \phi\left(r, \frac{\lambda s + (\mu - \lambda - 1)r}{\mu - 1}, \frac{\lambda^2 t + \lambda(2\mu - \lambda - 3)s + (\mu - \lambda - 1)(\mu - \lambda - 2)r}{(\mu - 1)(\mu - 2)};\ z\right).
\end{aligned}$$

By using the above equations we obtain

$$\psi\left(p(z), zp'(z), z^2 p''(z); z\right) = \phi\left(\mathbb{W}_{\lambda,\mu} f(z), \mathbb{W}_{\lambda,\mu-1} f(z), \mathbb{W}_{\lambda,\mu-2} f(z);\ z\right)$$

and hence (2.1) becomes

$$\psi\left(p(z), zp'(z), z^2 p''(z); z\right) \in \Omega.$$

The proof is completed, if it can be shown that the admissibility condition for $\phi \in \Phi_H[\Omega, q]$ is equivalent to the admissibility condition as given in Definition 2.1. On the other hand, we note that

$$\frac{t}{s} + 1 = \frac{(\mu - 1)(\mu - 2)w - (\mu - \lambda - 1)(\mu - \lambda - 2)u}{\lambda((\mu - 1)v - (\mu - \lambda - 1)u)} - \frac{2\mu - 3}{\lambda} + 2$$

and hence $\psi \in \Psi[\Omega, q]$. By Lemma 1.3, we obtain $p \prec q$. This completes the proof of Theorem 2.2. $\square$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω. In this case, the class $\Phi_H[h(\mathbb{D}), q]$ is written as $\Phi_H[h, q]$. The following result is an immediate consequence of Theorem 2.2.

Corollary 2.3. *Let $\lambda > -1$, $\mu - 2 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\phi \in \Phi_H[h, q]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\mathbb{W}_{\lambda,\mu} f(z), \mathbb{W}_{\lambda,\mu-1} f(z), \mathbb{W}_{\lambda,\mu-2} f(z); z\right) \prec h(z), \tag{2.2}$$

then

$$\mathbb{W}_{\lambda,\mu} f(z) \prec q(z), \quad z \in \mathbb{D}.$$

Our next result is an extension of Corollary 2.3 to the case when the behavior of q on $\partial\mathbb{D}$ is not known.

Corollary 2.4. *Let $\lambda > -1$, $\mu - 2 \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\Omega \subset \mathbb{C}$ and $q \in \mathcal{S}(\mathbb{D})$ with $q(0) = 0$. Let $\phi \in \Phi_H[\Omega, q_\rho]$ for some $\rho \in (0, 1)$, where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\mathbb{W}_{\lambda,\mu} f(z), \mathbb{W}_{\lambda,\mu-1} f(z), \mathbb{W}_{\lambda,\mu-2} f(z); z\right) \in \Omega,$$

then

$$\mathbb{W}_{\lambda,\mu} f(z) \prec q(z).$$

Proof. Corollary 2.3 yields $\mathbb{W}_{\lambda,\mu} f(z) \prec q_\rho(z)$. The result is now deduced from the relationship $q_\rho(z) \prec q(z)$, $z \in \mathbb{D}$. $\square$

The following main result is similar to Corollary 2.3.

Theorem 2.5. *Let $h, q \in \mathcal{H}(\mathbb{D})$ with $q(0) = 0$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\lambda > -1$, $\mu - 2 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfies one of the following conditions:*

(i) $\phi \in \Phi_H[h, q_\rho]$ for some $\rho \in (0, 1)$, or

(ii) there exist $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_H[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{A}$ satisfies (2.2), then

$$\mathbb{W}_{\lambda,\mu} f(z) \prec q(z), \quad z \in \mathbb{D}.$$

Proof. The proof is similar to the proof of [13, Theorem 2.3d, p. 30] and is therefore omitted. □

The next theorem yields the best dominant of the differential subordination.

Theorem 2.6. *Let $h \in \mathcal{S}(\mathbb{D})$, $\lambda > -1$, $\mu - 2 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), \frac{\lambda z q'(z) + (\mu - \lambda - 1)q(z)}{\mu - 1}, \frac{\lambda^2 z^2 q''(z) + \lambda(2\mu - \lambda - 3)zq'(z) + (\mu - \lambda - 1)(\mu - \lambda - 2)q(z)}{(\mu - 1)(\mu - 2)}; z\right) = h(z)$$

has a solution q with $q(0) = 0$ and satisfies one of the following conditions:

(i) $q \in \mathcal{Q}_0$ and $\phi \in \Phi_H[h, q]$,

(ii) $q \in \mathcal{S}(\mathbb{D})$ and $\phi \in \Phi_H[h, q_\rho]$, for some $\rho \in (0, 1)$, or

(iii) $q \in \mathcal{S}(\mathbb{D})$ and there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_H[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{A}$ satisfies (2.2), then

$$\mathbb{W}_{\lambda,\mu} f(z) \prec q(z), \quad z \in \mathbb{D}$$

and q is the best dominant.

Proof. By applying Corollary 2.3 and Theorem 2.6, we deduce that q is a dominant of (2.2). Since q satisfies the subordination $\mathbb{W}_{\lambda,\mu} f(z) \prec q(z)$, it is also a solution of (2.2) and therefore q will be dominated by all dominants of (2.2). Hence q is the best dominant. □

In the particular case $q(z) = Mz$, $M > 0$, and in view of Definition 2.1, the class of admissible functions $\Phi_H[\Omega, q]$ denoted by $\Phi_H[\Omega, M]$, is described below.

Definition 2.7. Let Ω be a set in $\mathbb{C}$ and $M > 0$. Suppose also that $\lambda > -1$, $\mu - 2 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $k \geq 1$. Then the class of admissible functions $\Phi_H[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ such that

(2.3)
$$\phi\left(Me^{i\theta}, \frac{\lambda k + \mu - \lambda - 1}{\mu - 1} Me^{i\theta}, \frac{\lambda^2 L + (k\lambda(2\mu - \lambda - 3) + (\mu - \lambda - 1)(\mu - \lambda - 2))\, Me^{i\theta}}{(\mu - 1)(\mu - 2)}; z\right) \notin \Omega,$$

whenever $z \in \mathbb{D}$ and $\Re(Le^{-i\theta}) \geq Mk(k-1)$ for all real θ.

Corollary 2.8. *Let $\lambda > -1$, $\mu - 2 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\phi \in \Phi_H(\Omega, M)$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\mathbb{W}_{\lambda,\mu}f(z), \mathbb{W}_{\lambda,\mu-1}f(z), \mathbb{W}_{\lambda,\mu-2}f(z); z\right) \in \Omega,$$

then

$$|\mathbb{W}_{\lambda,\mu}f(z)| < M, \quad z \in \mathbb{D}.$$

In the special case $\Omega = q(\mathbb{D}) = \{\omega : |\omega| < M\}$, the class $\Phi_H[\Omega, M]$ is simply denoted by $\Phi_H[M]$ and thus Corollary 2.8 can be written in the following form.

Corollary 2.9. *Let $\lambda > -1$, $\mu - 2 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\phi \in \Phi_H[M]$. If $f \in \mathcal{A}$ satisfies*

$$|\phi\left(\mathbb{W}_{\lambda,\mu}f(z), \mathbb{W}_{\lambda,\mu-1}f(z), \mathbb{W}_{\lambda,\mu-2}f(z);\, z\right)| < M,$$

then

$$|\mathbb{W}_{\lambda,\mu}f(z)| < M, \quad z \in \mathbb{D}.$$

Corollary 2.10. *Let $M > 0$, $\lambda > -1$, $\mu - 1 \in \mathbb{C}\backslash\mathbb{Z}_0^-$, and $\Re(\frac{\mu-1}{\lambda} + \frac{k-1}{2}) \geq 0$. If $f \in \mathcal{A}$ satisfies $|\mathbb{W}_{\lambda,\mu-1}f(z)| < M$, then $|\mathbb{W}_{\lambda,\mu}f(z)| < M$ for $z \in \mathbb{D}$.*

Proof. This follows from Corollary 2.9 by taking $\phi(u, v, w; z) = v$. □

Corollary 2.11. *Let $\lambda > -1$, $\mu - 1 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $M > 0$ and let $f \in \mathcal{A}$ satisfy*

$$\left|\mathbb{W}_{\lambda,\mu-1}f(z) + \left(\frac{\lambda}{\mu-1} - 1\right)\mathbb{W}_{\lambda,\mu}f(z)\right| < \left|\frac{\lambda}{\mu-1}\right| M,$$

then

$$|\mathbb{W}_{\lambda,\mu}f(z)| < M, \quad z \in \mathbb{D}.$$

Proof. Let $\phi(u, v, w; z) = v + \left(\frac{\lambda}{\mu-1} - 1\right)u$ and $\Omega = h(\mathbb{D})$, where

$$h(z) = \frac{M\lambda}{\mu-1}z \quad \text{and} \quad M > 0.$$

To use Corollary 2.8, we need to show that $\phi \in \Phi_H[\Omega, M]$. That is, the admissibility condition in Definition 2.7 is satisfied. This follows since

$$\begin{aligned}&\left|\phi\left(Me^{i\theta}, \frac{\lambda k + \mu - \lambda - 1}{\mu - 1}Me^{i\theta},\right.\right.\\ &\qquad\left.\left.\frac{L\lambda^2 + \left[k\lambda(2\mu - \lambda - 3) + (\mu - \lambda - 1)(\mu - \lambda - 2)\right]Me^{i\theta}}{(\mu-1)(\mu-2)}; z\right)\right|\\ &\qquad = \left|\frac{\lambda}{\mu-1}\right| kM \geq \left|\frac{\lambda}{\mu-1}\right| M,\end{aligned}$$

whenever $z \in \mathbb{D}, \theta \in \mathbb{R}$ and $k \geq 1$. The required result now follows from Corollary 2.8. Moreover, Theorem 2.6 shows that the result is sharp. The differential equation $zq'(z) = Mz$ has a univalent solution $q(z) = Mz$. It follows from Theorem 2.6 that $q(z) = Mz$ is the best dominant. □

Definition 2.12. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_1 \cap \mathcal{H}_1$. The class of admissible functions $\Phi_{H,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; z) \notin \Omega,$$

whenever

$$u = q(\zeta), \quad v = \frac{\lambda}{\mu-2}\left(\frac{k\zeta q'(\zeta)}{q(\zeta)} + \frac{\mu-1}{\lambda}q(\zeta) - \frac{1}{\lambda}\right)$$

and

$$\Re\left(\frac{v(\mu-2)((\mu-2)(w-v)+1-w)}{\lambda(v(\mu-2)-u(\mu-1)+1)} - \frac{2(\mu-1)}{\lambda}u + \frac{\mu-2}{\lambda}v + \frac{1}{\lambda}\right)$$
$$\geq k\,\Re\left(1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right),$$

where $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}\backslash E(q)$, $\lambda > -1$, $\mu - 3 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $k \geq 1$.

The corresponding main result to the above definition reads as follows.

Theorem 2.13. *Let $\lambda > -1$, $\mu - 3 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\phi \in \Phi_{H,1}[\Omega, q]$. If $f \in \mathcal{A}$ satisfies*

$$\left\{\phi\left(\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)}, \frac{\mathbb{W}_{\lambda,\mu-2}f(z)}{\mathbb{W}_{\lambda,\mu-1}f(z)}, \frac{\mathbb{W}_{\lambda,\mu-3}f(z)}{\mathbb{W}_{\lambda,\mu-2}f(z)}; z\right) : z \in \mathbb{D}\right\} \subset \Omega,$$

then

$$\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)} \prec q(z), \quad z \in \mathbb{D}.$$

Proof. Consider the analytic function $p : \mathbb{D} \to \mathbb{C}$, defined by

$$p(z) = \frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)}, \quad z \in \mathbb{D}.$$

Logarithmic differentiation gives

$$\frac{\mathbb{W}_{\lambda,\mu-2}f(z)}{\mathbb{W}_{\lambda,\mu-1}f(z)} = \frac{\lambda}{\mu-2}\left(\frac{zp'(z)}{p(z)} + \frac{\mu-1}{\lambda}p(z) - \frac{1}{\lambda}\right)$$

and

$$\frac{\mathbb{W}_{\lambda,\mu-3}f(z)}{\mathbb{W}_{\lambda,\mu-2}f(z)} = \frac{\lambda}{\mu-3}\left(\frac{zp'(z)}{p(z)} + \frac{\mu-1}{\lambda}p(z) - \frac{2}{\lambda}\right.$$
$$\left. + \frac{\frac{z^2p''(z)}{p(z)} + \frac{zp'(z)}{p(z)} - \left(\frac{zp'(z)}{p(z)}\right)^2 + \frac{\mu-1}{\lambda}zp'(z)}{\frac{zp'(z)}{p(z)} + \frac{\mu-1}{\lambda}p(z) - \frac{1}{\lambda}}\right).$$

Now, we define the transformation from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \quad v = \frac{\lambda}{\mu-2}\left(\frac{s}{r} + \frac{\mu-1}{\lambda}r - \frac{1}{\lambda}\right)$$

and

$$w=\frac{\lambda}{\mu-3}\left(\frac{s}{r}+\frac{\mu-1}{\lambda}r-\frac{2}{\lambda}+\frac{\frac{t}{r}+\frac{s}{r}-\left(\frac{s}{r}\right)^2+\frac{\mu-1}{\lambda}s}{\frac{s}{r}+\frac{\mu-1}{\lambda}r-\frac{1}{\lambda}}\right).$$

Let

$$\begin{aligned}\psi(r,s,t;z)&=\phi(u,v,w;z)\\&=\phi\Bigg(r,\ \frac{\lambda}{\mu-2}\left(\frac{s}{r}+\frac{\mu-1}{\lambda}r-\frac{1}{\lambda}\right.,\\&\qquad\qquad\frac{\lambda}{\mu-3}\left(\frac{s}{r}+\frac{\mu-1}{\lambda}r-\frac{2}{\lambda}+\frac{\frac{t}{r}+\frac{s}{r}-\left(\frac{s}{r}\right)^2+\frac{\mu-1}{\lambda}s}{\frac{s}{r}+\frac{\mu-1}{\lambda}r-\frac{1}{\lambda}}\right);z\Bigg).\end{aligned}$$

Using the above equations it follows that

$$\psi\left(p(z),zp'(z),z^2p''(z);z\right)=\phi\left(\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)},\frac{\mathbb{W}_{\lambda,\mu-2}f(z)}{\mathbb{W}_{\lambda,\mu-1}f(z)},\frac{\mathbb{W}_{\lambda,\mu-3}f(z)}{\mathbb{W}_{\lambda,\mu-2}f(z)};z\right)$$

and consequently we have

$$\psi\left(p(z),zp'(z),z^2p''(z);z\right)\in\Omega.$$

The proof is completed if it can be shown that the admissibility condition $\phi\in\Phi_{H,1}[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.1. Note that

$$\frac{t}{s}+1=\frac{v(\mu-2)((\mu-2)(w-v)+1-w)}{\lambda(v(\mu-2)-u(\mu-1)+1)}-\frac{2(\mu-1)}{\lambda}u+\frac{\mu-2}{\lambda}v+\frac{1}{\lambda}.$$

Hence $\psi\in\Psi[\Omega,q]$, and by Lemma 1.3 we have $p(z)\prec q(z)$, which completes the proof. ☐

In the case when $\Omega\neq\mathbb{C}$ is a simply connected domain with $\Omega=h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω, the class $\Phi_{H,1}[h(\mathbb{D}),q]$ is written as $\Phi_{H,1}[h,q]$. The following result is an immediate consequence of the above theorem.

Corollary 2.14. *Let $\lambda>-1$, $\mu-3\in\mathbb{C}\backslash\mathbb{Z}_0^-$ and $\phi\in\Phi_{H,1}[h,q]$ with $q(0)=1$. If $f\in\mathcal{A}$ satisfies*

$$\phi\left(\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)},\frac{\mathbb{W}_{\lambda,\mu-2}f(z)}{\mathbb{W}_{\lambda,\mu-1}f(z)},\frac{\mathbb{W}_{\lambda,\mu-3}f(z)}{\mathbb{W}_{\lambda,\mu-2}f(z)};z\right)\prec h(z),$$

then

$$\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)}\prec q(z),\quad z\in\mathbb{D}.$$

In the particular case $q(z)=1+Mz$, $M>0$, the class of admissible functions $\Phi_{H,1}[\Omega,q]$ is simply denoted by $\Phi_{H,1}[\Omega,M]$.

Definition 2.15. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{H,1}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ such that

$$\phi\left(1 + Me^{i\theta}, 1 + \frac{k\lambda + (\mu-1)(1+Me^{i\theta})}{(\mu-2)(1+Me^{i\theta})}Me^{i\theta}, 1 + \frac{k\lambda + (\mu-1)}{(\mu-3)(1+Me^{i\theta})}Me^{i\theta}\right.$$
$$\left. + \frac{\lambda(M+e^{-i\theta})\left(L\lambda e^{-i\theta} + kM(\lambda+\mu-1) + (\mu-1)kM^2e^{i\theta}\right) - \lambda k^2M^2}{(\mu-3)(M+e^{-i\theta})\left((\mu-2)e^{-i\theta} + (\mu-1)M^2e^{i\theta} + (k\lambda + 2(\mu-1)M - 1)M\right)}; z\right) \notin \Omega, \tag{2.4}$$

whenever $z \in \mathbb{D}$, $\lambda > -1$, $\mu - 3 \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\Re(Le^{-i\theta}) \geq (k-1)kM$ for all real θ and $k \geq 1$.

Corollary 2.16. *Let $\lambda > -1$, $\mu - 3 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\Phi \in \phi_{H,1}[\Omega, M]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)}, \frac{\mathbb{W}_{\lambda,\mu-2}f(z)}{\mathbb{W}_{\lambda,\mu-1}f(z)}, \frac{\mathbb{W}_{\lambda,\mu-3}f(z)}{\mathbb{W}_{\lambda,\mu-2}f(z)}; z\right) \in \Omega,$$

then

$$\left|\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)} - 1\right| < M, \quad z \in \mathbb{D}.$$

In the special case $\Omega = q(\mathbb{D}) = \{\omega : |\omega - 1| < M\}$ the class $\Phi_{H,1}[\Omega, M]$ is denoted by $\phi_{H,1}[M]$ and Corollary 2.16 takes the following form.

Corollary 2.17. *Let $\lambda > -1$, $\mu - 3 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\phi \in \Phi_{H,1}[M]$. If $f \in \mathcal{A}$ satisfies*

$$\left|\phi\left(\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)}, \frac{\mathbb{W}_{\lambda,\mu-2}f(z)}{\mathbb{W}_{\lambda,\mu-1}f(z)}, \frac{\mathbb{W}_{\lambda,\mu-3}f(z)}{\mathbb{W}_{\lambda,\mu-2}f(z)}; z\right) - 1\right| < M,$$

then

$$\left|\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)} - 1\right| < M, \quad z \in \mathbb{D}.$$

Corollary 2.18. *Let $\lambda > -1$, $\mu - 2 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $M > 0$. If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{\mathbb{W}_{\lambda,\mu-2}f(z)}{\mathbb{W}_{\lambda,\mu-1}f(z)} - \left(\frac{\mu-1}{\mu-2}\right)\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)} + \frac{1}{\mu-2}\right| < \frac{M}{1+M}\left|\frac{\lambda}{\mu-2}\right|,$$

then

$$\left|\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)} - 1\right| < M, \quad z \in \mathbb{D}.$$

Proof. This follows from Corollary 2.16 by taking $\phi(u, v, w; z) = v - \frac{\mu-1}{\mu-2}(u - 1) - 1$ and $\Omega = h(\mathbb{D})$, where $h(z) = \left|\frac{\lambda}{\mu-2}\right|\frac{Mz}{1+M}$, $M > 0$. To use Corollary 2.16, we need to show that $\phi \in \Phi_{H,1}[M]$, that is, the admissibility condition given in Definition 2.15 is satisfied. This follows since

$$\left|\phi\left(1 + Me^{i\theta}, 1 + \frac{k\lambda + (\mu-1)(1+Me^{i\theta})}{(\mu-2)(1+Me^{i\theta})}Me^{i\theta}, 1 + \frac{k\lambda + (\mu-1)}{(\mu-2)(1+Me^{i\theta})}Me^{i\theta}\right.\right.$$

$$+\frac{\lambda(M+e^{-i\theta})\left[L\lambda e^{-i\theta}+kM(\lambda+\mu-1)+(\mu-1)kM^2e^{i\theta}\right]-\lambda k^2M^2}{(\mu-3)(M+e^{-i\theta})\left[(\mu-2)e^{-i\theta}+(\mu-1)M^2e^{i\theta}+(k\lambda+2(\mu-1)M-\lambda)M\right]};z\Bigg)\Bigg|$$

$$=\left|1+\frac{k\lambda+(\mu-1)(1+Me^{i\theta})}{(\mu-2)(1+Me^{i\theta})}Me^{i\theta}-\frac{\mu-1}{\mu-2}Me^{i\theta}-1\right|$$

$$=\left|\frac{Mk\lambda}{(\mu-2)(1+Me^{i\theta})}\right| \geq \frac{M}{1+M}\left|\frac{\lambda}{\mu-2}\right|,$$

for $z\in\mathbb{D}$, $\theta\in\mathbb{R}$ and $k\geq 1$. □

Further, taking $\lambda=1$ and $f(z)=\dfrac{z}{1-z}$ in Corollary 2.18, we get the next result.

Corollary 2.19. *Let $\mu-3\in\mathbb{C}\backslash\mathbb{Z}_0^-$ and $M>0$. If the modified Bessel function I_μ satisfies*

$$\left|\frac{\sqrt{z}I_{\mu-3}(\sqrt{z})}{I_{\mu-2}(\sqrt{z})}-\frac{\sqrt{z}I_{\mu-2}(\sqrt{z})}{I_{\mu-1}(\sqrt{z})}+2\right|<\frac{2M}{M+1},\quad z\in\mathbb{D},$$

then

$$\left|\frac{\sqrt{z}I_{\mu-2}(\sqrt{z})}{I_{\mu-1}(\sqrt{z})}-2(\mu-1)\right|<2|\mu-1|M,\quad z\in\mathbb{D}.$$

Definition 2.20. Let Ω be a set in $\mathbb{C}$ and $q\in\mathcal{Q}_1\cap\mathcal{H}_1$. The class of admissible functions $\Phi_{H,2}[\Omega,q]$ consists of those functions $\phi:\mathbb{C}^3\times\mathbb{D}\to\mathbb{C}$ that satisfy the admissibility condition

$$\phi(u,v,w;z)\notin\Omega,$$

whenever

$$u=q(\zeta),\quad v=q(\zeta)+\frac{k\lambda\zeta}{\mu-1}q'(\zeta)$$

and

$$\Re\left(\frac{\mu-2}{\lambda}\left(\frac{w-u}{v-u}-2\right)-\frac{1}{\lambda}\right)\geq k\,\Re\left(1+\frac{\zeta q''(\zeta)}{q'(\zeta)}\right),$$

where $z\in\mathbb{D}$, $\zeta\in\partial\mathbb{D}\backslash E(q)$, $\lambda>-1$, $\mu-2\in\mathbb{C}\backslash\mathbb{Z}_0^-$ and $k\geq 1$.

Theorem 2.21. *Let $\lambda>-1$, $\mu-2\in\mathbb{C}\backslash\mathbb{Z}_0^-$ and $\phi\in\Phi_{H,2}[\Omega,q]$. If $f\in\mathcal{A}$ satisfies*

$$\left\{\phi\left(\frac{\mathbb{W}_{\lambda,\mu}f(z)}{z},\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{z},\frac{\mathbb{W}_{\lambda,\mu-2}f(z)}{z};z\right):\ z\in\mathbb{D}\right\}\subset\Omega,$$

then

$$\frac{\mathbb{W}_{\lambda,\mu}f(z)}{z}\prec q(z),\quad z\in\mathbb{D}.$$

Proof. Consider the analytic function $p : \mathbb{D} \to \mathbb{C}$, defined by $p(z) = \mathbb{W}_{\lambda,\mu} f(z)/z$. Differentiating logarithmically with respect to z we obtain

$$\frac{\mathbb{W}_{\lambda,\mu-1} f(z)}{z} = p(z) + \frac{\lambda}{\mu - 1} z p'(z)$$

and

$$\frac{\mathbb{W}_{\lambda,\mu-2} f(z)}{z} = \frac{\lambda^2 z^2 p''(z) + \lambda(2\mu + \lambda - 3) z p'(z) + (\mu - 1)(\mu - 2) p(z)}{(\mu - 1)(\mu - 2)}.$$

Now, define the transformation from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \qquad v = \frac{\lambda s + (\mu - 1) r}{\mu - 1}$$

and

$$w = \frac{\lambda^2 t + \lambda(2\mu + \lambda - 3) s + (\mu - 1)(\mu - 2) r}{(\mu - 1)(\mu - 2)}.$$

Let

$$\begin{aligned} \psi(r, s, t; z) &= \phi(u, v, w; z) \\ &= \phi\left(r, \ \frac{\lambda s + (\mu - 1) r}{\mu - 1}, \frac{\lambda^2 t + (2\mu + \lambda - 3) s + (\mu - 1)(\mu - 2) r}{(\mu - 1)(\mu - 2)}; z\right). \end{aligned}$$

By using the above equations it follows that

$$\psi\left(p(z), z p'(z), z^2 p''(z); \ z\right) = \phi\left(\frac{\mathbb{W}_{\lambda,\mu} f(z)}{z}, \frac{\mathbb{W}_{\lambda,\mu-1} f(z)}{z}, \frac{\mathbb{W}_{\lambda,\mu-2} f(z)}{z}; \ z\right)$$

and consequently we have that

$$\psi\left(p(z), z p'(z), z^2 p''(z); z\right) \in \Omega.$$

Thus, the proof is completed if it can be shown that the admissibility condition $\phi \in \Phi_{H,2}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.1. Note that

$$\frac{t}{s} + 1 = \frac{\mu - 2}{\lambda}\left(\frac{w - u}{v - u} - 2\right) - \frac{1}{\lambda}.$$

Hence $\psi \in \Psi[\Omega, q]$ and by Lemma 1.3 we get that $p(z) \prec q(z)$. □

In the case when $\Omega \neq \mathbb{C}$ is a simply connected domain with $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω, the class $\Phi_{H,2}[h(\mathbb{D}), q]$ is written as $\Phi_{H,2}[h, q]$. The following result is an immediate consequence of the above theorem.

Corollary 2.22. *Let $\lambda > -1$, $\mu - 2 \in \mathbb{C} \backslash \mathbb{Z}_0^-$ and $\phi \in \Phi_{H,2}[h, q]$ with $q(0) = 1$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\frac{\mathbb{W}_{\lambda,\mu} f(z)}{z}, \frac{\mathbb{W}_{\lambda,\mu-1} f(z)}{z}, \frac{\mathbb{W}_{\lambda,\mu-2} f(z)}{z}; z\right) \prec h(z),$$

then

$$\frac{\mathbb{W}_{\lambda,\mu} f(z)}{z} \prec q(z), \quad z \in \mathbb{D}.$$

In the particular case $q(z) = 1 + Mz$, $M > 0$ and in view of Definition 2.20 the class of admissible functions $\Phi_{H,2}[\Omega, q]$ is simply denoted by $\Phi_{H,2}[\Omega, M]$ as described below.

Definition 2.23. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\phi_{H,2}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ such that

$$\phi\Bigg(1 + Me^{i\theta}, 1 + \frac{k\lambda + \mu - 1}{\mu - 1}Me^{i\theta},$$
$$1 + \frac{L\lambda^2 + [k\lambda(2\mu + \lambda - 3) + (\mu - 1)(\mu - 2)]Me^{i\theta}}{(\mu - 1)(\mu - 2)}; z\Bigg) \notin \Omega,$$

whenever $z \in \mathbb{D}$, $\lambda > -1$, $\mu - 2 \in \mathbb{C} \backslash \mathbb{Z}_0^-$, $\Re(Le^{-i\theta}) \geq (k-1)kM$ for all real θ and $k \geq 1$.

Corollary 2.24. *Let $\lambda > -1$, $\mu - 2 \in \mathbb{C} \backslash \mathbb{Z}_0^-$ and $\phi \in \phi_{H,2}[\Omega, M]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\frac{\mathbb{W}_{\lambda,\mu} f(z)}{z}, \frac{\mathbb{W}_{\lambda,\mu-1} f(z)}{z}, \frac{\mathbb{W}_{\lambda,\mu-2} f(z)}{z} : z\right) \in \Omega,$$

then

$$\left|\frac{\mathbb{W}_{\lambda,\mu} f(z)}{z} - 1\right| < M, \quad z \in \mathbb{D}.$$

In the special case $\Omega = q(\mathbb{D}) = \{\omega : |\omega - 1| < M\}$, the class $\phi_{H,2}[\Omega, M]$ is denoted by $\phi_{H,2}[M]$ and the above corollary takes the following form.

Corollary 2.25. *Let $\lambda > -1$, $\mu - 2 \in \mathbb{C} \backslash \mathbb{Z}_0^-$ and $\phi \in \phi_{H,2}[M]$. If $f \in \mathcal{A}$ satisfies*

$$\left|\phi\left(\frac{\mathbb{W}_{\lambda,\mu} f(z)}{z}, \frac{\mathbb{W}_{\lambda,\mu-1} f(z)}{z}, \frac{\mathbb{W}_{\lambda,\mu-2} f(z)}{z}; z\right) - 1\right| < M,$$

then

$$\left|\frac{\mathbb{W}_{\lambda,\mu-1} f(z)}{z} - 1\right| < M, \quad z \in \mathbb{D}.$$

Corollary 2.26. *Let $\lambda > -1$, $\mu - 1 \in \mathbb{C} \backslash \mathbb{Z}_0^-$ and $M > 0$. If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{\mathbb{W}_{\lambda,\mu-1} f(z)}{z} - \frac{\mathbb{W}_{\lambda,\mu} f(z)}{z}\right| < \left|\frac{\lambda}{\mu - 1}\right| M,$$

then

$$\left|\frac{\mathbb{W}_{\lambda,\mu} f(z)}{z} - 1\right| < M, \quad z \in \mathbb{D}.$$

Proof. This follows from Corollary 2.22 by taking $\phi(u, v, w; z) = v - u$ and $\Omega = h(\mathbb{D})$ where $h(z) = \frac{\lambda M}{\mu - 1} z$. $\square$

Further, taking $\lambda = 1$ and $f(z) = \dfrac{z}{1-z}$ in Corollary 2.26, we get the next result.

Corollary 2.27. *Let $\mu - 1 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $M > 0$. If the modified Bessel function I_μ satisfies*

$$\left| \frac{I_{\mu-2}(\sqrt{z})}{z^{\frac{\mu-2}{2}}} - \frac{2\mu I_{\mu-2}(\sqrt{z})}{z^{\frac{\mu-1}{2}}} \right| < \frac{M}{|\Gamma(\mu)|2^{\mu-2}}, \quad z \in \mathbb{D},$$

then

$$\left| \frac{\Gamma(\mu)2^{\mu-1}I_{\mu-1}(\sqrt{z})}{z^{\frac{\mu-1}{2}}} - 1 \right| < M, \quad z \in \mathbb{D}.$$

3. Superordination results for the operator $\mathbb{W}_{\lambda,\mu}$

The dual problem of differential subordination, the differential superordination of the integral operator $\mathbb{W}_{\lambda,\mu}$ is investigated in this section. For this purpose, a new class of admissible functions is given in the following definition.

Definition 3.1. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}_0$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_H[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; \zeta) \in \Omega,$$

whenever

$$u = q(z), \quad v = \frac{\lambda zq'(z) + (\mu - \lambda - 1)mq(z)}{m(\mu - 1)}$$

and

$$\Re\left(\frac{(\mu-1)(\mu-2)w - (\mu-\lambda-1)(\mu-\lambda-2)u}{\lambda[(\mu-1)v - (\mu-\lambda-1)u]} - \frac{2\mu-3}{\lambda} + 2 \right) \leq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)} + 1 \right),$$

for $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}$, $m \geq 1$, $\lambda > -1$, $\mu - 2 \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Our first main result of this section reads as follows.

Theorem 3.2. *Let $\lambda > -1$, $\mu - 2 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\phi \in \Phi'_H[\Omega, q]$. If $f \in \mathcal{A}$, $\mathbb{W}_{\lambda,\mu}f(z) \in \mathcal{Q}_0$ and $\phi\left(\mathbb{W}_{\lambda,\mu}f(z), \mathbb{W}_{\lambda,\mu-1}f(z), \mathbb{W}_{\lambda,\mu-2}f(z); z\right) \in \mathcal{S}(\mathbb{D})$, then*

$$\Omega \subset \left\{ \phi\left(\mathbb{W}_{\lambda,\mu}f(z), \mathbb{W}_{\lambda,\mu-1}f(z), \mathbb{W}_{\lambda,\mu-2}f(z); z\right) \,:\, z \in \mathbb{D} \right\}$$

implies

$$q(z) \prec \mathbb{W}_{\lambda,\mu}f(z), \quad z \in \mathbb{D}.$$

Proof. With $p(z) = \mathbb{W}_{\lambda,\mu}f(z)$ and

$$\begin{aligned} &\psi(r, s, t; z) = \phi(u, v, w; z), \\ &= \phi\left(r, \frac{s\lambda + (\mu-\lambda-1)r}{\mu-1}, \frac{t\lambda^2 + \lambda(2\mu-\lambda-3)s + (\mu-\lambda-1)(\mu-\lambda-2)r}{(\mu-1)(\mu-2)}; z \right) \end{aligned}$$

we have

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in \mathbb{D}\right\}.$$

Since

$$\frac{t}{s}+1 = \frac{(\mu-1)(\mu-2)w-(\mu-\lambda-1)(\mu-\lambda-2)u}{\lambda((\mu-1)v-(\mu-\lambda-1)u)} - \frac{2\mu-3}{\lambda}+2,$$

the admissibility condition for $\phi \in \Phi'_H[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence $\psi \in \Psi'[\Omega, q]$ and by Lemma 1.4 we have that $q(z) \prec p(z)$. □

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω, and then the class $\Phi'_H[h(\mathbb{D}), q]$ is written as $\Phi'_H[h, q]$. Proceeding as in the previous section, the following result is an immediate consequence of Theorem 3.2.

Corollary 3.3. *Let $\lambda > -1$, $\mu-2 \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $q \in \mathcal{H}_0$, $h \in \mathcal{S}(\mathbb{D})$ and $\phi \in \Phi'_H[h, q]$. If $f \in \mathcal{A}$, $\mathbb{W}_{\lambda,\mu}f(z) \in \mathcal{Q}_0$ and $\phi\left(\mathbb{W}_{\lambda,\mu}f(z), \mathbb{W}_{\lambda,\mu-1}f(z), \mathbb{W}_{\lambda,\mu-2}f(z); z\right) \in \mathcal{S}(\mathbb{D})$, then*

$$h(z) \prec \phi\left(\mathbb{W}_{\lambda,\mu}f(z), \mathbb{W}_{\lambda,\mu-1}f(z), \mathbb{W}_{\lambda,\mu-2}f(z); z\right) \tag{3.1}$$

implies

$$q(z) \prec \mathbb{W}_{\lambda,\mu}f(z), \quad z \in \mathbb{D}.$$

The following theorem proves the existence of the best subordination for an appropriate ϕ.

Theorem 3.4. *Let $\lambda > -1$, $\mu-2 \in \mathbb{C}\backslash\mathbb{Z}_0^-$, h be analytic in $\mathbb{D}$ and $\phi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\Bigg(q(z), \frac{\lambda zq'(z)+(\mu-\lambda-1)q(z)}{\mu-1},$$
$$\frac{\lambda^2z^2q''(z)+(2\mu-\lambda-3)\lambda zq'(z)+(\mu-\lambda-1)(\mu-\lambda-2)q(z)}{(\mu-1)(\mu-2)}; z\Bigg) = h(z)$$

has a solution $q(z) \in \mathcal{Q}_0$. If $\phi \in \Phi'_H[h, q]$, $f \in \mathcal{A}$, $\mathbb{W}_{\lambda,\mu}f(z) \in \mathcal{Q}_0$ and

$$\phi\left(\mathbb{W}_{\lambda,\mu}f(z), \mathbb{W}_{\lambda,\mu-1}f(z), \mathbb{W}_{\lambda,\mu-2}f(z); z\right) \in \mathcal{S}(\mathbb{D}),$$

then

$$h(z) \prec \phi\left(\mathbb{W}_{\lambda,\mu}f(z), \mathbb{W}_{\lambda,\mu-1}f(z), \mathbb{W}_{\lambda,\mu-2}f(z); z\right)$$

implies

$$q(z) \prec \mathbb{W}_{\lambda,\mu}f(z), \quad z \in \mathbb{D}$$

and q is the best subordinant.

Proof. The proof is similar to the proof of Theorem 2.6 and is therefore omitted. □

By combining Corollary 2.3 and Corollary 3.3, we obtain the following sandwich-type result.

Corollary 3.5. *Let* $\lambda > -1$, $\mu - 2 \in \mathbb{C}\backslash\mathbb{Z}_0^-$, h_1 *and* q_1 *be analytic in* $\mathbb{D}$, $h_2 \in \mathcal{S}(\mathbb{D})$, $q_2 \in \mathcal{Q}_0$ *with* $q_1(0) = q_2(0) = 0$ *and* $\phi \in \Phi_H[h_2, q_2] \cap \Phi'_H[h_1, q_1]$. *If* $f \in \mathcal{A}$,

$$\mathbb{W}_{\lambda,\mu}f(z) \in \mathcal{H}_0 \cap \mathcal{Q}_0 \quad \textit{and} \quad \phi\left(\mathbb{W}_{\lambda,\mu}f(z), \mathbb{W}_{\lambda,\mu-1}f(z), \mathbb{W}_{\lambda,\mu-2}f(z);\, z\right) \in \mathcal{S}(\mathbb{D}),$$

then

$$h_1(z) \prec \phi\left(\mathbb{W}_{\lambda,\mu}f(z), \mathbb{W}_{\lambda,\mu-1}f(z), \mathbb{W}_{\lambda,\mu-2}f(z);\, z\right) \prec h_2(z)$$

implies

$$q_1(z) \prec \mathbb{W}_{\lambda,\mu}f(z) \prec q_2(z), \quad z \in \mathbb{D}.$$

Definition 3.6. Let Ω be a set in $\mathbb{C}$ and $q(z) \in \mathcal{H}_1$ with $q(z) \neq 0$, $zq'(z) \neq 0$. The class of admissible functions $\Phi'_{H,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; \zeta) \in \Omega,$$

whenever

$$u = q(z), \quad v = \frac{1}{\mu - 2}\left((\mu - 1)q(z) + \frac{\lambda z q'(z)}{m q(z)} - 1\right) \quad (q(z) \neq 0)$$

and

$$\Re\left(\frac{v(\mu-2)((\mu-2)(w-v)+1-w)}{\lambda(v(\mu-2)-u(\mu-1)+1)} - \frac{2(\mu-1)}{\lambda}u + \frac{\mu-2}{\lambda}v + \frac{1}{\lambda}\right) \leq \frac{1}{m}\Re\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

for $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}$, $\lambda > -1$, $\mu - 3 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $m \geq 1$.

Now, we give another dual result of differential superordinations, concerning Theorem 2.13.

Theorem 3.7. *Let* $\lambda > -1$, $\mu - 3 \in \mathbb{C}\backslash\mathbb{Z}_0^-$ *and* $\phi \in \Phi'_{H,1}[\Omega, q]$. *If* $f \in \mathcal{A}$,

$$\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)} \in \mathcal{Q}_1 \quad \textit{and} \quad \phi\left(\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)}, \frac{\mathbb{W}_{\lambda,\mu-2}f(z)}{\mathbb{W}_{\lambda,\mu-1}f(z)}, \frac{\mathbb{W}_{\lambda,\mu-3}f(z)}{\mathbb{W}_{\lambda,\mu-2}f(z)};\, z\right) \in \mathcal{S}(\mathbb{D}),$$

then

$$\Omega \subset \left\{\phi\left(\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)}, \frac{\mathbb{W}_{\lambda,\mu-2}f(z)}{\mathbb{W}_{\lambda,\mu-1}f(z)}, \frac{\mathbb{W}_{\lambda,\mu-3}f(z)}{\mathbb{W}_{\lambda,\mu-2}f(z)};\, z\right) : z \in \mathbb{D}\right\}$$

implies

$$q(z) \prec \frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)}, \quad z \in \mathbb{D}.$$

Proof. In view of the proof of Theorem 2.13 we have

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z); z\right) :\, z \in \mathbb{D}\right\}$$

and we can see that the admissibility condition for $\phi \in \Phi'_{H,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence $\psi \in \Psi'[\Omega, q]$ and by Lemma 1.4, we get $q \prec p$. $\square$

If $\Omega \neq \mathbb{C}$ is a simply connected domain and $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω, then the class $\Phi'_{H,1}[h(\mathbb{D}), q]$ is written as $\Phi'_{H,1}[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.7.

Corollary 3.8. *Let $\lambda > -1$, $\mu - 3 \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $q \in \mathcal{H}_1$, $h \in \mathcal{H}(\mathbb{D})$ and $\phi \in \Phi'_{H,1}[h, q]$. If $f \in \mathcal{A}$,*

$$\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)} \in \mathcal{Q}_1 \quad \text{and} \quad \phi\left(\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)}, \frac{\mathbb{W}_{\lambda,\mu-2}f(z)}{\mathbb{W}_{\lambda,\mu-1}f(z)}, \frac{\mathbb{W}_{\lambda,\mu-3}f(z)}{\mathbb{W}_{\lambda,\mu-2}f(z)}; z\right) \in \mathcal{S}(\mathbb{D}),$$

then

$$h(z) \prec \phi\left(\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)}, \frac{\mathbb{W}_{\lambda,\mu-2}f(z)}{\mathbb{W}_{\lambda,\mu-1}f(z)}, \frac{\mathbb{W}_{\lambda,\mu-3}f(z)}{\mathbb{W}_{\lambda,\mu-2}f(z)}; z\right)$$

implies

$$q(z) \prec \frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)}, \quad z \in \mathbb{D}.$$

Combining Theorem 2.13 and Corollary 3.8, we obtain the following sandwich-type theorem.

Corollary 3.9. *Let $\lambda > -1$, $\mu-3 \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $h_1, q_1 \in \mathcal{H}(\mathbb{D})$, $h_2 \in \mathcal{S}(\mathbb{D})$, $q_2 \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{H,1}[h_2, q_2] \cap \Phi'_{H,1}[h_1, q_1]$. If $f \in \mathcal{A}$,*

$$\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)} \in \mathcal{H}_1 \cap \mathcal{Q}_1 \quad \text{and} \quad \phi\left(\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)}, \frac{\mathbb{W}_{\lambda,\mu-2}f(z)}{\mathbb{W}_{\lambda,\mu-1}f(z)}, \frac{\mathbb{W}_{\lambda,\mu-3}f(z)}{\mathbb{W}_{\lambda,\mu-2}f(z)}; z\right) \in \mathcal{S}(\mathbb{D}),$$

then

$$h_1(z) \prec \phi\left(\frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)}, \frac{\mathbb{W}_{\lambda,\mu-2}f(z)}{\mathbb{W}_{\lambda,\mu-1}f(z)}, \frac{\mathbb{W}_{\lambda,\mu-3}f(z)}{\mathbb{W}_{\lambda,\mu-2}f(z)}; z\right) \prec h_2(z)$$

implies

$$q_1(z) \prec \frac{\mathbb{W}_{\lambda,\mu-1}f(z)}{\mathbb{W}_{\lambda,\mu}f(z)} \prec q_2(z), \quad z \in \mathbb{D}.$$

References

[1] R. Aghalary, S. B. Joshi, R. N. Mohapatra and V. Ravichandran, Subordination for analytic functions defined by the Dziok-Srivastava linear operator, *Appl. Math. Comput.* **187** (2007), no. 1, 13–19.

[2] R. M. Ali, V. Ravichandran and N. Seenivasagan, Subordination and superordination of the Liu-Srivastava linear operator on meromorphic functions, *Bull. Malays. Math. Sci. Soc. (2)* **31** (2008), no. 2, 192–207.

[3] R. M. Ali, V. Ravichandran and N. Seenivasagan, Differential subordination and superordination of analytic functions defined by the Dziok-Srivastava operator, *J. Franklin Inst.* **347** (2010), no. 9, 1762–1781.

[4] Á. Baricz, Geometric properties of generalized Bessel functions, *Publ. Math. Debrecen* **73** (2008) 155–178.

[5] Á. Baricz, Generalized Bessel Functions of the First Kind, Lecture Notes in Math. Springer, New York, 2010.

[6] Á. Baricz, E. Deniz, M. Çağlar and H. Orhan, Differential subordinations involving generalized Bessel functions, *Bull. Malays. Math. Sci. Soc.* **38** (2015), no. 3, 1255–1280.

[7] Á. Baricz, P. A. Kupán and R. Szász, The radius of starlikeness of normalized Bessel functions of the first kind, *Proc. Amer. Math. Soc.* **142** (2014), no. 6, 2019–2025.

[8] Á. Baricz and S. Ponnusamy, Starlikeness and convexity of generalized Bessel functions, *Integral Transforms Spec. Funct.* **21** (2010), no. 9, 641–653.

[9] R. Gorenflo, Y. Luchko and F. Mainardi, Analytic properties and applications of Wright functions, *Fract. Calc. Appl. Anal.* **2** (1999), no. 4, 383–414.

[10] Y. C. Kim and H. M. Srivastava, Inequalities involving certain families of integral and convolution operators, *Math. Inequal. Appl.* **7** (2004), no. 2, 227–234.

[11] V. Kiryakova, Generalized Fractional Calculus and Applications, Pitman Research Notes in Mathematics Series, 301, Longman Scientific & Technical, Harlow, UK, 1994.

[12] F. Mainardi, A. Mura and G. Pagnini, The M-Wright function in time-fractional diffusion processes: a tutorial survey, *Int. J. Differ. Equ.* (2010), Article 104505, 29 pages.

[13] S. S. Miller and P. T. Mocanu, Differential Subordinations, Theory and Applications, New York and Basel, Marcel Dekker, 2000.

[14] S. S. Miller and P. T. Mocanu, Subordinants of differential superordinations, *Complex Var. Theory Appl.* **48** (2003), no. 10, 815–826.

[15] S. R. Mondal and A. Swaminathan, Geometric properties of generalized Bessel functions, *Bull. Malays. Math. Soc. (2)* **35** (2012), no. 1, 179–194.

[16] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[17] J. K. Prajapat, Certain geometric properties of normalized Bessel functions, *Appl. Math. Lett.* **24** (2011), no. 12, 2133–2139.

[18] J. K. Prajapat, Certain geometric properties of the Wright functions, *Integral Transforms Spec. Funct.* **26** (2015), no. 3, 203–212.

[19] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, New York, 1993.

[20] A. Soni, S. Kant and J. K. Prajapat, Differential subordinations and superordinations for multivalently meromorphic functions involving the Hurwitz-Lerch Zeta functions, *Proc. Nat. Acad. Sci. India Sect. A* **85** (2015), no. 3, 385–393.

[21] R. Szász and P. A. Kupán, About the univalence of the Bessel functions, *Stud. Univ. Babes-Bolyai Math.* **54** (2009), no. 1, 127–132.

[22] E. M. Wright, On the coefficients of power series having exponential singularities, *J. London Math. Soc.* **8** (1933) 71–79.

[23] R. G. Xiang, Z. G. Wang and M. Darus, A family of integral operators preserving subordination and superordination, *Bull. Malays. Math. Sci. Soc. (2)* **33** (2010), no. 1, 121–131.

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