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ON TIME-DEPENDENT NEUTRAL STOCHASTIC EVOLUTION EQUATIONS WITH A FRACTIONAL BROWNIAN MOTION AND INFINITE DELAYS

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ABSTRACT. In this paper, we consider a class of time-dependent neutral stochastic evolution equations with the infinite delay and a fractional Brownian motion in a Hilbert space. We establish the existence and uniqueness of mild solutions for these equations under non-Lipschitz conditions with Lipschitz conditions which is being considered as a special case. An example is provided to illustrate the theory.
Keywords: Stochastic neutral evolution equations, fractional Brownian motion, infinite delay, non-Lipschitz condition.
MSC(2010): Primary: 60H15; Secondary: 60G15, 60H05.

1. Introduction

It is known that stochastic partial differential equations (SPDEs) play a very important role in formulation and analysis of many phenomena in economic and finance, in physics, mechanics, electric and control engineering, etc. There is much current interest in studying qualitative properties for SPDEs (see, e.g., Da Prato and Zabczyk [8], Liu [12, 13], Wei and Wang [24], Luo and Liu [15], Zhou et al. [25], Jahanipur [11], and references therein).

One solution for many SDEs is a semimartingale as well a Markov process. However, many objects in real world are not always such processes since they have long-range aftereffects. Since the work of Mandelbrot and Van Ness [16], there is an increasing interest in stochastic models based on the fractional Brownian motion. A fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$ is a centered Gaussian process $B^H = \{B^H(t), t \geq 0\}$ with the covariance function

$$R_H(t,s) = \mathbb{E}(B^H(t)B^H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

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When $H = 1/2$, the fBm becomes the standard Brownian motion, and the fBm B^H neither is a semimartingale nor a Markov process if $H \neq 1/2$. However, the fBm B^H, $H > 1/2$ is a long-memory process and presents an aggregation behavior. The long-memory property make the fBm as a potential candidate to model noise in mathematical finance (see [6,17]); in biology (see [4,7]); in communication networks (see, for instance [23]); the analysis of global temperature anomaly [21] and electricity markets [22] etc.

Recently, stochastic partial functional differential equations driven by a fractional Brownian motion have attracted the interest of many researchers. For example, under the global Lipschitz condition, Caraballo et al. [5] showed the existence, uniqueness and stability of mild solutions for SPDEs with finite delays driven by a fBm; under the global Lipschitz condition, Boufoussi and Hajji [2] considered the existence and uniqueness of mild solutions to neutral SPDEs with finite delays driven by a fBm; Boufoussi et at. [3] obtained the existence and uniqueness result of mild solutions to a class of of time-dependent stochastic functional differential equations driven by a fBm; Ren et at. [20] proved the existence and uniqueness of the mild solution for a class of time-dependent stochastic evolution equations with finite delays driven by a standard cylindrical Wiener process and an independent cylindrical fractional Brownian motion. Huang et al. [10] studied a class of stochastic modified Boussinesq approximation equations driven by a cylindrical fractional Brownian motion.

On the other hand, it is well known that stochastic equations with infinite delays have wide application in many areas (see, e.g. [9, 19]). However, to the best of our knowledge, there is no result on stochastic partial differential equations with infinite delays driven by a fBm. To close the gap, we will make the first attempt to study such problem in this paper. We aim to derive the existence and uniqueness of mild solutions under some local conditions.

The rest of this paper is organized as follows. In Section 2, we introduce some necessary notations and preliminaries. In Section 3, the existence and uniqueness of mild solutions are discussed. An example is presented in Section 4 to illustrate the theory.

2. Preliminaries

In this section we collect some notions, conceptions and lemmas on Wiener integrals with respect to an infinite dimensional fractional Brownian motion and recall some basic results which will be used throughout the whole of this paper.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all P-null sets.

Now we aim at introducing the Wiener integral with respect to the one-dimensional fBm B^H. Consider a time interval $[0, T]$ with arbitrary fixed horizon T and let $\{B^H(t), t \in [0, T]\}$ be the one-dimensional fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. This means B^H has the following Wiener integral representation:

$$B^H(t) = \int_0^t K_H(t,s)dB(s),$$

where $B = \{B(t) : t \in [0,T]\}$ is a standard Brownian motion, and $K_H(t,s)$ is the kernel given by

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

for $t > s$, where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and $\beta(\cdot,\cdot)$ denotes the Beta function. We put $K_H(t,s) = 0$ if $t \leq s$.

We will denote by $\mathcal{H}$ the reproducing kernel Hilbert space of the fBm. In fact $\mathcal{H}$ is the closure of the linear space of indicator functions $\{I_{[0,t]}, t \in [0,T]\}$ with respect to the scalar product

$$\langle I_{[0,t]}, I_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s).$$

The mapping $I_{[0,t]} \to B^H(t)$ can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos and we will denote by $B^H(\varphi)$ the image of φ by the such isometry.

We recall that for $\psi, \varphi \in \mathcal{H}$ their scalar product in $\mathcal{H}$ is given by

$$\langle \psi, \varphi \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T \psi(s)\varphi(t)|t-s|^{2H-2} ds dt.$$

Let us consider the operator K_H^* from $\mathcal{H}$ to $L^2([0,T])$ defined by

$$(K_H^* \varphi)(s) = \int_s^T \varphi(r) \frac{\partial K}{\partial r}(r,s) dr.$$

We refer to [18] for the proof of the fact that K_H^* is an isometry between $\mathcal{H}$ and $L^2([0,T])$. Moreover, for any $\varphi \in \mathcal{H}$, we have

$$B^H(\varphi) = \int_0^T (K_H^* \varphi)(t) dB(t).$$

Also denoting $L^2_{\mathcal{H}}([0,T]) = \{\varphi \in \mathcal{H}, K_H^* \varphi \in L^2([0,T])\}$, since $H > 1/2$, we have the following useful Lemma 2.1 (see [18]).

Lemma 2.1.

$$L^2([0,T]) \subseteq L^{1/H}([0,T]) \subseteq L^2_{\mathcal{H}}([0,T]) \subseteq \mathcal{H},$$

and for any $\psi \in L^{1/H}([0,T])$*, we have*

$$H(2H-1) \int_0^T \int_0^T |\psi(s)||\psi(t)||s-t|^{2H-2} ds dt \leq c_H \|\psi(s)\|^2_{L^{1/H}([0,T])}.$$

Let $(X, \|\cdot\|, \langle\cdot,\cdot\rangle)$ and $(Y, \|\cdot\|_Y, \langle\cdot,\cdot\rangle_Y)$ be two real, separable Hilbert spaces and let $\mathcal{L}(Y, X)$ be the space of bounded linear operator from Y to X. Let $Q \in \mathcal{L}(Y, Y)$ be a non-negative self-adjoint operator. Consider the following series

$$\sum_{n=1}^{\infty} B_n^H(t) e_n, \quad t \geq 0,$$

where $\{B_n^H(t)\}_{n\in\mathbb{N}}$ is a sequence of two-sided one dimensional standard fBm mutually in dependent and $\{e_n\}_{n\in\mathbb{N}}$ is a complete orthonormal basis in Y, the series does not necessarily converge in the space Y. Therefore, we consider a Y-valued stochastic process $B_Q^H(t)$ given by the following series:

$$B_Q^H(t) = \sum_{n=1}^{\infty} B_n^H(t) Q^{\frac{1}{2}} e_n, \quad t \geq 0.$$

Moreover, if Q is a non-negative self-adjoint trace class operator, then this series converges in the space Y, that is, $B_Q^H(t) \in L^2(\Omega, Y)$. Then, we say that the above $B_Q^H(t)$ is a Y-valued Q-cylindrical fBm with covariance operator Q. For example, if $\{\lambda_n\}_{n\in\mathbb{N}}$ is a bounded sequence of non-negative real numbers such that $Qe_n = \lambda_n e_n$, assuming that Q is a nuclear operator in Y (that is, $\sum_{n=1}^{\infty} \lambda_n < \infty$), then the stochastic process

$$B_Q^H(t) = \sum_{n=1}^{\infty} B_n^H(t) Q^{\frac{1}{2}} e_n = \sum_{n=1}^{\infty} \sqrt{\lambda_n} B_n^H(t) e_n, \quad t \geq 0,$$

is well-defined as a Y-valued Q-cylindrical fBm.

In order to define Wiener integrals with respect to the Q-fBm, we introduce the space $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y, X)$ of all Q-Hilbert-Schmidt operators $\psi : Y \to X$. We recall that $\psi \in \mathcal{L}(Y, X)$ is called a Q-Hilbert-Schmidt operator, if

$$\|\psi\|_{\mathcal{L}_2^0}^2 := \sum_{n=1}^{\infty} \left\| \sqrt{\lambda_n} \psi e_n \right\|^2 < \infty,$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product $\langle \varphi, \psi \rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n \rangle$ is a separable Hilbert space.

Let $\phi : [0, T] \to \mathcal{L}_2^0(Y, X)$ such that

$$\sum_{n=1}^{\infty} \|K_H^*(\phi Q^{1/2} e_n)\|_{L^2([0,T];X)} < \infty. \tag{2.1}$$

The Wiener integral of ϕ with respect to the B_Q^H is defined by

$$\begin{aligned}\int_0^t \phi(s) dB_Q^H(s) &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \phi(s) Q^{\frac{1}{2}} e_n dB_n^H(s) \\ &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} (K_H^*(\phi Q^{\frac{1}{2}} e_n))(s) dB_n(s),\end{aligned}$$

where B_n is the standard Brownian motion used to present B_n^H.

Notice that if

$$\sum_{n=1}^{\infty} \|\phi Q^{1/2} e_n\|^2_{L^{1/H}([0,T];X)} < \infty, \tag{2.2}$$

in particular (2.1) holds, which follows immediately from Lemma 2.1.

Next, we introduce the notion of evolution family.

Definition 2.2. A set $\{U(t,s) : 0 \le s \le t \le T\}$ of bounded linear operators on X is called an evolution family if

(a) $U(t,s)U(s,r) = U(t,r)$, $U(s,s) = I$ for $0 \le r \le s \le t \le T$, where I is the identity operator;

(b) $(t,s) \to U(t,s)$ is strongly continuous for $0 \le s \le t \le T$.

Let $\{A(t), t \in [0,T]\}$ be a family of closed densely defined linear operators on Hilbert space X and with domain $D(A(t))$ independent of t, subject to the following hypothesis introduced by Acquistapace and Terreni in [1].

There exist constants $\lambda_0 \ge 0$, $\theta \in (\frac{\pi}{2}, \pi)$, $L, K \ge 0$, and $\mu, \nu \in (0,1]$ with $\mu + \nu > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \quad \|R(\lambda, A(t) - \lambda_0)\| \le \frac{K}{1 + |\lambda|}, \tag{2.3}$$

and

$$\|(A(t)-\lambda_0)R(\lambda, A(t)-\lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \le L|t-s|^\mu |\lambda|^{-\nu}, \tag{2.4}$$

for $t, s \in \mathbb{R}$, $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C} - \{0\} : |arg\lambda| \le \theta\}$.

This assumption implies that there exists a unique evolution family $\{U(t,s) : 0 \le s \le t \le T\}$ on X such that $(t,s) \to U(t,s) \in \mathcal{L}(X)$ is continuous for $t > s$, $U(\cdot, s) \in C^1((s,\infty), \mathcal{L}(X))$, $\partial_t U(t,s) = A(t)U(t,s)$, and

$$\|A(t)^k U(t,s)\| \le C(t-s)^{-k}, \tag{2.5}$$

for $0 < t - s \le 1$, $k = 0, 1$, $0 \le \alpha < \mu$, $x \in D((\lambda_0 - A(s))^\alpha)$, and a constant C depending only on the constants in (2.3)-(2.4). Moreover, $\partial_s^+ U(t,s)x = -U(t,s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$. We say that $A(\cdot)$ generates $\{U(t,s) : 0 \le s \le t \le T\}$. Note that $\{U(t,s) : 0 \le s \le t \le T\}$ is exponentially bounded by (2.5) with $k = 0$.

Remark 2.3. If $A(t)$, $t \ge 0$ is a second order differential operator A, i.e. $A(t) = A$ for each $t \ge 0$. Then, A generates a C_0-semigroup $\{e^{At}, t \ge 0\}$.

For additional details on evolution families, we refer the reader to the book by Lunardi [14].

In this paper, $\mathfrak{B}((-\infty, 0]; L^2(\Omega, X))$ (denoted by $\mathfrak{B}$ simply) denotes the family of all $\mathcal{F}_0$-measurable bounded continuous functions $\varphi : (-\infty, 0] \to L^2(\Omega, X)$ endowed with the norm $\|\varphi\|_t^2 = \sup_{-\infty < \theta \le t} \mathbb{E}\|\varphi(\theta)\|^2$. Let $\mathfrak{B}^B_{\mathcal{F}_0}((-\infty, 0]; X)$ denote the family of all almost surely bounded, $\mathcal{F}_0$-measurable, $\mathfrak{B}$-valued random variables. Moreover, let $\mathcal{B}_T$ denote Banach space of all $\mathcal{F}_t$ adapted processes $\varphi(t,\omega)$ which are almost surely continuous in t for fixed $\omega \in \Omega$ with the norm

$$\|\varphi\|_{\mathcal{B}_T} = \Big(\sup_{0 \le t \le T} \|\varphi\|_t^2 \Big)^{1/2}.$$

Consider the following neutral stochastic partial differential equations with a fractional Brownian motion and infinite delays in the form:

$$\text{(2.6)} \quad \begin{cases} d[x(t) + G(t, x_t)] = [A(t)x(t) + f(t, x_t)]dt + g(t, x_t)dw(t) \\ \qquad\qquad\qquad\qquad\qquad +\sigma(t)dB_Q^H(t),\ 0 \le t \le T, \\ x(t) = \varphi(t) \in \mathfrak{B}, \qquad t \le 0, \end{cases}$$

where $x_t = x(t+\theta) : -\infty < \theta \le 0$ can be regarded as a $\mathfrak{B}$-valued stochastic process. Assume that

$$f, G : [0,T] \times \mathfrak{B} \longrightarrow X, \quad g : [0,T] \times \mathfrak{B} \longrightarrow \mathcal{L}_2^0(Y, X), \quad \sigma : [0,T] \longrightarrow \mathcal{L}_2^0(Y, X),$$

are appropriate mappings specified later. w is a standard Wiener process on a real and separable Hilbert space Y. The initial value $\varphi = \{\varphi(\theta) : -\infty < \theta \le 0\}$ is an $\mathcal{F}_0$-measurable $\mathfrak{B}$-valued random variable independent of the fBm B_Q^H and Wiener process w with finite second moment. Now we present the definition of the mild solution for (2.6).

Definition 2.4. An $\mathcal{F}_t$-adapted X-valued stochastic process $x(t)$ defined on $-\infty < t \le T$ is called the mild solution for (2.6) if

(i) $x(t)$ is continuous and $\{x_t : 0 \le t \le T\}$ is a $\mathfrak{B}$-valued stochastic process;
(ii) for arbitrary $t \in [0,T]$, $x(t)$ satisfied the following integral equation:

$$\text{(2.7)} \quad \begin{cases} x(t) = U(t,0)(\varphi(0) + G(0,\varphi)) - \int_0^t A(s)U(t,s)G(s, x_s)ds \\ \qquad + \int_0^t U(t,s)f(s,x_s)ds + \int_0^t U(t,s)g(s,x_s)dw(s) \\ \qquad + \int_0^t U(t,s)\sigma(s)dB_Q^H(s) - G(t, x_t), \\ x_0 = \varphi \in \mathfrak{B}. \end{cases}$$

3. **Existence and uniqueness**

In this section, we present our main results on the existence and uniqueness of the mild solution of (2.6). We first introduce the following assumptions.

(H1) (a) The evolution family is exponentially stable, that is, there exist two constants $\beta > 0$ and $M > 0$ such that

$$\|U(t,s)\| \le Me^{-\beta(t-s)}, \qquad \text{for any} \quad t \ge s,$$

(b) There exists a constant $M_* > 0$ such that

$$\|A^{-1}(t)\| \le M_*, \qquad \text{for any} \quad t \in [0,T].$$

(H2) The function $\sigma : [0, +\infty) \to \mathcal{L}_2^0(Y, X)$ satisfies the following conditions: for the complete orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ in Y, we have

(σ_1) $\sum_{n=1}^{\infty} \|\sigma Q^{1/2} e_n\|_{L^2([0,T];X)} < \infty$.

(σ_2) $\sum_{n=1}^{\infty} \|\sigma(t) Q^{1/2} e_n\|$ is uniformly convergent for $t \in [0, T]$.

(H3) There exists a constant $L_* < \frac{1}{M_*}$ such that

$$\|A(t)G(t, x_t) - A(t)G(t, y_t)\| \leq L_* \|x - y\|_t,$$

for any $t \in [0, T]$, $x, y \in \mathfrak{B}$. Moreover, we assume that $G(t, 0) = 0$.

(H4) (a) There exist a function $F : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $F(t, u)$ is locally integrable in t for any fixed $u \geq 0$ and is continuous, nondecreasing, and concave in u for each fixed $t \in [0, T]$. Moreover, for any $t \in [0, T]$, $x_t \in \mathfrak{B}$, the following inequality holds:

$$\|f(t, x_t)\|^2 + \|g(t, x_t)\|^2_{\mathcal{L}_2^0} \leq F(t, \|x\|_t^2).$$

(b)The differential equation

$$\frac{du}{dt} = F(t, u),$$

has a global solution for any initial value u_0.

(H5) (global conditions)

(a) There exists a function $Z : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $Z(t, u)$ is locally integrable in t for any fixed $u \geq 0$ and is continuous, nondecreasing, and concave in u for each fixed $t \in [0, T]$, $Z(t, 0) = 0$ for any $t \in [0, T]$. Moreover, for any $t \in [0, T]$, $x_t, y_t \in \mathfrak{B}$, the following inequality holds:

$$\|f(t, x_t) - f(t, y_t)\|^2 + \|g(t, x_t) - g(t, y_t)\|^2_{\mathcal{L}_2^0} \leq Z(t, \|x - y\|_t^2).$$

(b) For any constant $D > 0$, if a nonnegative function $u(t)$ satisfies

$$u(t) \leq D \int_0^t Z(s, u(s)) ds, \qquad t \in [0, T],$$

then $u(t) \equiv 0$ for any $t \in [0, T]$.

(H5') (local conditions)

(a) For any integer $N > 0$, there exists a function $Z_N : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $Z_N(t, u)$ is locally integrable in t for any fixed $u \geq 0$ and is continuous, nondecreasing, and concave in u for each fixed $t \in [0, T]$, $Z_N(t, 0) = 0$ for any $t \in [0, T]$. Moreover, for any $t \in [0, T]$, $x_t, y_t \in \mathfrak{B}$ with $\|x\|_t \leq N$, $\|y\|_t \leq N$, the following inequality holds:

$$\|f(t, x_t) - f(t, y_t)\|^2 + \|g(t, x_t) - g(t, y_t)\|^2_{\mathcal{L}_2^0} \leq Z_N(t, \|x - y\|_t^2).$$

(b) For any constant $D > 0$, if a nonnegative function $u(t)$ satisfies

$$u(t) \leq D \int_0^t Z_N(s, u(s)) ds, \qquad t \in [0, T],$$

then $u(t) \equiv 0$ for any $t \in [0,T]$.

Remark 3.1. Let $Z(t,u) = L(t)\overline{Z}(u)$, $t \in [0,T]$, where $L(t) \geq 0$ is locally integrable and $\overline{Z}(u)$ is a concave nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\overline{Z}(0) = 0$, $\overline{Z}(u) > 0$ for $u > 0$ and $\int_{0^+} \frac{1}{\overline{Z}(u)} du = \infty$. Then, by the comparison theorem of differential equations we know that assumption (H5-b) holds.

Now let us give some concrete examples of the function $\overline{Z}(u)$. Let $\zeta > 0$ and let $\delta \in (0,1)$ be sufficiently small. Define

$$\overline{Z_1}(u) = \zeta u, \ u \geq 0,$$

$$\overline{Z_2}(u) = \begin{cases} u\log(u^{-1}), & 0 \leq u \leq \delta, \\ \delta\log(\delta^{-1}) + \overline{Z_2}'(\delta-)(u-\delta), & u > \delta, \end{cases}$$

where $\overline{Z_2}'$ denotes the derivative of function $\overline{Z_2}$. They are all concave nondecreasing functions satisfying $\int_{0^+} \frac{1}{\overline{Z_i}(u)} du = \infty (i = 1,2)$.

Now, we establish the following lemma, which is useful to prove our results.

Lemma 3.2. *Suppose that $\psi : [0,T] \to \mathcal{L}_2^0(Y,X)$ such that (2.2) and (H1) hold. Then, for any $t \in [0,T]$ we have*

$$\mathbb{E}\Big\| \int_0^t U(t,s)\psi(s) dB_Q^H(s) \Big\|^2 \leq c_H H(2H-1) M^2 t^{2H-1} \sum_{n=1}^{\infty} \int_0^t \|\psi(s) Q^{\frac{1}{2}} e_n\|^2 ds.$$

If, in addition,

$$\sum_{n=1}^{\infty} \|\psi(t) Q^{\frac{1}{2}} e_n\| \quad \textit{is uniformly convergent for } \ t \in [0,T], \tag{3.1}$$

then

$$\mathbb{E}\Big\| \int_0^t U(t,s)\psi(s) dB_Q^H(s) \Big\|^2 \leq c_H H(2H-1) M^2 t^{2H-1} \int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0(Y,X)} ds.$$

Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be the complete orthonormal basis of Y introduced in Section 2. Thus, applying Lemma 2.1 we obtain

$$\begin{aligned}
&\mathbb{E}\Big\| \int_0^t U(t,s)\psi(s)dB_Q^H(s)\Big\|^2 \\
=&\sum_{n=1}^{\infty}\mathbb{E}\Big\| \int_0^t U(t,s)\psi(s)Q^{\frac{1}{2}}e_n dB_n^H(s)\Big\|^2 \\
=&\sum_{n=1}^{\infty} H(2H-1)\int_0^t\int_0^t \langle U(t,s)\psi(s)Q^{\frac{1}{2}}e_n, U(t,r)\psi(r)Q^{\frac{1}{2}}e_n\rangle |s-r|^{2H-2}dsdr \\
\leq& c_H H(2H-1)\sum_{n=1}^{\infty}\Big(\int_0^t \|U(t,s)\psi(s)Q^{\frac{1}{2}}e_n\|^{1/H}ds\Big)^{2H} \\
\leq& c_H H(2H-1)M^2t^{2H-1}\sum_{n=1}^{\infty}\int_0^t \|\psi(s)Q^{\frac{1}{2}}e_n\|^2 ds.
\end{aligned}$$

The second assertion is an immediate consequence of the Weierstrass M-test. □

Remark 3.3. If $\{\sigma_n\}_{n\in\mathbb{N}}$ is a bounded sequence of non-negative real numbers such that the nuclear operator Q satisfies $Qe_n = \sigma_n e_n$, assuming that there exists a positive constant k_ψ such that

$$\|\psi(s)\|^2_{\mathcal{L}_2^0(Y,X)} \leq k_\psi, \quad \text{uniformly in} \quad [0,T],$$

then (3.1) holds automatically.

Theorem 3.4. *Let (H1)-(H5) hold. Then (2.6) admits a unique mild solution $x(t)\in\mathcal{B}_T$ provided that*

$$6L_*^2M_*^2\beta^2 + 6M^2 < \beta^2. \tag{3.2}$$

Proof. In order to prove the existence of the solution to (2.6), let $x^0(t) = U(t,0)\varphi(0)$, $t\in[0,T]$ and $x^n(t) = \varphi(t)$, $n\geq 1$ for $-\infty < t \leq 0$, define the following successive approximations procedure for $t\in[0,T]$ and $n\geq 1$:

$$\begin{aligned}
x^n(t) =& U(t,0)(\varphi(0)+G(0,\varphi)) - G(t,x_t^n) - \int_0^t A(s)U(t,s)G(s,x_s^n)ds \\
&+ \int_0^t U(t,s)f(s,x_s^{n-1})ds + \int_0^t U(t,s)g(s,x_s^{n-1})dw(s) \\
&+ \int_0^t U(t,s)\sigma(s)dB_Q^H(s).
\end{aligned}\tag{3.3}$$

The proof is divided into the following three steps.

Step 1. For all $s\in(-\infty,t]$, we claim that the sequence $x^n\in\mathcal{B}_T, n\geq 0$ is bounded. It is obvious that $x^0(t) \in \mathcal{B}_T$. From (3.3), for $0 \le t \le T$,

$$
\begin{aligned}
&\mathbb{E}\|x^n(s)\|^2 \\
&\le 6\mathbb{E}\|U(s,0)(\varphi(0)+G(0,\varphi))\|^2 + 6\mathbb{E}\Big\|\int_0^s A(r)U(s,r)G(r,x_r^n)dr\Big\|^2 \\
&\quad + 6\mathbb{E}\Big\|\int_0^s U(s,r)f(r,x_r^{n-1})dr\Big\|^2 + 6\mathbb{E}\Big\|\int_0^s U(s,r)g(r,x_r^{n-1})dw(r)\Big\|^2 \\
&\quad + 6\mathbb{E}\Big\|\int_0^s U(s,r)\sigma(r)dB_Q^H(r)\Big\|^2 + 6\mathbb{E}\|G(s,x_s^n)\|^2 \\
&=: 6\sum_{i=1}^{6} I_i.
\end{aligned}
\tag{3.4}
$$

By (H1) and (H3), one has

$$
I_1 \le 2M^2(1+L_*^2M_*^2)\mathbb{E}\|\varphi(0)\|^2,\ \ I_6 \le L_*^2M_*^2\mathbb{E}\|x^n\|_s^2 \tag{3.5}
$$

and

$$
\begin{aligned}
I_2 \le &\mathbb{E}\Big(\int_0^s \|A(r)U(s,r)G(r,x_r^n)\|dr\Big)^2 \\
\le &\mathbb{E}\Big(\int_0^s Me^{-\beta(s-r)}\|x^n\|_r dr\Big)^2 \\
\le &\frac{M^2}{\beta^2}\mathbb{E}\|x^n\|_s^2.
\end{aligned}
\tag{3.6}
$$

By (H4), we obtain

$$
I_3 \le M^2\mathbb{E}\int_0^s F(r,\|x^{n-1}\|_r^2)dr. \tag{3.7}
$$

By (H4) and Doob martingale inequality, there exists a positive constant C_1 such that

$$
I_4 \le C_1\mathbb{E}\int_0^s F(r,\|x^{n-1}\|_r^2)dr. \tag{3.8}
$$

For I_5, by Lemma 3.2, we can obtain

$$
I_5 \le c_H H(2H-1)M^2T^{2H-1}\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0(Y,X)}ds. \tag{3.9}
$$

Hence, substituting (3.5)-(3.9) into (3.4) yields

$$
\mathbb{E}\|x^n(s)\|^2 \le C_2 + C_3\mathbb{E}\|x^n\|_s^2 + C_4\mathbb{E}\int_0^s F(r,\|x^{n-1}\|_r^2)dr,
$$

where

$$
C_2 = 12M^2(1+L_*^2M_*^2)\mathbb{E}\|\varphi(0)\|^2 + 6c_H H(2H-1)M^2T^{2H-1}\int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2^0(Y,X)}ds,
$$

$C_3 = 6L_*^2M_*^2 + 6M^2/\beta^2$ and $C_4 = C_1 + M^2$.
Then

$$
\begin{aligned}
\mathbb{E}\|x^n\|_t^2 = & \sup_{-\infty<s\leq t} \mathbb{E}\|x^n(s)\|^2 \\
\leq & \mathbb{E} \sup_{-\infty<\theta\leq 0} \|\varphi(\theta)\|^2 + C_2 + C_3\mathbb{E}\|x^n\|_t^2 + C_4\mathbb{E}\int_0^t F(s, \|x^{n-1}\|_s^2)ds \\
\leq & C_5 + C_6\mathbb{E}\int_0^t F(s, \|x^{n-1}\|_s^2)ds
\end{aligned}
$$

where $C_5 = \frac{C_2+\mathbb{E}\sup_{-\infty<\theta\leq 0}\|\varphi(\theta)\|^2}{1-C_3}$ and $C_6 = \frac{C_4}{1-C_3}$.
Thus, by Jensen inequality we have

$$
\mathbb{E}\|x^n\|_t^2 \leq C_5 + C_6\int_0^t F(s, \mathbb{E}\|x^{n-1}\|_s^2)ds.
$$

Assumption (H4-b) indicates that there is a solution u_t satisfying

$$
u_t = 2C_5 + 2C_6\int_0^t F(s, u_r)ds.
$$

Since $\mathbb{E}\|x^0(t)\| \leq M^2\|\varphi(0)\| < \infty$, we have $\mathbb{E}\|x^n\|_t^2 \leq u_t \leq u_T < \infty$, which shows that the boundedness of $\{x^n(t), n \geq 0\}$.
Step 2. We claim that $\{x^n(t), n \geq 0\}$ is a Cauchy sequence. For all $n, m \geq 0$ and $t \in [0, T]$, from (3.3), (H5) and Step 1, we have

$$
\begin{aligned}
& \mathbb{E}\|x^{n+1}(s) - x^{m+1}(s)\|^2 \\
\leq & 4\mathbb{E}\|G(s, x_s^{n+1}) - G(s, x_s^{m+1}))\|^2 \\
& + 4\mathbb{E}\Big\|\int_0^s A(r)U(s,r)(G(r, x_r^{n+1}) - G(r, x_r^{m+1}))dr\Big\|^2 \\
& + 4\mathbb{E}\Big\|\int_0^s U(s,r)(f(r, x_r^n) - f(r, x_r^m))dr\Big\|^2 \\
& + 4\mathbb{E}\Big\|\int_0^s U(s,r)(g(r, x_r^n) - g(r, x_r^m))dw(r)\Big\|^2.
\end{aligned}
$$

By (H5) and Burkhölder-Davis-Gundy inequality, there exists a positive constant C_7 such that

$$
\begin{aligned}
& \mathbb{E}\|x^{n+1}(s) - x^{m+1}(s)\|^2 \\
\leq & 4(L_*^2M_*^2 + \frac{M^2}{\beta^2})\mathbb{E}\|x^{n+1} - x^{m+1}\|_s^2 + (4M^2 + C_7)\mathbb{E}\int_0^s Z(r, \|x^n - x^m\|_r^2)dr.
\end{aligned}
$$

Then, by Jensen inequality we have

$$\begin{aligned}\mathbb{E}\|x^{n+1}-x^{m+1}\|_t^2 &= \sup_{-\infty<s\le t}\mathbb{E}\|x^{n+1}(s)-x^{m+1}(s)\|^2\\ &\le \frac{4M^2+C_7}{1-4L_*^2M_*^2-4M^2/\beta^2}\int_0^t Z(s,\mathbb{E}\|x^n-x^m\|_s^2)ds.\end{aligned}\tag{3.10}$$

By (3.10) and Fatou lemma, we have

$$\begin{aligned}&\limsup_{n,m\to\infty}\Big(\sup_{0\le s\le t}\mathbb{E}\|x^{n+1}(s)-x^{m+1}(s)\|^2\Big)\\ \le& C_8\int_0^t Z(s,\limsup_{n,m\to\infty}(\sup_{0\le\theta\le s}\mathbb{E}\|x^{n+1}(\theta)-x^{m+1}(\theta)\|^2))ds,\end{aligned}$$

where $C_8=\frac{4M^2+C_7}{1-4L_*^2M_*^2-4M^2/\beta^2}$. By assumption (H5-b) we obtain

$$\lim_{n,m\to\infty}\sup_{0\le s\le T}\mathbb{E}\|x^{n+1}(s)-x^{m+1}(s)\|^2=0.$$

This implies that $\{x^n,n\ge 0\}$ is Cauchy in $\mathcal{B}_T$.
Step 3. We claim the existence and uniqueness of the solution to (2.6). The completeness of $\mathcal{B}_T$ guarantees the existence of a process $x\in\mathcal{B}_T$ such that

$$\lim_{n\to\infty}\sup_{-\infty\le s\le T}\mathbb{E}\|x^n(s)-x(s)\|^2=0.$$

Hence, letting $n\to\infty$ and taking limits on both sides of (3.3), we obtain that $x(t)$ is a solution to (2.6). This shows the existence. And the uniqueness of the solutions could be obtained by the same procedure as step 2. The proof is complete. □

Theorem 3.5. *Let (H1)-(H4) and (H5') hold. Then (2.6) admits a unique mild solution $x(t)\in\mathcal{B}_T$ provided that*

$$L:=6L_*^2M_*^2+6M^2/\beta^2<1.\tag{3.11}$$

Proof. Let N be a positive integer and $T_0\in(0,T)$. We introduce the sequence of the functions $f^N(t,u_t)$ and $g^N(t,u_t)$, $(t,u_t)\in[0,T]\times\mathfrak{B}$ as follows:

$$f^N(t,u)=\begin{cases} f(t,u_t), & \|u\|_t\le N;\\ f(t,\frac{Nu}{\|u\|_t}), & \|u\|_t>N,\end{cases}$$

$$g^N(t,u_t)=\begin{cases} g(t,u_t), & \|u\|_t\le N;\\ g(t,\frac{Nu}{\|u\|_t}), & \|u\|_t>N.\end{cases}$$

Then the functions $\{f^N(t,u_t)\}$ and $\{g^N(t,u_t)\}$ satisfy assumption (H4), and for any $x,y\in\mathfrak{B}$, $t\in[0,T]$, the following inequality holds:

$$\|f^N(t,x_t)-f^N(t,y_t)\|^2+\|g^N(t,x_t)-g^N(t,y_t)\|_{\mathcal{L}_2^0}^2\le Z_N\Big(t,\|x-y\|_t^2\Big).$$

As a consequence of Theorem 3.4 , there exist the unique mild solutions $x^N(t)$ and $x^{N+1}(t)$, respectively, to the following integral equations:

$$(3.12)\begin{cases} x^N(t) = U(t,0)(\varphi(0)+G(0,\varphi)) - G(t,x_t^N) \\ \qquad + \int_0^t A(s)U(t,s)G(s,x_s^N)ds + \int_0^t U(t,s)f^N(s,x_s^N)ds \\ \qquad + \int_0^t U(t,s)g^N(s,x_s^N)dw(s) \\ \qquad + \int_0^t U(t-s)\sigma(s)dB_Q^H(s), \quad t\in[0,T], \\ x^N(t) = \varphi(t), \quad t\le 0. \end{cases}$$

$$(3.13)\qquad \begin{cases} x^{N+1}(t) = U(t,0)(\varphi(0)+G(0,\varphi)) - G(t,x_t^{N+1}) \\ \qquad - \int_0^t A(s)U(t,s)G(s,x_s^{N+1})ds \\ \qquad + \int_0^t U(t,s)f^{N+1}(s,x_s^{N+1})ds \\ \qquad + \int_0^t U(t,s)g^{N+1}(s,x_s^{N+1})dw(s) \\ \qquad + \int_0^t U(t-s)\sigma(s)dB_Q^H(s), \quad t\in[0,T], \\ x^{N+1}(t) = \varphi(t), \quad t\le 0. \end{cases}$$

Define the stopping time

$$\delta_N := T_0 \wedge \inf\{t\in[0,T] : \|x^N\|_t \ge N\},$$

$$\delta_{N+1} := T_0 \wedge \inf\{t\in[0,T] : \|x^{N+1}\|_t \ge N+1\},$$

$$\tau_N := \delta_N \wedge \delta_{N+1}.$$

In view of (3.12) and (3.13), we obtain

$$\begin{aligned} &\mathbb{E}\|x^{N+1}(s) - x^N(s)\|^2 \\ \le& 4\mathbb{E}\|G(s,x_s^{N+1}) - G(s,x_s^N))\|^2 \\ &+ 4\mathbb{E}\Big\|\int_0^s A(r)U(s,r)(G(r,x_r^{N+1}) - G(r,x_r^N))dr\Big\|^2 \\ &+ 4\mathbb{E}\Big\|\int_0^s U(s,r)(f^{N+1}(r,x_r^{N+1}) - f^N(r,x_r^N))dr\Big\|^2 \\ &+ 4\mathbb{E}\Big\|\int_0^s U(s,r)(g^{N+1}(r,x_r^{N+1}) - g^N(r,x_r^N))dw(r)\Big\|^2 \\ =& 4\sum_{i=1}^4 I_i, \end{aligned}$$

where we have used the fact that for $0\le u\le \tau_N$,

$$f^{N+1}(u,x_u^N) = f^N(u,x_u^N), \qquad g^{N+1}(u,x_u^N) = g^N(u,x_u^N).$$

By assumptions (H1) and (H3), we have

$$\sup_{0\le s\le t\wedge\tau_N} I_1 \le L_*^2 M_*^2 \sup_{0\le s\le t\wedge\tau_N} \mathbb{E}\|x^{N+1} - x^N\|_s^2$$

and

$$\sup_{0\le s\le t\wedge\tau_N} I_2 \le \frac{M^2}{\beta^2}\sup_{0\le s\le t\wedge\tau_N}\mathbb{E}\|x^{N+1}-x^N\|_s^2.$$

Employing assumption (H5'), Hölder inequality, and Jensen inequality, it follows that

$$\begin{aligned}
&\sup_{0\le s\le t\wedge\tau_N} I_3\\
\le& M^2\sup_{0\le s\le t\wedge\tau_N}\mathbb{E}\Big(\int_0^s e^{-2\beta(s-u)}du\times\int_0^s\|f^{N+1}(u,x^{N+1})-f^N(u,x^N))\|_u^2du\Big)\\
\le&\frac{M^2}{2\beta}\sup_{0\le s\le t\wedge\tau_N}\mathbb{E}\Big(\int_0^s Z_{N+1}(u,\|x^{N+1}-x^N\|_u^2)du\Big).
\end{aligned}$$

Combining Burkhölder-Davis-Gundy inequality with Jensen inequality, there exists a positive constant K such that

$$\begin{aligned}
\sup_{0\le s\le t\wedge\tau_N} I_4 &\le K\sup_{0\le s\le t\wedge\tau_N}\mathbb{E}\Big(\int_0^s\|g^{N+1}(u,x_u^{N+1})-g^N(u,x_u^N))\|_{\mathcal{L}_2^0}^2du\Big)\\
&\le K\sup_{0\le s\le t\wedge\tau_N}\mathbb{E}\Big(\int_0^s Z_{N+1}(u,\|x^{N+1}-x^N\|_u^2)du\Big).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sup_{0\le s\le t\wedge\tau_N}\mathbb{E}\|x^{N+1}(s)-x^N(s)\|^2\\
\le&4(M^2/2\beta+K)\sup_{0\le s\le t\wedge\tau_N}\mathbb{E}\Big(\int_0^s Z_{N+1}(u,\|x^{N+1}-x^N\|_u^2)du\Big)\\
&+L\sup_{0\le s\le t\wedge\tau_N}\mathbb{E}\|x^{N+1}-x^N\|_s^2.
\end{aligned}$$

Then, for all $t\in[0,T_0]$, by Jensen inequality we have

$$\begin{aligned}
&\sup_{-\infty<s\le t\wedge\tau_N}\mathbb{E}\|x^{N+1}(s)-x^N(s)\|^2\\
\le&\frac{2M^2/\beta+4K}{1-L}\int_0^t Z_{N+1}(s\wedge\tau_N,\|x^{N+1}-x^N\|_{s\wedge\tau_N}^2)ds.
\end{aligned}\tag{3.14}$$

The assumption (H5') indicates that

$$\sup_{-\infty\le s\le t\wedge\tau_N}\mathbb{E}\|x^{N+1}(s)-x^N(s)\|^2=0.$$

Thus, for a.e. ω,

$$x^{N+1}(t)=x^N(t),\qquad \text{for } 0\le t\le T_0\wedge\tau_N.$$

Note that for each $\omega\in\Omega$, there exists an $N_0(\omega)>0$ such that $0<T_0\le\tau_{N_0}$. Define $x(t)$ by

$$x(t)=x^{N_0}(t),\qquad \text{for } t\in[0,T_0].$$

Since $x(t\wedge\tau_N)=x^N(t\wedge\tau_N)$, it holds that

$$\begin{aligned}
x(t\wedge\tau_N)=&U(t\wedge\tau_N,0)(\varphi(0)+G(0,\varphi))-G(t\wedge\tau_N,x^N_{t\wedge\tau_N})\\
&-\int_0^{t\wedge\tau_N}A(s)U(t\wedge\tau_N,s)G(s,x_s^N)ds\\
&+\int_0^{t\wedge\tau_N}U(t\wedge\tau_N,s)f^N(s,x_s^N)ds\\
&+\int_0^{t\wedge\tau_N}U(t\wedge\tau_N,s)g^N(s,x_s^N)dw(s)\\
&+\int_0^{t\wedge\tau_N}U(t\wedge\tau_N,s)\sigma(s)dB_Q^H(s)\\
=&U(t\wedge\tau_N,0)(\varphi(0)+G(0,\varphi))-G(t\wedge\tau_N,x_{t\wedge\tau_N})\\
&-\int_0^{t\wedge\tau_N}A(s)U(t\wedge\tau_N,s)G(s,x_s)ds\\
&+\int_0^{t\wedge\tau_N}U(t\wedge\tau_N,s)f(s,x_s)ds+\int_0^{t\wedge\tau_N}U(t\wedge\tau_N,s)g(s,x_s)dw(s)\\
&+\int_0^{t\wedge\tau_N}U(t\wedge\tau_N,s)\sigma(s)dB_Q^H(s).
\end{aligned}$$

Taking $N\to\infty$, we have

$$\begin{aligned}
x(t)=&U(t,0)(\varphi(0)+G(0,\varphi))-G(t,x_t)-\int_0^t A(s)U(t,s)G(s,x_s)ds\\
&+\int_0^t U(t,s)f(s,x_s)ds+\int_0^t U(t,s)g(s,x_s)dw(s)\\
&+\int_0^t U(t,s)\sigma(s)dB_Q^H(s),
\end{aligned}$$

which completes the proof. □

4. An example

In this section, an example is provided to illustrate the theory obtained.

Let $Y=L^2(0,\pi)$ and $e_n=\sqrt{\frac{2}{\pi}}\sin(nx)$, $n\in\mathbb{N}$. Then $\{e_n\}_{n\in\mathbb{N}}$ is a complete orthonormal basis in Y. Let $X=L^2(0,\pi)$ and $A=\frac{\partial^2}{\partial x^2}$ with domain $D(A)=L_0^1(0,\pi)\cap L^2(0,\pi)$. Then, it is well known that $Au=-\sum_{n=1}^\infty n^2\langle u,e_n\rangle_U e_n$ for any $u\in X$, and A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $S(t):X\to X$, where $S(t)u=\sum_{n=1}^\infty e^{-n^2t}\langle u,e_n\rangle e_n$ and for $\forall t\geq 0$, $\|S(t)\|\leq e^{-t}$. In order to define the operator $Q:Y\to Y$, we choose a sequence $\{\sigma_n\}_{n\geq 1}\subset\mathbb{R}^+$ and set

$Qe_n = \sigma_n e_n$, and assume that $tr(Q) = \sum_{n=1}^{\infty} \sqrt{\sigma_n} < \infty$. Define the process $B_Q^H(s)$ by

$$B_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\sigma_n} B_n^H(t) e_n,$$

where $H \in (1/2, 1)$ and $\{B_n^H\}_{n\in\mathbb{N}}$ is a sequence of two-sided one-dimensional fractional Brownian motions mutually independent.

Then we consider the following stochastic evolution equation:

$$(4.1) \quad \begin{cases} d[u(t,x) - h(t, u(t-r, x))] \\ = \left[\frac{\partial^2}{\partial x^2} u(t,x) + F(t, u(t-r,x))\right] dt + H(t, u(t-r,x)) dw(t) \\ \quad +\Theta(t) dB_Q^H(t),\ t \in [0,T],\ x \in [0,\pi],\ r > 0, \\ u(t,0) = u(t,\pi) = 0,\ \ t \in [0,T], \\ u(t,x) = \varphi(t,x),\ \ t \in (-\infty, 0],\ \ x \in [0,\pi]. \end{cases}$$

Thus, assuming that $h, F : [0,T] \times \mathfrak{B} \to X$, $H : [0,T] \times \mathfrak{B} \to \mathcal{L}_2^0(Y,X)$ by $G(t,z)(\cdot) = h(t, z(\cdot))$, $f(t,z)(\cdot) = F(t, z(\cdot))$, $g(t,z)(\cdot) = H(t, z(\cdot))$, $\sigma(t) = \Theta(t)$, then, system (4.1) can be rewritten as the abstract form as system (2.6). Further, If all the conditions of Theorem 3.1 have been fulfilled (for instance we can take $F(t,u) = L(t)u$ in (H4) and $Z(t,u) = L(t)u^{\alpha}$, where for any $L(t) \geq 0$ is locally integrable and $\frac{1}{2} < \alpha \leq 1$), then we can conclude that there exists a unique mild solution to (4.1).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Each of the authors, Zhi Li, Liping Xu and Xiong Li, contributed to each part of this study equally and read and approved the final version of the manuscript.

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References

[1] P. Acquistapace and B. Terreni, A unified approach to abstract linear nonautonomous parabolic equations, *Rend. Semin. Mat. Univ. Padova* **78** (1987) 47–107.

[2] B. Boufoussi and S. Hajji, Neutral stochastic functional differential equations driven by a fractional Brownian motion in a Hilbert space, *Statist. Probab. Lett.* **82** (2012), no. 8, 1549–1558.

[3] B. Boufoussi, S. Hajji and E. H. Lakhel, Time-dependent neutral stochastic functional differential equation driven by a fractional Brownian motion in a Hlibert space, ArXiv:1401.2555 (2014).

[4] S. Boudrahem and P. R. Rougier, Relation between postural control assessment with eyes open and centre of pressure visual feed back effects in healthy individuals, *Exp. Brain Res.* **195** (2009) 145–152.

[5] T. Caraballo, M. J. Garrido-Atienza and T. Taniguchi, The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion, *Nonlinear Anal.* **74** (2011), no. 11, 3671–3684.

[6] F. Comte and E. Renault, Long memory continuous time models, *J. Econometrics* **73** (1996), no. 1, 101–149.

[7] F. De La, A. L. Perez-Samartin, L. Matnez, M. A. Garcia and A. Vera-Lopez, Long-range correlations in rabbit brain neural activity, *Ann. Biomed Eng.* **34** (2006) 295–299.

[8] G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimensions, Encyclopedia Math. Appl. 44, Cambridge Univ. Press, Cambridge, 1992.

[9] M. E. Gurtin and A. C. Pipkin, A general theory of heat conduction with finite wave speeds, *Arch. Ration. Mech. Anal.* **31** (1968), no. 2, 113–126.

[10] J. Huang, J. Li and T. Shen, Dynamics of stochastic modified Boussinesq approximation equation driven by fractional Brownian motion, *Dyn. Partial Differ. Equ.* **11** (2014), no. 2, 183–209.

[11] R. Jahanipur, Nonlinear functional differential equations of monotone-type in Hilbert spaces, *Nonlinear Anal.* **72** (2010), no. 3-4 1393–1408.

[12] K. Liu, Lyapunov functionals and asymptotic stability of stochastic delay evolution equations, *Stochastics* **63** (1998), no. 1-2, 1–26.

[13] K. Liu, Stability of Ininite Dimensional Stochastic Diferential Equations with Applications, Monographs and Surveys in Pure and Applied Mathematics, 135, Chapman and Hall/CRC, London, 2006.

[14] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, PNLDE, 16, Birkhäuser Verlag, Basel, 1995.

[15] J. Luo and K. Liu, Stability of infinite dimensional stochastic evolution equations with memory and Markovian jumps, *Stochastic Process. Appl.* **118** (2008), no. 5, 864–895.

[16] B. B. Mandelbrot and J. Van Ness, Fractional Brownian motion, fractional noises and applications, *SIAM Rev.* **10** (1968) 422–437.

[17] O. Naghshineh and B. Z. Zangeneh, Existence and measurability of the solution of the stochastic differential equations driven by fractional Brownian motion, *Bull. Iranian Math. Soc.* **35** (2009), no. 2, 47–68.

[18] D. Nualart, The Malliavin Calculus and Related Topics, Springer-Verlag, 2nd edition, Berlin, 2006.

[19] J. W. Nunziato, On heat conduction in materials with memory, *Quart. Appl. Math.* **29** (1971) 187–204.

[20] Y. Ren, X. Chen and R. Sakthivel, On time-dependent stochastic evolution equations driven by fractional Brownian motion in a Hilbert space with finite delay, *Math. Methods Appl. Sci.* **37** (2014), no. 14, 2177–2184.

[21] M. Rypdal and K. Rypdal, Testing hypotheses about sun-climate complexity linking, *Phys. Rev. Lett.* **104** (2010) 128–151.

[22] I. Simonsen, Measuring anti-correlations in the nordic electricity spot market by wavelets, *Phys. A* **322** (2003) 597–606.

[23] W. Willinger, W. Leland, M. Taqqu and D. Wilson, On self-similar nature of ethernet traffic, *IEEE/ACM Trans. Netw.* **2** (1994) 1–15.

[24] F. Wei and K. Wang, The existence and uniqueness of the solution for stochastic functional differential equations with infinite delay, *J. Math. Anal. Appl.* **331** (2007), no. 1, 516–531.

[25] S. Zhou, Z. Wang and D. Feng, Stochastic functional differential equations with infinite delay, *J. Math. Anal. Appl.* **357** (2009), no. 2, 416–426.

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