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**On time-dependent neutral stochastic evolution equations with a fractional Brownian motion and infinite delays**

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## ON TIME-DEPENDENT NEUTRAL STOCHASTIC EVOLUTION EQUATIONS WITH A FRACTIONAL BROWNIAN MOTION AND INFINITE DELAYS

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**ABSTRACT.** In this paper, we consider a class of time-dependent neutral stochastic evolution equations with the infinite delay and a fractional Brownian motion in a Hilbert space. We establish the existence and uniqueness of mild solutions for these equations under non-Lipschitz conditions with Lipschitz conditions which is being considered as a special case. An example is provided to illustrate the theory.

**Keywords:** Stochastic neutral evolution equations, fractional Brownian motion, infinite delay, non-Lipschitz condition.

**MSC(2010):** Primary: 60H15; Secondary: 60G15, 60H05.

### 1. Introduction

It is known that stochastic partial differential equations (SPDEs) play a very important role in formulation and analysis of many phenomena in economic and finance, in physics, mechanics, electric and control engineering, etc. There is much current interest in studying qualitative properties for SPDEs (see, e.g., Da Prato and Zabczyk [8], Liu [12, 13], Wei and Wang [24], Luo and Liu [15], Zhou et al. [25], Jahanipur [11], and references therein).

One solution for many SDEs is a semimartingale as well a Markov process. However, many objects in real world are not always such processes since they have long-range aftereffects. Since the work of Mandelbrot and Van Ness [16], there is an increasing interest in stochastic models based on the fractional Brownian motion. A fractional Brownian motion (fBm) of Hurst parameter  $H \in (0, 1)$  is a centered Gaussian process  $B^H = \{B^H(t), t \geq 0\}$  with the covariance function

$$R_H(t, s) = \mathbb{E}(B^H(t)B^H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

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When  $H = 1/2$ , the fBm becomes the standard Brownian motion, and the fBm  $B^H$  neither is a semimartingale nor a Markov process if  $H \neq 1/2$ . However, the fBm  $B^H$ ,  $H > 1/2$  is a long-memory process and presents an aggregation behavior. The long-memory property make the fBm as a potential candidate to model noise in mathematical finance (see [6,17]); in biology (see [4,7]); in communication networks (see, for instance [23]); the analysis of global temperature anomaly [21] and electricity markets [22] etc.

Recently, stochastic partial functional differential equations driven by a fractional Brownian motion have attracted the interest of many researchers. For example, under the global Lipschitz condition, Caraballo et al. [5] showed the existence, uniqueness and stability of mild solutions for SPDEs with finite delays driven by a fBm; under the global Lipschitz condition, Boufoussi and Hajji [2] considered the existence and uniqueness of mild solutions to neutral SPDEs with finite delays driven by a fBm; Boufoussi et al. [3] obtained the existence and uniqueness result of mild solutions to a class of of time-dependent stochastic functional differential equations driven by a fBm; Ren et al. [20] proved the existence and uniqueness of the mild solution for a class of time-dependent stochastic evolution equations with finite delays driven by a standard cylindrical Wiener process and an independent cylindrical fractional Brownian motion. Huang et al. [10] studied a class of stochastic modified Boussinesq approximation equations driven by a cylindrical fractional Brownian motion.

On the other hand, it is well known that stochastic equations with infinite delays have wide application in many areas (see, e.g. [9,19]). However, to the best of our knowledge, there is no result on stochastic partial differential equations with infinite delays driven by a fBm. To close the gap, we will make the first attempt to study such problem in this paper. We aim to derive the existence and uniqueness of mild solutions under some local conditions.

The rest of this paper is organized as follows. In Section 2, we introduce some necessary notations and preliminaries. In Section 3, the existence and uniqueness of mild solutions are discussed. An example is presented in Section 4 to illustrate the theory.

## 2. Preliminaries

In this section we collect some notions, conceptions and lemmas on Wiener integrals with respect to an infinite dimensional fractional Brownian motion and recall some basic results which will be used throughout the whole of this paper.

Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$  be a filtered complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and  $\mathcal{F}_0$  contains all  $P$ -null sets.

Now we aim at introducing the Wiener integral with respect to the one-dimensional fBm  $B^H$ . Consider a time interval  $[0, T]$  with arbitrary fixed horizon  $T$  and let  $\{B^H(t), t \in [0, T]\}$  be the one-dimensional fractional Brownian

motion with Hurst parameter  $H \in (1/2, 1)$ . This means  $B^H$  has the following Wiener integral representation:

$$B^H(t) = \int_0^t K_H(t, s) dB(s),$$

where  $B = \{B(t) : t \in [0, T]\}$  is a standard Brownian motion, and  $K_H(t, s)$  is the kernel given by

$$K_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

for  $t > s$ , where  $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$  and  $\beta(\cdot, \cdot)$  denotes the Beta function. We put  $K_H(t, s) = 0$  if  $t \leq s$ .

We will denote by  $\mathcal{H}$  the reproducing kernel Hilbert space of the fBm. In fact  $\mathcal{H}$  is the closure of the linear space of indicator functions  $\{I_{[0,t]}, t \in [0, T]\}$  with respect to the scalar product

$$\langle I_{[0,t]}, I_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).$$

The mapping  $I_{[0,t]} \rightarrow B^H(t)$  can be extended to an isometry between  $\mathcal{H}$  and the first Wiener chaos and we will denote by  $B^H(\varphi)$  the image of  $\varphi$  by the such isometry.

We recall that for  $\psi, \varphi \in \mathcal{H}$  their scalar product in  $\mathcal{H}$  is given by

$$\langle \psi, \varphi \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T \psi(s)\varphi(t)|t-s|^{2H-2} ds dt.$$

Let us consider the operator  $K_H^*$  from  $\mathcal{H}$  to  $L^2([0, T])$  defined by

$$(K_H^* \varphi)(s) = \int_s^T \varphi(r) \frac{\partial K}{\partial r}(r, s) dr.$$

We refer to [18] for the proof of the fact that  $K_H^*$  is an isometry between  $\mathcal{H}$  and  $L^2([0, T])$ . Moreover, for any  $\varphi \in \mathcal{H}$ , we have

$$B^H(\varphi) = \int_0^T (K_H^* \varphi)(t) dB(t).$$

Also denoting  $L_{\mathcal{H}}^2([0, T]) = \{\varphi \in \mathcal{H}, K_H^* \varphi \in L^2([0, T])\}$ , since  $H > 1/2$ , we have the following useful Lemma 2.1 (see [18]).

**Lemma 2.1.**

$$L^2([0, T]) \subseteq L^{1/H}([0, T]) \subseteq L_{\mathcal{H}}^2([0, T]) \subseteq \mathcal{H},$$

and for any  $\psi \in L^{1/H}([0, T])$ , we have

$$H(2H-1) \int_0^T \int_0^T |\psi(s)| |\psi(t)| |s-t|^{2H-2} ds dt \leq c_H \|\psi\|_{L^{1/H}([0, T])}^2.$$

Let  $(X, \|\cdot\|, \langle \cdot, \cdot \rangle)$  and  $(Y, \|\cdot\|_Y, \langle \cdot, \cdot \rangle_Y)$  be two real, separable Hilbert spaces and let  $\mathcal{L}(Y, X)$  be the space of bounded linear operator from  $Y$  to  $X$ . Let  $Q \in \mathcal{L}(Y, Y)$  be a non-negative self-adjoint operator. Consider the following series

$$\sum_{n=1}^{\infty} B_n^H(t)e_n, \quad t \geq 0,$$

where  $\{B_n^H(t)\}_{n \in \mathbb{N}}$  is a sequence of two-sided one dimensional standard fBm mutually in dependent and  $\{e_n\}_{n \in \mathbb{N}}$  is a complete orthonormal basis in  $Y$ , the series does not necessarily converge in the space  $Y$ . Therefore, we consider a  $Y$ -valued stochastic process  $B_Q^H(t)$  given by the following series:

$$B_Q^H(t) = \sum_{n=1}^{\infty} B_n^H(t)Q^{\frac{1}{2}}e_n, \quad t \geq 0.$$

Moreover, if  $Q$  is a non-negative self-adjoint trace class operator, then this series converges in the space  $Y$ , that is,  $B_Q^H(t) \in L^2(\Omega, Y)$ . Then, we say that the above  $B_Q^H(t)$  is a  $Y$ -valued  $Q$ -cylindrical fBm with covariance operator  $Q$ . For example, if  $\{\lambda_n\}_{n \in \mathbb{N}}$  is a bounded sequence of non-negative real numbers such that  $Qe_n = \lambda_n e_n$ , assuming that  $Q$  is a nuclear operator in  $Y$  (that is,  $\sum_{n=1}^{\infty} \lambda_n < \infty$ ), then the stochastic process

$$B_Q^H(t) = \sum_{n=1}^{\infty} B_n^H(t)Q^{\frac{1}{2}}e_n = \sum_{n=1}^{\infty} \sqrt{\lambda_n}B_n^H(t)e_n, \quad t \geq 0,$$

is well-defined as a  $Y$ -valued  $Q$ -cylindrical fBm.

In order to define Wiener integrals with respect to the  $Q$ -fBm, we introduce the space  $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y, X)$  of all  $Q$ -Hilbert-Schmidt operators  $\psi : Y \rightarrow X$ . We recall that  $\psi \in \mathcal{L}(Y, X)$  is called a  $Q$ -Hilbert-Schmidt operator, if

$$\|\psi\|_{\mathcal{L}_2^0}^2 := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty,$$

and that the space  $\mathcal{L}_2^0$  equipped with the inner product  $\langle \varphi, \psi \rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n \rangle$  is a separable Hilbert space.

Let  $\phi : [0, T] \rightarrow \mathcal{L}_2^0(Y, X)$  such that

$$(2.1) \quad \sum_{n=1}^{\infty} \|K_H^*(\phi Q^{1/2}e_n)\|_{L^2([0, T]; X)} < \infty.$$

The Wiener integral of  $\phi$  with respect to the  $B_Q^H$  is defined by

$$\begin{aligned} \int_0^t \phi(s)dB_Q^H(s) &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\phi(s)Q^{\frac{1}{2}}e_n dB_n^H(s) \\ &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}(K_H^*(\phi Q^{\frac{1}{2}}e_n))(s)dB_n(s), \end{aligned}$$

where  $B_n$  is the standard Brownian motion used to present  $B_n^H$ .

Notice that if

$$(2.2) \quad \sum_{n=1}^{\infty} \|\phi Q^{1/2} e_n\|_{L^{1/H}([0,T];X)}^2 < \infty,$$

in particular (2.1) holds, which follows immediately from Lemma 2.1.

Next, we introduce the notion of evolution family.

**Definition 2.2.** A set  $\{U(t, s) : 0 \leq s \leq t \leq T\}$  of bounded linear operators on  $X$  is called an evolution family if

- (a)  $U(t, s)U(s, r) = U(t, r)$ ,  $U(s, s) = I$  for  $0 \leq r \leq s \leq t \leq T$ , where  $I$  is the identity operator;
- (b)  $(t, s) \rightarrow U(t, s)$  is strongly continuous for  $0 \leq s \leq t \leq T$ .

Let  $\{A(t), t \in [0, T]\}$  be a family of closed densely defined linear operators on Hilbert space  $X$  and with domain  $D(A(t))$  independent of  $t$ , subject to the following hypothesis introduced by Acquistapace and Terreni in [1].

There exist constants  $\lambda_0 \geq 0$ ,  $\theta \in (\frac{\pi}{2}, \pi)$ ,  $L, K \geq 0$ , and  $\mu, \nu \in (0, 1]$  with  $\mu + \nu > 1$  such that

$$(2.3) \quad \Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0), \quad \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{K}{1 + |\lambda|},$$

and

$$(2.4) \quad \|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq L|t - s|^\mu |\lambda|^{-\nu},$$

for  $t, s \in \mathbb{R}$ ,  $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C} - \{0\} : |\arg \lambda| \leq \theta\}$ .

This assumption implies that there exists a unique evolution family  $\{U(t, s) : 0 \leq s \leq t \leq T\}$  on  $X$  such that  $(t, s) \rightarrow U(t, s) \in \mathcal{L}(X)$  is continuous for  $t > s$ ,  $U(\cdot, s) \in C^1((s, \infty), \mathcal{L}(X))$ ,  $\partial_t U(t, s) = A(t)U(t, s)$ , and

$$(2.5) \quad \|A(t)^k U(t, s)\| \leq C(t - s)^{-k},$$

for  $0 < t - s \leq 1$ ,  $k = 0, 1$ ,  $0 \leq \alpha < \mu$ ,  $x \in D((\lambda_0 - A(s))^\alpha)$ , and a constant  $C$  depending only on the constants in (2.3)-(2.4). Moreover,  $\partial_s^+ U(t, s)x = -U(t, s)A(s)x$  for  $t > s$  and  $x \in D(A(s))$  with  $A(s)x \in \overline{D(A(s))}$ . We say that  $A(\cdot)$  generates  $\{U(t, s) : 0 \leq s \leq t \leq T\}$ . Note that  $\{U(t, s) : 0 \leq s \leq t \leq T\}$  is exponentially bounded by (2.5) with  $k = 0$ .

*Remark 2.3.* If  $A(t), t \geq 0$  is a second order differential operator  $A$ , i.e.  $A(t) = A$  for each  $t \geq 0$ . Then,  $A$  generates a  $C_0$ -semigroup  $\{e^{At}, t \geq 0\}$ .

For additional details on evolution families, we refer the reader to the book by Lunardi [14].

In this paper,  $\mathfrak{B}((-\infty, 0]; L^2(\Omega, X))$  (denoted by  $\mathfrak{B}$  simply) denotes the family of all  $\mathcal{F}_0$ -measurable bounded continuous functions  $\varphi : (-\infty, 0] \rightarrow L^2(\Omega, X)$  endowed with the norm  $\|\varphi\|_t^2 = \sup_{-\infty < \theta \leq t} \mathbb{E}\|\varphi(\theta)\|^2$ . Let  $\mathfrak{B}_{\mathcal{F}_0}^B((-\infty, 0]; X)$  denote the family of all almost surely bounded,  $\mathcal{F}_0$ -measurable,  $\mathfrak{B}$ -valued random



(H2) The function  $\sigma : [0, +\infty) \rightarrow \mathcal{L}_2^0(Y, X)$  satisfies the following conditions: for the complete orthonormal basis  $\{e_n\}_{n \in \mathbb{N}}$  in  $Y$ , we have

$$(\sigma_1) \sum_{n=1}^{\infty} \|\sigma Q^{1/2} e_n\|_{L^2([0, T]; X)} < \infty.$$

$$(\sigma_2) \sum_{n=1}^{\infty} \|\sigma(t) Q^{1/2} e_n\| \text{ is uniformly convergent for } t \in [0, T].$$

(H3) There exists a constant  $L_* < \frac{1}{M_*}$  such that

$$\|A(t)G(t, x_t) - A(t)G(t, y_t)\| \leq L_* \|x - y\|_t,$$

for any  $t \in [0, T]$ ,  $x, y \in \mathfrak{B}$ . Moreover, we assume that  $G(t, 0) = 0$ .

(H4) (a) There exist a function  $F : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $F(t, u)$  is locally integrable in  $t$  for any fixed  $u \geq 0$  and is continuous, nondecreasing, and concave in  $u$  for each fixed  $t \in [0, T]$ . Moreover, for any  $t \in [0, T]$ ,  $x_t \in \mathfrak{B}$ , the following inequality holds:

$$\|f(t, x_t)\|^2 + \|g(t, x_t)\|_{\mathcal{L}_2^0}^2 \leq F(t, \|x\|_t^2).$$

(b) The differential equation

$$\frac{du}{dt} = F(t, u),$$

has a global solution for any initial value  $u_0$ .

(H5) (global conditions)

(a) There exists a function  $Z : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $Z(t, u)$  is locally integrable in  $t$  for any fixed  $u \geq 0$  and is continuous, nondecreasing, and concave in  $u$  for each fixed  $t \in [0, T]$ ,  $Z(t, 0) = 0$  for any  $t \in [0, T]$ . Moreover, for any  $t \in [0, T]$ ,  $x_t, y_t \in \mathfrak{B}$ , the following inequality holds:

$$\|f(t, x_t) - f(t, y_t)\|^2 + \|g(t, x_t) - g(t, y_t)\|_{\mathcal{L}_2^0}^2 \leq Z(t, \|x - y\|_t^2).$$

(b) For any constant  $D > 0$ , if a nonnegative function  $u(t)$  satisfies

$$u(t) \leq D \int_0^t Z(s, u(s)) ds, \quad t \in [0, T],$$

then  $u(t) \equiv 0$  for any  $t \in [0, T]$ .

(H5') (local conditions)

(a) For any integer  $N > 0$ , there exists a function  $Z_N : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $Z_N(t, u)$  is locally integrable in  $t$  for any fixed  $u \geq 0$  and is continuous, nondecreasing, and concave in  $u$  for each fixed  $t \in [0, T]$ ,  $Z_N(t, 0) = 0$  for any  $t \in [0, T]$ . Moreover, for any  $t \in [0, T]$ ,  $x_t, y_t \in \mathfrak{B}$  with  $\|x\|_t \leq N$ ,  $\|y\|_t \leq N$ , the following inequality holds:

$$\|f(t, x_t) - f(t, y_t)\|^2 + \|g(t, x_t) - g(t, y_t)\|_{\mathcal{L}_2^0}^2 \leq Z_N(t, \|x - y\|_t^2).$$

(b) For any constant  $D > 0$ , if a nonnegative function  $u(t)$  satisfies

$$u(t) \leq D \int_0^t Z_N(s, u(s)) ds, \quad t \in [0, T],$$



then  $u(t) \equiv 0$  for any  $t \in [0, T]$ .

*Remark 3.1.* Let  $Z(t, u) = L(t)\bar{Z}(u)$ ,  $t \in [0, T]$ , where  $L(t) \geq 0$  is locally integrable and  $\bar{Z}(u)$  is a concave nondecreasing function from  $\mathbb{R}_+$  to  $\mathbb{R}_+$  such that  $\bar{Z}(0) = 0$ ,  $\bar{Z}(u) > 0$  for  $u > 0$  and  $\int_{0^+} \frac{1}{\bar{Z}(u)} du = \infty$ . Then, by the comparison theorem of differential equations we know that assumption (H5-b) holds.

Now let us give some concrete examples of the function  $\bar{Z}(u)$ . Let  $\zeta > 0$  and let  $\delta \in (0, 1)$  be sufficiently small. Define

$$\bar{Z}_1(u) = \zeta u, \quad u \geq 0,$$

$$\bar{Z}_2(u) = \begin{cases} u \log(u^{-1}), & 0 \leq u \leq \delta, \\ \delta \log(\delta^{-1}) + \bar{Z}_2'(\delta-)(u - \delta), & u > \delta, \end{cases}$$

where  $\bar{Z}_2'$  denotes the derivative of function  $\bar{Z}_2$ . They are all concave nondecreasing functions satisfying  $\int_{0^+} \frac{1}{\bar{Z}_i(u)} du = \infty$  ( $i = 1, 2$ ).

Now, we establish the following lemma, which is useful to prove our results.

**Lemma 3.2.** *Suppose that  $\psi : [0, T] \rightarrow \mathcal{L}_2^0(Y, X)$  such that (2.2) and (H1) hold. Then, for any  $t \in [0, T]$  we have*

$$\mathbb{E} \left\| \int_0^t U(t, s) \psi(s) dB_Q^H(s) \right\|^2 \leq c_H H(2H - 1) M^2 t^{2H-1} \sum_{n=1}^{\infty} \int_0^t \|\psi(s) Q^{\frac{1}{2}} e_n\|^2 ds.$$

If, in addition,

$$(3.1) \quad \sum_{n=1}^{\infty} \|\psi(t) Q^{\frac{1}{2}} e_n\| \quad \text{is uniformly convergent for } t \in [0, T],$$

then

$$\mathbb{E} \left\| \int_0^t U(t, s) \psi(s) dB_Q^H(s) \right\|^2 \leq c_H H(2H - 1) M^2 t^{2H-1} \int_0^t \|\psi(s)\|_{\mathcal{L}_2^0(Y, X)}^2 ds.$$

*Proof.* Let  $\{e_n\}_{n \in \mathbb{N}}$  be the complete orthonormal basis of  $Y$  introduced in Section 2. Thus, applying Lemma 2.1 we obtain

$$\begin{aligned} & \mathbb{E} \left\| \int_0^t U(t, s) \psi(s) dB_Q^H(s) \right\|^2 \\ &= \sum_{n=1}^{\infty} \mathbb{E} \left\| \int_0^t U(t, s) \psi(s) Q^{\frac{1}{2}} e_n dB_n^H(s) \right\|^2 \\ &= \sum_{n=1}^{\infty} H(2H - 1) \int_0^t \int_0^t \langle U(t, s) \psi(s) Q^{\frac{1}{2}} e_n, U(t, r) \psi(r) Q^{\frac{1}{2}} e_n \rangle |s - r|^{2H-2} ds dr \\ &\leq c_H H(2H - 1) \sum_{n=1}^{\infty} \left( \int_0^t \|U(t, s) \psi(s) Q^{\frac{1}{2}} e_n\|^{1/H} ds \right)^{2H} \\ &\leq c_H H(2H - 1) M^2 t^{2H-1} \sum_{n=1}^{\infty} \int_0^t \|\psi(s) Q^{\frac{1}{2}} e_n\|^2 ds. \end{aligned}$$

The second assertion is an immediate consequence of the Weierstrass M-test. □

*Remark 3.3.* If  $\{\sigma_n\}_{n \in \mathbb{N}}$  is a bounded sequence of non-negative real numbers such that the nuclear operator  $Q$  satisfies  $Qe_n = \sigma_n e_n$ , assuming that there exists a positive constant  $k_\psi$  such that

$$\|\psi(s)\|_{\mathcal{L}_2^0(Y, X)}^2 \leq k_\psi, \quad \text{uniformly in } [0, T],$$

then (3.1) holds automatically.

**Theorem 3.4.** *Let (H1)-(H5) hold. Then (2.6) admits a unique mild solution  $x(t) \in \mathcal{B}_T$  provided that*

$$(3.2) \quad 6L_*^2 M_*^2 \beta^2 + 6M^2 < \beta^2.$$

*Proof.* In order to prove the existence of the solution to (2.6), let  $x^0(t) = U(t, 0)\varphi(0)$ ,  $t \in [0, T]$  and  $x^n(t) = \varphi(t)$ ,  $n \geq 1$  for  $-\infty < t \leq 0$ , define the following successive approximations procedure for  $t \in [0, T]$  and  $n \geq 1$ :

$$\begin{aligned} (3.3) \quad x^n(t) &= U(t, 0)(\varphi(0) + G(0, \varphi)) - G(t, x_t^n) - \int_0^t A(s)U(t, s)G(s, x_s^n) ds \\ &+ \int_0^t U(t, s)f(s, x_s^{n-1}) ds + \int_0^t U(t, s)g(s, x_s^{n-1}) dw(s) \\ &+ \int_0^t U(t, s)\sigma(s) dB_Q^H(s). \end{aligned}$$

The proof is divided into the following three steps.

Step 1. For all  $s \in (-\infty, t]$ , we claim that the sequence  $x^n \in \mathcal{B}_T, n \geq 0$  is

bounded. It is obvious that  $x^0(t) \in \mathcal{B}_T$ . From (3.3), for  $0 \leq t \leq T$ ,

$$\begin{aligned}
 & \mathbb{E}\|x^n(s)\|^2 \\
 & \leq 6\mathbb{E}\|U(s,0)(\varphi(0) + G(0,\varphi))\|^2 + 6\mathbb{E}\left\|\int_0^s A(r)U(s,r)G(r,x_r^n)dr\right\|^2 \\
 (3.4) \quad & + 6\mathbb{E}\left\|\int_0^s U(s,r)f(r,x_r^{n-1})dr\right\|^2 + 6\mathbb{E}\left\|\int_0^s U(s,r)g(r,x_r^{n-1})dw(r)\right\|^2 \\
 & + 6\mathbb{E}\left\|\int_0^s U(s,r)\sigma(r)dB_Q^H(r)\right\|^2 + 6\mathbb{E}\|G(s,x_s^n)\|^2 \\
 & = : 6 \sum_{i=1}^6 I_i.
 \end{aligned}$$

By (H1) and (H3), one has

$$(3.5) \quad I_1 \leq 2M^2(1 + L_*^2 M_*^2)\mathbb{E}\|\varphi(0)\|^2, \quad I_6 \leq L_*^2 M_*^2 \mathbb{E}\|x^n\|_s^2$$

and

$$\begin{aligned}
 (3.6) \quad I_2 & \leq \mathbb{E}\left(\int_0^s \|A(r)U(s,r)G(r,x_r^n)\|dr\right)^2 \\
 & \leq \mathbb{E}\left(\int_0^s M e^{-\beta(s-r)} \|x^n\|_r dr\right)^2 \\
 & \leq \frac{M^2}{\beta^2} \mathbb{E}\|x^n\|_s^2.
 \end{aligned}$$

By (H4), we obtain

$$(3.7) \quad I_3 \leq M^2 \mathbb{E} \int_0^s F(r, \|x^{n-1}\|_r^2) dr.$$

By (H4) and Doob martingale inequality, there exists a positive constant  $C_1$  such that

$$(3.8) \quad I_4 \leq C_1 \mathbb{E} \int_0^s F(r, \|x^{n-1}\|_r^2) dr.$$

For  $I_5$ , by Lemma 3.2, we can obtain

$$(3.9) \quad I_5 \leq c_H H(2H - 1)M^2 T^{2H-1} \int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0(Y,X)}^2 ds.$$

Hence, substituting (3.5)-(3.9) into (3.4) yields

$$\mathbb{E}\|x^n(s)\|^2 \leq C_2 + C_3 \mathbb{E}\|x^n\|_s^2 + C_4 \mathbb{E} \int_0^s F(r, \|x^{n-1}\|_r^2) dr,$$

where

$$C_2 = 12M^2(1 + L_*^2 M_*^2)\mathbb{E}\|\varphi(0)\|^2 + 6c_H H(2H - 1)M^2 T^{2H-1} \int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0(Y,X)}^2 ds,$$

$$C_3 = 6L_*^2 M_*^2 + 6M^2/\beta^2 \text{ and } C_4 = C_1 + M^2.$$

Then

$$\begin{aligned} \mathbb{E}\|x^n\|_t^2 &= \sup_{-\infty < s \leq t} \mathbb{E}\|x^n(s)\|^2 \\ &\leq \mathbb{E} \sup_{-\infty < \theta \leq 0} \|\varphi(\theta)\|^2 + C_2 + C_3 \mathbb{E}\|x^n\|_t^2 + C_4 \mathbb{E} \int_0^t F(s, \|x^{n-1}\|_s^2) ds \\ &\leq C_5 + C_6 \mathbb{E} \int_0^t F(s, \|x^{n-1}\|_s^2) ds \end{aligned}$$

where  $C_5 = \frac{C_2 + \mathbb{E} \sup_{-\infty < \theta \leq 0} \|\varphi(\theta)\|^2}{1 - C_3}$  and  $C_6 = \frac{C_4}{1 - C_3}$ .

Thus, by Jensen inequality we have

$$\mathbb{E}\|x^n\|_t^2 \leq C_5 + C_6 \int_0^t F(s, \mathbb{E}\|x^{n-1}\|_s^2) ds.$$

Assumption (H4-b) indicates that there is a solution  $u_t$  satisfying

$$u_t = 2C_5 + 2C_6 \int_0^t F(s, u_r) ds.$$

Since  $\mathbb{E}\|x^0(t)\| \leq M^2 \|\varphi(0)\| < \infty$ , we have  $\mathbb{E}\|x^n\|_t^2 \leq u_t \leq u_T < \infty$ , which shows that the boundedness of  $\{x^n(t), n \geq 0\}$ .

Step 2. We claim that  $\{x^n(t), n \geq 0\}$  is a Cauchy sequence. For all  $n, m \geq 0$  and  $t \in [0, T]$ , from (3.3), (H5) and Step 1, we have

$$\begin{aligned} &\mathbb{E}\|x^{n+1}(s) - x^{m+1}(s)\|^2 \\ &\leq 4\mathbb{E}\|G(s, x_s^{n+1}) - G(s, x_s^{m+1})\|^2 \\ &\quad + 4\mathbb{E}\left\| \int_0^s A(r)U(s,r)(G(r, x_r^{n+1}) - G(r, x_r^{m+1}))dr \right\|^2 \\ &\quad + 4\mathbb{E}\left\| \int_0^s U(s,r)(f(r, x_r^n) - f(r, x_r^m))dr \right\|^2 \\ &\quad + 4\mathbb{E}\left\| \int_0^s U(s,r)(g(r, x_r^n) - g(r, x_r^m))dw(r) \right\|^2. \end{aligned}$$

By (H5) and Burkholder-Davis-Gundy inequality, there exists a positive constant  $C_7$  such that

$$\begin{aligned} &\mathbb{E}\|x^{n+1}(s) - x^{m+1}(s)\|^2 \\ &\leq 4(L_*^2 M_*^2 + \frac{M^2}{\beta^2})\mathbb{E}\|x^{n+1} - x^{m+1}\|_s^2 + (4M^2 + C_7)\mathbb{E} \int_0^s Z(r, \|x^n - x^m\|_r^2) dr. \end{aligned}$$

Then, by Jensen inequality we have

$$(3.10) \quad \begin{aligned} \mathbb{E}\|x^{n+1} - x^{m+1}\|_t^2 &= \sup_{-\infty < s \leq t} \mathbb{E}\|x^{n+1}(s) - x^{m+1}(s)\|^2 \\ &\leq \frac{4M^2 + C_7}{1 - 4L_*^2 M_*^2 - 4M^2/\beta^2} \int_0^t Z(s, \mathbb{E}\|x^n - x^m\|_s^2) ds. \end{aligned}$$

By (3.10) and Fatou lemma, we have

$$\begin{aligned} &\limsup_{n,m \rightarrow \infty} \left( \sup_{0 \leq s \leq t} \mathbb{E}\|x^{n+1}(s) - x^{m+1}(s)\|^2 \right) \\ &\leq C_8 \int_0^t Z(s, \limsup_{n,m \rightarrow \infty} \left( \sup_{0 \leq \theta \leq s} \mathbb{E}\|x^{n+1}(\theta) - x^{m+1}(\theta)\|^2 \right)) ds, \end{aligned}$$

where  $C_8 = \frac{4M^2 + C_7}{1 - 4L_*^2 M_*^2 - 4M^2/\beta^2}$ . By assumption (H5-b) we obtain

$$\lim_{n,m \rightarrow \infty} \sup_{0 \leq s \leq T} \mathbb{E}\|x^{n+1}(s) - x^{m+1}(s)\|^2 = 0.$$

This implies that  $\{x^n, n \geq 0\}$  is Cauchy in  $\mathcal{B}_T$ .

Step 3. We claim the existence and uniqueness of the solution to (2.6). The completeness of  $\mathcal{B}_T$  guarantees the existence of a process  $x \in \mathcal{B}_T$  such that

$$\lim_{n \rightarrow \infty} \sup_{-\infty \leq s \leq T} \mathbb{E}\|x^n(s) - x(s)\|^2 = 0.$$

Hence, letting  $n \rightarrow \infty$  and taking limits on both sides of (3.3), we obtain that  $x(t)$  is a solution to (2.6). This shows the existence. And the uniqueness of the solutions could be obtained by the same procedure as step 2. The proof is complete.  $\square$

**Theorem 3.5.** *Let (H1)-(H4) and (H5') hold. Then (2.6) admits a unique mild solution  $x(t) \in \mathcal{B}_T$  provided that*

$$(3.11) \quad L := 6L_*^2 M_*^2 + 6M^2/\beta^2 < 1.$$

*Proof.* Let  $N$  be a positive integer and  $T_0 \in (0, T)$ . We introduce the sequence of the functions  $f^N(t, u_t)$  and  $g^N(t, u_t)$ ,  $(t, u_t) \in [0, T] \times \mathfrak{B}$  as follows:

$$\begin{aligned} f^N(t, u) &= \begin{cases} f(t, u_t), & \|u\|_t \leq N; \\ f(t, \frac{Nu}{\|u\|_t}), & \|u\|_t > N, \end{cases} \\ g^N(t, u_t) &= \begin{cases} g(t, u_t), & \|u\|_t \leq N; \\ g(t, \frac{Nu}{\|u\|_t}), & \|u\|_t > N. \end{cases} \end{aligned}$$

Then the functions  $\{f^N(t, u_t)\}$  and  $\{g^N(t, u_t)\}$  satisfy assumption (H4), and for any  $x, y \in \mathfrak{B}$ ,  $t \in [0, T]$ , the following inequality holds:

$$\|f^N(t, x_t) - f^N(t, y_t)\|^2 + \|g^N(t, x_t) - g^N(t, y_t)\|_{\mathcal{L}_2^0}^2 \leq Z_N \left( t, \|x - y\|_t^2 \right).$$

As a consequence of Theorem 3.4, there exist the unique mild solutions  $x^N(t)$  and  $x^{N+1}(t)$ , respectively, to the following integral equations:

$$(3.12) \left\{ \begin{array}{l} x^N(t) = U(t, 0)(\varphi(0) + G(0, \varphi)) - G(t, x_t^N) \\ \quad + \int_0^t A(s)U(t, s)G(s, x_s^N)ds + \int_0^t U(t, s)f^N(s, x_s^N)ds \\ \quad + \int_0^t U(t, s)g^N(s, x_s^N)dw(s) \\ \quad + \int_0^t U(t-s)\sigma(s)dB_Q^H(s), \quad t \in [0, T], \\ x^N(t) = \varphi(t), \quad t \leq 0. \end{array} \right.$$

$$(3.13) \left\{ \begin{array}{l} x^{N+1}(t) = U(t, 0)(\varphi(0) + G(0, \varphi)) - G(t, x_t^{N+1}) \\ \quad - \int_0^t A(s)U(t, s)G(s, x_s^{N+1})ds \\ \quad + \int_0^t U(t, s)f^{N+1}(s, x_s^{N+1})ds \\ \quad + \int_0^t U(t, s)g^{N+1}(s, x_s^{N+1})dw(s) \\ \quad + \int_0^t U(t-s)\sigma(s)dB_Q^H(s), \quad t \in [0, T], \\ x^{N+1}(t) = \varphi(t), \quad t \leq 0. \end{array} \right.$$

Define the stopping time

$$\delta_N := T_0 \wedge \inf\{t \in [0, T] : \|x^N\|_t \geq N\},$$

$$\delta_{N+1} := T_0 \wedge \inf\{t \in [0, T] : \|x^{N+1}\|_t \geq N + 1\},$$

$$\tau_N := \delta_N \wedge \delta_{N+1}.$$

In view of (3.12) and (3.13), we obtain

$$\begin{aligned} & \mathbb{E}\|x^{N+1}(s) - x^N(s)\|^2 \\ & \leq 4\mathbb{E}\|G(s, x_s^{N+1}) - G(s, x_s^N)\|^2 \\ & \quad + 4\mathbb{E}\left\| \int_0^s A(r)U(s, r)(G(r, x_r^{N+1}) - G(r, x_r^N))dr \right\|^2 \\ & \quad + 4\mathbb{E}\left\| \int_0^s U(s, r)(f^{N+1}(r, x_r^{N+1}) - f^N(r, x_r^N))dr \right\|^2 \\ & \quad + 4\mathbb{E}\left\| \int_0^s U(s, r)(g^{N+1}(r, x_r^{N+1}) - g^N(r, x_r^N))dw(r) \right\|^2 \\ & = 4 \sum_{i=1}^4 I_i, \end{aligned}$$

where we have used the fact that for  $0 \leq u \leq \tau_N$ ,

$$f^{N+1}(u, x_u^N) = f^N(u, x_u^N), \quad g^{N+1}(u, x_u^N) = g^N(u, x_u^N).$$

By assumptions (H1) and (H3), we have

$$\sup_{0 \leq s \leq t \wedge \tau_N} I_1 \leq L_*^2 M_*^2 \sup_{0 \leq s \leq t \wedge \tau_N} \mathbb{E}\|x^{N+1} - x^N\|_s^2$$

and

$$\sup_{0 \leq s \leq t \wedge \tau_N} I_2 \leq \frac{M^2}{\beta^2} \sup_{0 \leq s \leq t \wedge \tau_N} \mathbb{E} \|x^{N+1} - x^N\|_s^2.$$

Employing assumption (H5'), Hölder inequality, and Jensen inequality, it follows that

$$\begin{aligned} & \sup_{0 \leq s \leq t \wedge \tau_N} I_3 \\ & \leq M^2 \sup_{0 \leq s \leq t \wedge \tau_N} \mathbb{E} \left( \int_0^s e^{-2\beta(s-u)} du \times \int_0^s \|f^{N+1}(u, x^{N+1}) - f^N(u, x^N)\|_u^2 du \right) \\ & \leq \frac{M^2}{2\beta} \sup_{0 \leq s \leq t \wedge \tau_N} \mathbb{E} \left( \int_0^s Z_{N+1}(u, \|x^{N+1} - x^N\|_u^2) du \right). \end{aligned}$$

Combining Burkholder-Davis-Gundy inequality with Jensen inequality, there exists a positive constant  $K$  such that

$$\begin{aligned} \sup_{0 \leq s \leq t \wedge \tau_N} I_4 & \leq K \sup_{0 \leq s \leq t \wedge \tau_N} \mathbb{E} \left( \int_0^s \|g^{N+1}(u, x_u^{N+1}) - g^N(u, x_u^N)\|_{\mathcal{L}_2^0}^2 du \right) \\ & \leq K \sup_{0 \leq s \leq t \wedge \tau_N} \mathbb{E} \left( \int_0^s Z_{N+1}(u, \|x^{N+1} - x^N\|_u^2) du \right). \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \sup_{0 \leq s \leq t \wedge \tau_N} \mathbb{E} \|x^{N+1}(s) - x^N(s)\|^2 \\ & \leq 4(M^2/2\beta + K) \sup_{0 \leq s \leq t \wedge \tau_N} \mathbb{E} \left( \int_0^s Z_{N+1}(u, \|x^{N+1} - x^N\|_u^2) du \right) \\ & \quad + L \sup_{0 \leq s \leq t \wedge \tau_N} \mathbb{E} \|x^{N+1} - x^N\|_s^2. \end{aligned}$$

Then, for all  $t \in [0, T_0]$ , by Jensen inequality we have

$$(3.14) \quad \begin{aligned} & \sup_{-\infty < s \leq t \wedge \tau_N} \mathbb{E} \|x^{N+1}(s) - x^N(s)\|^2 \\ & \leq \frac{2M^2/\beta + 4K}{1-L} \int_0^t Z_{N+1}(s \wedge \tau_N, \|x^{N+1} - x^N\|_{s \wedge \tau_N}^2) ds. \end{aligned}$$

The assumption (H5') indicates that

$$\sup_{-\infty \leq s \leq t \wedge \tau_N} \mathbb{E} \|x^{N+1}(s) - x^N(s)\|^2 = 0.$$

Thus, for a.e.  $\omega$ ,

$$x^{N+1}(t) = x^N(t), \quad \text{for } 0 \leq t \leq T_0 \wedge \tau_N.$$

Note that for each  $\omega \in \Omega$ , there exists an  $N_0(\omega) > 0$  such that  $0 < T_0 \leq \tau_{N_0}$ .

Define  $x(t)$  by

$$x(t) = x^{N_0}(t), \quad \text{for } t \in [0, T_0].$$

Since  $x(t \wedge \tau_N) = x^N(t \wedge \tau_N)$ , it holds that

$$\begin{aligned}
 x(t \wedge \tau_N) &= U(t \wedge \tau_N, 0)(\varphi(0) + G(0, \varphi)) - G(t \wedge \tau_N, x_{t \wedge \tau_N}^N) \\
 &\quad - \int_0^{t \wedge \tau_N} A(s)U(t \wedge \tau_N, s)G(s, x_s^N)ds \\
 &\quad + \int_0^{t \wedge \tau_N} U(t \wedge \tau_N, s)f^N(s, x_s^N)ds \\
 &\quad + \int_0^{t \wedge \tau_N} U(t \wedge \tau_N, s)g^N(s, x_s^N)dw(s) \\
 &\quad + \int_0^{t \wedge \tau_N} U(t \wedge \tau_N, s)\sigma(s)dB_Q^H(s) \\
 &= U(t \wedge \tau_N, 0)(\varphi(0) + G(0, \varphi)) - G(t \wedge \tau_N, x_{t \wedge \tau_N}) \\
 &\quad - \int_0^{t \wedge \tau_N} A(s)U(t \wedge \tau_N, s)G(s, x_s)ds \\
 &\quad + \int_0^{t \wedge \tau_N} U(t \wedge \tau_N, s)f(s, x_s)ds + \int_0^{t \wedge \tau_N} U(t \wedge \tau_N, s)g(s, x_s)dw(s) \\
 &\quad + \int_0^{t \wedge \tau_N} U(t \wedge \tau_N, s)\sigma(s)dB_Q^H(s).
 \end{aligned}$$

Taking  $N \rightarrow \infty$ , we have

$$\begin{aligned}
 x(t) &= U(t, 0)(\varphi(0) + G(0, \varphi)) - G(t, x_t) - \int_0^t A(s)U(t, s)G(s, x_s)ds \\
 &\quad + \int_0^t U(t, s)f(s, x_s)ds + \int_0^t U(t, s)g(s, x_s)dw(s) \\
 &\quad + \int_0^t U(t, s)\sigma(s)dB_Q^H(s),
 \end{aligned}$$

which completes the proof.  $\square$

#### 4. An example

In this section, an example is provided to illustrate the theory obtained.

Let  $Y = L^2(0, \pi)$  and  $e_n = \sqrt{\frac{2}{\pi}} \sin(nx)$ ,  $n \in \mathbb{N}$ . Then  $\{e_n\}_{n \in \mathbb{N}}$  is a complete orthonormal basis in  $Y$ . Let  $X = L^2(0, \pi)$  and  $A = \frac{\partial^2}{\partial x^2}$  with domain  $D(A) = L_0^1(0, \pi) \cap L^2(0, \pi)$ . Then, it is well known that  $Au = -\sum_{n=1}^{\infty} n^2 \langle u, e_n \rangle_U e_n$  for any  $u \in X$ , and  $A$  is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators  $S(t) : X \rightarrow X$ , where  $S(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} \langle u, e_n \rangle e_n$  and for  $\forall t \geq 0$ ,  $\|S(t)\| \leq e^{-t}$ . In order to define the operator  $Q : Y \rightarrow Y$ , we choose a sequence  $\{\sigma_n\}_{n \geq 1} \subset \mathbb{R}^+$  and set



$Qe_n = \sigma_n e_n$ , and assume that  $tr(Q) = \sum_{n=1}^{\infty} \sqrt{\sigma_n} < \infty$ . Define the process  $B_Q^H(s)$  by

$$B_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\sigma_n} B_n^H(t) e_n,$$

where  $H \in (1/2, 1)$  and  $\{B_n^H\}_{n \in \mathbb{N}}$  is a sequence of two-sided one-dimensional fractional Brownian motions mutually independent.

Then we consider the following stochastic evolution equation:

$$(4.1) \quad \begin{cases} d[u(t, x) - h(t, u(t - r, x))] \\ = \left[ \frac{\partial^2}{\partial x^2} u(t, x) + F(t, u(t - r, x)) \right] dt + H(t, u(t - r, x)) dw(t) \\ + \Theta(t) dB_Q^H(t), \quad t \in [0, T], \quad x \in [0, \pi], \quad r > 0, \\ u(t, 0) = u(t, \pi) = 0, \quad t \in [0, T], \\ u(t, x) = \varphi(t, x), \quad t \in (-\infty, 0], \quad x \in [0, \pi]. \end{cases}$$

Thus, assuming that  $h, F : [0, T] \times \mathfrak{B} \rightarrow X$ ,  $H : [0, T] \times \mathfrak{B} \rightarrow \mathcal{L}_2^0(Y, X)$  by  $G(t, z)(\cdot) = h(t, z(\cdot))$ ,  $f(t, z)(\cdot) = F(t, z(\cdot))$ ,  $g(t, z)(\cdot) = H(t, z(\cdot))$ ,  $\sigma(t) = \Theta(t)$ , then, system (4.1) can be rewritten as the abstract form as system (2.6). Further, If all the conditions of Theorem 3.1 have been fulfilled ( for instance we can take  $F(t, u) = L(t)u$  in (H4) and  $Z(t, u) = L(t)u^\alpha$ , where for any  $L(t) \geq 0$  is locally integrable and  $\frac{1}{2} < \alpha \leq 1$ ), then we can conclude that there exists a unique mild solution to (4.1).

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

Each of the authors, Zhi Li, Liping Xu and Xiong Li, contributed to each part of this study equally and read and approved the final version of the manuscript.

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