ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 42 (2016), No. 6, pp. 1523–1534

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 6, pp. 1523–1534
Online ISSN: 1735-8515

A REMARK ON REMAINDERS OF HOMOGENEOUS SPACES IN SOME COMPACTIFICATIONS

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(Communicated by Fariborz Azarpanah)

Abstract. We prove that a remainder Y of a non-locally compact rectifiable space X is locally a p-space if and only if either X is a Lindelöf p-space or X is σ-compact, which improves two results by Arhangel'skii. We also show that if a non-locally compact rectifiable space X that is locally paracompact has a remainder Y which has locally a G_δ-diagonal, then both X and Y are separable and metrizable, which improves another Arhangel'skii's result. It is proved that if a non-locally compact paratopological group G has a locally developable remainder Y, then either G and Y are separable and metrizable, or G is a σ-compact space with a countable network, which improves a result by Wang-He.
Keywords: Remainder, rectifiable space, p-space, paratopological group.
MSC(2010): Primary: 54D40; Secondary: 54E35, 22A05.

1. Introduction

Throughout this paper a space always means a Tychonoff topological space. A remainder of a space X is the subspace $bX \setminus X$ of a Hausdorff compactification bX of X. Remainders of topological groups and rectifiable spaces have been studied extensively in the literature (see [2–4, 6–8, 13–15]).

Recall that a topological group is a group G with a topology such that multiplication on G considered as a map of $G \times G$ to G is jointly continuous and the inversion in G is continuous. A paratopological group is a group G with a topology such that multiplication on G is jointly continuous. Clearly every topological group is a paratopological group and every paratopological group is a homogeneous space.

A space X is of countable type if every compact subspace P of X is contained in a compact subspace $F \subseteq X$ which has a countable base of open

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Article electronically published on December 18, 2016.
Received: 2 December 2014, Accepted: 27 September 2015.
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neighborhoods in X. All metrizable spaces, and all locally compact Hausdorff spaces, as well as all Čech-complete spaces, are of countable type.

A space X is Ohio complete [2], if in every compactification bX of X there exists a G_δ-subset $Z \subseteq bX$ such that $X \subset Z$ and every $y \in Z \setminus X$ is separated from X by a G_δ-subset of bX. According to [2], every Tychonoff space with a G_δ-diagonal and every p-space are Ohio complete.

Say that X is locally Ohio complete, if for each $x \in X$, there exists a closed neighbourhood U of x such that U is Ohio complete.

A space X is called a p-space, if in any (in some) compactification bX of X there exists a countable family $\{\gamma_n \mid n \in \omega\}$ of families γ_n of open subsets of bX such that $x \in \bigcap\{St_{\gamma_n}(x) \mid n \in \omega\} \subset X$ for each $x \in X$. It is well known that every p-space is of countable type, and every metrizable space is a p-space.

The following two results on remainders of topological spaces are well known.

Theorem 1.1. [13] *A Tychonoff space X is of countable type if and only if the remainder in any (or some) Hausdorff compactification of X is Lindelöf.*

Theorem 1.2. [2] *If X is a Lindelöf p-space, then any remainder of X is a Lindelöf p-space.*

In this paper, we mainly investigate some local properties of remainders of homogeneous spaces (rectifiable spaces, topological groups, paratopological groups). These local properties include locally Ohio complete spaces, local p-spaces and spaces with locally G_δ-diagonal.

In [2, Theorem 4.5] ([7, Theorem 3.8]), Arhangel'skii proved that a remainder Y of a non-locally compact topological group (rectifiable space) G is a p-space if and only if either G is a Lindelöf p-space or G is σ-compact.

We will show that a stronger result holds for rectifiable spaces, i.e., we prove that:

- A remainder Y of a non-locally compact rectifiable space X is locally a p-space if and only if either G is a Lindelöf p-space or G is σ-compact.

In [7, Theorem 3.9], Arhangel'skii proved that if a non-locally compact paracompact rectifiable space X has a remainder Y with a G_δ-diagonal, then both X and Y are separable and metrizable. In this paper, we will prove the following strengthened result:

- If a non-locally compact locally paracompact rectifiable space X has a remainder Y with locally a G_δ-diagonal, then both X and Y are separable and metrizable.

Some other results obtained in this paper on remainders are as follows .

- Let X be a homogeneous space with a compactification bX such that the remainder $Y = bX \setminus X$ is locally Ohio complete. Then at least one of the following conditions holds:
  (1) X is of point-countable type;

(2) X is locally σ-compact.

- Let G be a topological group of countable cellularity and let bG be a compactification of G. Then the following conditions are equivalent:
  (1) $Y = bG \setminus G$ is locally Ohio complete;
  (2) G is a Lindelöf p-space, or σ-compact.
- Suppose that X and Y are rectifiable spaces. If X is a p-space r-equivalent to Y, then Y is a p-space.
- If bX is a compactification of a p-space X with no isolated points, then the cellularity of $Y = bX \setminus X$ is greater than ω.
- Suppose that bX is a compactification of a topological p-group X. If X is non-discrete, then $bX \setminus X$ is pseudocompact and $bX \setminus X$ is compact if X is discrete.

We will denote $c(X)$ the cellularity or Souslin number of a space X. $\overline{A}^X$ and $\text{int}_X A$ stand for the closure and the interior of A in X respectively. For other terms and symbols we refer to [10].

2. **Main results**

Firstly we consider the conditions of a homogeneous space with a remainder that is locally Ohio complete.

Theorem 2.1. *Let X be a homogeneous space with a compactification bX such that the remainder $Y = bX \setminus X$ is locally Ohio complete. Then at least one of the following conditions holds:*
(1) X is of point-countable type;
(2) X is locally σ-compact.

Proof. If X is locally compact, then X is of countable type.

If X is non-locally compact, then X is nowhere locally compact since X is a homogeneous space, which implies that $Y = bX \setminus X$ is dense in bX. Hence, bX is also a compactification of $bX \setminus X$. Since Y is locally Ohio complete, we can fix an open subset O of Y such that $F = \overline{O}^Y$ is an Ohio complete space. Put $K = \overline{O}^{bX}$. Then there exists a G_δ-set Z of K such that: (a) $F \subset Z$; (b) for each $x \in Z \setminus F$, there is a G_δ-set P of K such that $x \in P$ and $P \cap F = \emptyset$. We consider the following two cases:

(1) $Z = F$. Then $K \setminus F$ is a σ-compact subset of K. Since $K \setminus F \subset X$, $K \setminus F$ is a σ-compact subset of X. Observe that $\text{int}_X(K \setminus F) \neq \emptyset$. Therefore it follows from X being homogeneous that X is locally σ-compact.

(2) $Z \setminus F \neq \emptyset$. Notice that $U = \text{int}_X(K \setminus F)$ is a dense subset of K. Now we consider the next two subcases:

(2a) $Z \cap U = \emptyset$, i.e., $U \subset (K \setminus Z)$. Since $K \setminus Z$ is a σ-compact subset of X, it follows from the regularity of X that U is locally σ-compact. Therefore X is locally σ-compact.

(2b) $Z \cap U \neq \emptyset$. Fix a point $x \in Z \cap U$. Then there is a G_δ-set P of K such that $x \in P \subset Z \setminus F$. Take a sequence $\{P_n : n \in \omega\}$ of open subsets of K such that $P = \cap\{P_n : n \in \omega\}$. Fix a sequence $\{W_n : n \in \omega\}$ of open neighbourhoods of x in bX such that $W_0 \subset K$ and $\overline{W_{n+1}}^{bX} \subset W_n \cap P_n$ for each $n \in \omega$. It is easy to see that $\cap\{W_n : n \in \omega\}$, contained in X, is a compact subset of bX and $\{W_n : n \in \omega\}$ is a base of $\cap\{W_n : n \in \omega\}$ in bX. Hence, $\cap\{W_n : n \in \omega\}$ has a countable outer base in X. Since X is a homogeneous space, X is of point-countable type. □

Corollary 2.2. *Assume that X is homogeneous space with locally a G_δ-diagonal. If X has a compactification bX such that $Y = bX \setminus X$ is locally Ohio complete, then at least one of the following conditions holds:*

(1) X is first-countable;

(2) X has a locally countable network.

Proof. By Theorem 2.1, we need to consider the following two cases.

Case 1: X is of point-countable type. Since X has locally a G_δ-diagonal, the pseudocharacter of X is countable. Then it follows that X is first-countable.

Case 2: X is locally σ-compact. By [10], every compact space with a G_δ-diagonal has a countable base. It follows that a compact space with locally a G_δ-diagonal has a countable base. Since X is locally σ-compact and has locally a G_δ-diagonal, X has a locally countable network. □

Since every topological group of point-countable type is a paracompact p-space [5, Theorem 4.3.35] and each paratopological group of point-countable type is of countable type [7], by Theorem 2.1, the following result is obvious.

Corollary 2.3. *Suppose that X is a paratopological group (or a topological group). If bX is a compactification of X such that $Y = bX \setminus X$ is locally Ohio complete, then at least one of the following conditions holds:*

(1) X is of countable type (or a paracompact p-space);

(2) X is locally σ-compact.

The next result is about a topological group with a remainder that is locally Ohio complete.

Theorem 2.4. *Let G be a topological group of countable cellularity and let bG be a compactification of G. Then the following conditions are equivalent:*

(1) $Y = bG \setminus G$ is locally Ohio complete;

(2) G is either a Lindelöf p-space, or σ-compact.

Proof. We consider two cases.

Case 1: G is locally compact. Since G is dense in bG, it follows that G is an open subset of bG. Hence Y is compact, which implies that Y is locally Ohio complete. Since G is a locally compact topological group, it follows that

G is a paracompact p-space. Noticing that G has countable cellularity one can conclude that G is a Lindelöf p-space.

Case 2: G is non-locally compact. Then G is nowhere locally compact since G is a homogeneous space.

(2)⇒ (1): If G is a Lindelöf p-space, then Y is a Lindelöf p-space by Theorem 1.2. Since every closed subspace of a p-space is also a p-space and every p-space is Ohio complete [2], it follows that Y is locally Ohio complete. If G is σ-compact, then Y is Čech-complete, which implies that Y is locally Čech-complete. Therefore Y is locally Ohio complete.

(1)⇒ (2): Since Y is locally Ohio complete, by Corollary 2.3, G is a paracompact p-space or locally σ-compact. If G is a paracompact p-space, then G is a Lindelöf p-space by $c(G) = \omega$. If G is locally σ-compact, we can take an open subset of O of G such that $F = \overline{O}^G$ is σ-compact. Let H denote the subgroup of G generated by F. Then H is σ-compact. By [5, Corollary 1.3.3], H is an open subgroup G. Thus, all left cosets of H in G form a cover of G consisting of disjoint open subsets of G. By $c(G) = \omega$, the family $\{xH : x \in G\}$ is countable. Since each xH is homeomorphic to H, it follows that G is σ-compact. □

Corollary 2.5. *Suppose that G is a hereditarily separable topological group, and bG is a compactification of G such that $Y = bG\setminus G$ is locally Ohio complete. Then G is either σ-compact, or has a countable base.*

Proof. From the condition that G is separable we know that G has countable cellularity. By Theorem 2.4, G is either a Lindelöf p-space, or σ-compact. Assume that G is a Lindelöf p-space. Then G is of countable type. Fix a compact subspace K of G that has a countable outer base in G. Since K is hereditarily separable, K has countable tightness. By [17], every compact space of countable tightness has countable π-character, which implies that K has countable π-character. Noticing that K has a countable outer base in G, one can conclude that G has a countable π-base at each point of K. Since G is a homogeneous space, it follows that G has countable π-character. This implies that G has countable character by [5, Proposition 5.2.6]. Since G is a topological group, it follows from [5, Theorem 3.3.12] that G is metrizable. Now the separability of G guarantees that G has a countable base. □

Let X be a Tychonoff space. It is well known that the function space $C_p(X)$ with the topology of pointwise convergence is a topological group. Now we investigate conditions such that the remainder of the topological group $C_p(X)$ is Ohio complete.

Theorem 2.6. *For an arbitrary infinite Tychonoff space X, let bG be a compactification of the topological group $G = C_p(X)$ and $Y = bG \setminus G$, where $G = C_p(X)$ has the topology of pointwise convergence. Then the following conditions are equivalent:*

(1) Y is Ohio complete;
(2) Y is a p-space;
(3) Y is locally a p-space;
(4) Y is a Lindelöf p-space;
(5) $|X| = \omega$.

Proof. Since $G = C_p(X)$ is dense in R^X, it follows from X being infinite that G is not locally compact. Since G is a homogeneous space, G is nowhere locally compact, which implies that Y is dense in bG. Clearly, the cellularity of $G = C_p(X)$ is countable since $G = C_p(X)$ is dense in R^X whose cellularity is countable.

(2)⇒(3) is obvious since the p-property is hereditary to every closed subspace.

(3)⇒(4). Since G is a topological group, it follows from Theorem 2.4 that G is either a Lindelöf p-space or a σ-compact space. If $G = C_p(X)$ is σ-compact, X is finite by [1, I.2.4], which is a contradiction.

(4)⇒(5). Since Y is Lindelöf, $G = C_p(X)$ is of countable type by Theorem 1.1. By [1, I.3.2], X is countable.

(5)⇒(1). From the condition that X is countable we know that G is separable and metrizable. Hence, Y is a Lindelöf p-space by Theorem 1.2, which implies that Y is Ohio complete.

(1)⇒(2). Similarly as in (3)⇒(4), the group G is a Lindelöf p-space. Therefore, Y is a Lindelöf p-space by Theorem 1.2. □

In [18], a paratopological group with developable remainders was studied. The following result improves [18, Theorem 3.2].

Theorem 2.7. *Assume that G is a non-locally compact paratopological group with a compactification bG such that $Y = bG \setminus G$ is locally developable. Then at least one of the following conditions holds:*
(1) both G and Y are separable and metrizable;
(2) G is a σ-compact space with a countable network.

Proof. Since every developable space is a p-space, Y is locally Ohio complete. By Corollary 2.3, G is either of countable type or locally σ-compact.

Case 1: G is of countable type. By Theorem 1.1, Y is a Lindelöf space. Then, from the condition that Y is locally developable, one can conclude that Y has a countable base. Thus G is a Lindelöf p-space by Theorem 1.2. Let $\mathcal{B}_1 = \{B_n : n \in \omega\}$ be a countable base of Y, and fix a sequence $\mathcal{B} = \{O_n : O_n \cap Y = B_n, n \in \omega\}$ consisting of open subsets of bG. Put $\mathcal{B}_2 = \{O_n \cap G : O_n \in \mathcal{B}\}$. Since both G and Y are dense in bG, it follows that $\mathcal{B}_2$ is a countable π-base of G. Particularly, G has countable π-character. Then G has a G_δ-diagonal by [5, Corollary 5.7.5]. Since G is a Lindelöf p-space with a G_δ-diagonal, G is separable and metrizable.

Case 2: G is locally σ-compact. Since Y is locally developable, it is locally of countable type. By [18, Lemma 2.2], every space that is locally of countable type is of countable type. Therefore Y is of countable type, hence G is a Lindelöf space by Theorem 1.1. Then G is σ-compact by the condition that G is locally σ-compact. Hence Y is Čech-complete. By [16], each Čech-complete space contains a dense subspace which is paracompact and Čech-complete. Let Z be such a dense subspace of Y. According to [5, Corollary 5.7.12], every σ-compact paratopological group has countable cellularity, so $c(G) = \omega$. Since both G and Z are dense in bG, it follows that $c(Z) = \omega$. Therefore Z is a Lindelöf p-space. Since every developable space has a G_δ-diagonal, the space Y must have a G_δ-diagonal. Hence Z has a G_δ-diagonal. Then it follows that Z has a countable base. Since both G and Z are dense in bG, G has a countable π-base. By Corollary 5.7.5 in [5], G has a G_δ-diagonal. Then from the condition that G is σ-compact we can conclude that G has a countable network. □

Corollary 2.8. *Let G be a non-locally compact paratopological group with a compactification bG such that $Y = bG \setminus G$ is locally metrizable. Then bG is separable and metrizable.*

Proof. By Theorem 2.7, we consider two cases.

(1) Both G and Y are separable and metrizable. In this case, bG is a compact space with a countable network. Therefore, bG is separable and metrizable.

(2) G is a σ-compact space with a countable network. It follows that the cellularities of G and Y are countable. Then Y is locally separable by the condition that Y is locally metrizable. It follows from Theorem 1.2 and the homogeneity of G that G is locally a Lindelöf p-space. Since G has a countable network, it has a G_δ-diagonal. Therefore G has locally a countable base. Then the Lindelöfness of G shows that G has a countable base. It follows that Y is a Lindelöf p-space. Since Y is also locally developable, it has a countable base. Therefore bG is separable and metrizable. □

Theorem 2.10 below generalizes the result of Arhangel'skii and Choban proved in [7, Theorem 3.9]. At first we recall a proposition in [18].

Lemma 2.9. [18] *Suppose that X is a nowhere locally compact space with locally a G_δ-diagonal, and Y is a remainder of X in some compactification bX such that Y is a paracompact p-space. Then $w(X) = \omega$.*

Theorem 2.10. *Suppose that X is a non-locally compact locally paracompact rectifiable space and bX is a compactification of X. If $Y = bX \setminus X$ has locally a G_δ-diagonal, then X and Y are separable and metrizable.*

Proof. By [2], every space with a G_δ-diagonal is Ohio complete, hence Y is locally Ohio complete. Since each rectifiable space is homogeneous, X is of point-countable type or locally σ-compact by Theorem 2.1.

(1) X is of point-countable type. By [7], every rectifiable space of point-countable type is a p-space. Hence X is locally a paracompact p-space. Since X is homogeneous, we can take an open subset U of bX such that $\overline{U}^{bX} \cap Y$ has a G_δ-diagonal and $\overline{U}^{bX} \cap X$ is a paracompact p-space. By Lemma 2.9, $\overline{U}^{bX} \cap Y$ is separable and metrizable. Since both $\overline{U}^{bX} \cap X$ and $\overline{U}^{bX} \cap Y$ are dense in $\overline{U}^{bX}$, $\overline{U}^{bX} \cap X$ has a countable π-base. Thus $U \cap X$ has a countable π-base since $U \cap X$ is dense in $\overline{U}^{bX} \cap X$. Particularly, $U \cap X$ has countable π-character. It implies that X has countable π-character. According to [12], X is metrizable. Therefore X and Y are separable and metrizable by [8].

(2) X is locally σ-compact. Noticing that X is homogeneous, one can fix an open subset O of bX such that $\overline{O \cap X}^X$ is σ-compact and $\overline{O \cap Y}^Y$ has a G_δ-diagonal. Since $\overline{O \cap X}^X$ and $\overline{O \cap Y}^Y$ are dense in $\overline{O}^{bX}$, it follows from the fact that $\overline{O \cap X}^X$ is a Lindelöf space that $\overline{O \cap Y}^Y$ is of countable type. Then the condition that $\overline{O \cap Y}^Y$ has a G_δ-diagonal makes $\overline{O \cap Y}^Y$ first-countable. By [9], every countably compact space with a G_δ-diagonal is compact. Then $\overline{O \cap Y}^Y$ is not countably compact since $\overline{O \cap Y}^Y$ is not compact. Therefore $\overline{O \cap Y}^Y$ contains a countable infinite subset A which is closed and discrete in Y. Since bX is compact, there exists a point $x \in bX$ such that $x \in \overline{A}^{bX}$. Since $\overline{O \cap Y}^Y$ is first-countable and is dense in $\overline{O}^{bX}$, it follows that $\overline{O}^{bX}$ has countable character at each point of $\overline{O \cap Y}^Y$. For each $a \in A$ fix a countable base η_a of $\overline{O}^{bX}$ at a, and put $\eta = \{U \cap X : U \in \bigcup_{a \in A} \eta_a\}$. Since $O \cap X$ is dense in $\overline{O}^{bX}$, η is a countable π-base of $O \cap X$ at x. Obviously, η is also a countable π-base of X at x. Since X is homogeneous, it follows that X has countable π-character. Therefore X and Y are separable and metrizable. □

Corollary 2.11. *Assume that X is a non-locally compact rectifiable space and bX is a compactification of X such that $Y = bX \setminus X$ has locally a G_δ-diagonal. If X has countable tightness or countable pseudocharacter, then X and Y are separable and metrizable.*

Proof. By Theorem 2.1, X is of point-countable type or locally σ-compact.

If X is locally σ-compact, X and Y are separable and metrizable by Theorem 2.10.

Let X be of point-countable type. If X has countable tightness, then X has countable π-character. If X has countable pseudocharacter, then X has countable character. Therefore, X and Y are separable and metrizable. □

By Corollary 2.11, the following result is obvious.

Corollary 2.12. *Assume that X is a non-locally compact rectifiable space and bX is a compactification of X such that $Y = bX \setminus X$ has locally a G_δ-diagonal. If one of the following conditions is satisfied:*

(1) X is hereditarily separable;
(2) X has locally a G_δ-diagonal;
(3) X is a sequential space,
then X and bX are separable and metrizable.

The theorem that follows generalizes the results proved in [2, Theorem 4.5] and [7, Theorem 3.8].

Theorem 2.13. *Let X be a non-locally compact rectifiable space and let bX be a compactification of X. Then $Y = bX \setminus X$ is locally a p-space if and only if at least one of the following conditions holds:*
(1) X is a Lindelöf p-space;
(2) X is σ-compact;

Proof. Sufficiency. If X is a Lindelöf p-space, then Y is a Lindelöf p-space. If X is σ-compact, then Y is Čech-complete. Therefore Y is locally a p-space.

Necessity. Since Y is locally a p-space, it is locally of countable type, which implies that Y is of countable type by [18]. Then X is a Lindelöf space. Since Y is locally Ohio complete, by Theorem 2.1, X is of point-countable type or locally σ-compact.

If X is of point-countable type, then X is a p-space by [7]. Therefore X is a Lindelöf p-space.

If X is locally σ-compact, then X is σ-compact since it is Lindelöf. □

Corollary 2.14. *Let X be a non-locally compact rectifiable space with a compactification bX such that $Y = bX \setminus X$ is locally developable. Then X and Y are separable and metrizable.*

Proof. By Theorem 2.13, X is a Lindelöf p-space or a σ-compact space. Since every developable space has a G_δ-diagonal, it follows from Theorem 2.10 that X and Y are separable and metrizable. □

Corollary 2.15. *Let X be a non-locally compact rectifiable space with a compactification bX such that $Y = bX \setminus X$ is locally a p-space. If X has locally a G_δ-diagonal, then X has a countable network.*

Proof. By Theorem 2.13, X is either a Lindelöf p-space or σ-compact. If X is a Lindelöf p-space, then X is locally separable and locally metrizable since X has locally a G_δ-diagonal. Then the Lindelöfness of X makes X have a countable base. Now we consider the case that X is σ-compact. Observe that every compact space with locally a G_δ-diagonal is separable and metrizable, and hence has a countable network. Therefore, X has a countable network. □

Corollary 2.16. *Suppose that X is a non-locally compact rectifiable space, and bX is a compactification of X such that $Y = bX \setminus X$ is locally a paracompact p-space. If X has locally a G_δ-diagonal, then X is separable and metrizable.*

Proof. By Corollary 2.15, X has a countable network. Then the cellularity of X is countable, which implies that the cellularity of Y is also countable. Since Y is locally a paracompact p-space, Y is locally a Lindelöf p-space. By Theorem 1.2 it follows from the homogeneity of X that X is locally a Lindelöf p-space. Then the fact that X has a countable network guarantees that X is separable and metrizable. □

Two topological spaces X and Y are called r-equivalent [6], if there exists compactifications aX and bY of X and Y respectively such that $aX \setminus X$ and $bY \setminus Y$ are homeomorphic.

Theorem 2.17. *Suppose that X and Y are rectifiable spaces. If X is a p-space r-equivalent to Y, then Y is a p-space.*

Proof. Assume that aX and bY are compactifications of X and Y respectively such that $aX \setminus X$ and $bY \setminus Y$ are homeomorphic. Since every p-space is of countable type, $aX \setminus X$ is a Lindelöf space. Therefore $bY \setminus Y$ is a Lindelöf space. It follows that Y is of countable type. Hence Y is a p-space by [7]. □

Say that X is a p-space, if for each $x \in X$, the intersection of any countable neighbourhoods of x is also a neighbourhood of x. A topological group whose underlying space is a p-space is called a topological p-group. Next we give several results about remainders of p-spaces.

Theorem 2.18. *If bX is a compactification of a p-space X with no isolated points, then the cellularity of $Y = bX \setminus X$ is greater than ω.*

Proof. Since X is a p-space with no isolated points, it follows that X is nowhere locally compact. Fix a point $x \in X$. For each $y \in Y$, take a local base $\mathcal{B}_y$ of y in Y such that $x \notin \overline{B}^{bX}$ for $B \in \mathcal{B}_y$. By Zorn's Lemma, we can take a maximal disjoint family $\mathcal{O}$ of open subsets of Y from $\bigcup_{y \in Y} \mathcal{B}_y$. Since $\bigcup_{y \in Y} \mathcal{B}_y$ is a base of Y, $\bigcup \mathcal{O}$ is dense in Y. $\bigcup \mathcal{O}$ is dense in bX since Y is dense in bX.

Claim: $|\mathcal{O}| > \omega$, which implies that $c(Y) > \omega$.

Suppose that $|\mathcal{O}| = \omega$, and let $\mathcal{O} = \{O_n : n \in \omega\}$. For $n \in \omega$, fix an open subset V_n of bX such that $O_n = V_n \cap Y$. Since $x \notin \overline{O_n}^{bX}$ for each $n \in \omega$, $bX \setminus \overline{O_n}^{bX}$ is a neighbourhood of x. Observe that $F = (\bigcap_{n \in \omega} bX \setminus \overline{O_n}^{bX}) \cap X$ is a neighbourhood of x in X by the condition that X is a p-space. For each $n \in \omega$, $F \subset bX \setminus \overline{O_n}^{bX} = bX \setminus \overline{V_n}^{bX}$, so that $F \cap V_n = \emptyset$. Hence $\overline{F}^{bX} \cap O_n \subset \overline{F}^{bX} \cap V_n = \emptyset$. Fix an open subset U of bX such that $U \cap X = F$. Then $U \cap O_n \subset \overline{U}^{bX} \cap O_n = \overline{F}^{bX} \cap V_n = \emptyset$ for $n \in \omega$. This contradicts the fact that $\bigcup \mathcal{O}$ is dense in bX. □

Corollary 2.19. *Suppose that bX is a compactification of a homogeneous p-space X. If $Y = bX \setminus X$ is separable, then X is discrete.*

Theorem 2.20. *If bX is a compactification of a topological p-group X, then at least one of the following holds:*

(1) X is discrete;

(2) $Y = bX \setminus X$ is pseudocompact.

Proof. Suppose that X is non-discrete. Then it follows from X being a P-space that X is non-locally compact. By [4], a remainder of a topological group in any compactification is either pseudocompact or Lindelöf, which implies that Y is either pseudocompact or Lindelöf.

Claim: Y is pseudocompact.

Suppose to the contrary that Y is Lindelöf. Then X is of countable type by Theorem 1.1. Fix a compact subset F of X such that F has a countable outer base in X. Since X is a p-space, F is an open subset of X. Hence, X is locally compact. This is a contradiction. □

Acknowledgements

The authors would like to thank the reviewer for the detailed list of corrections, suggestions to the paper, and all her/his efforts in order to improve the paper.

This research is supported by Shandong Natural Science Foundation (Grant No. ZR2014AL002) and by the National Natural Science Foundation of China (Grant No. 11571175).

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