ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 42 (2016), No. 6, pp. 1535–1546

Title:

Spacetimes admitting quasi-conformal curvature tensor

Author(s):

S. Mallick, P. Zhao and U. C. De

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 6, pp. 1535–1546
Online ISSN: 1735-8515

SPACETIMES ADMITTING QUASI-CONFORMAL CURVATURE TENSOR

S. MALLICK*, P. ZHAO AND U. C. DE

(Communicated by Mohammad Bagher Kashani)

Abstract. The object of the present paper is to study spacetimes admitting quasi-conformal curvature tensor. At first we prove that a quasi-conformally flat spacetime is Einstein and hence it is of constant curvature and the energy momentum tensor of such a spacetime satisfying Einstein's field equation with cosmological constant is covariant constant. Next, we prove that if the perfect fluid spacetime with vanishing quasi-conformal curvature tensor obeys Einstein's field equation without cosmological constant, then the spacetime has constant energy density and isotropic pressure and the perfect fluid always behave as a cosmological constant and also such a spacetime is infinitesimally spatially isotropic relative to the unit timelike vector field U. Moreover, it is shown that in a purely electromagnetic distribution the spacetime with vanishing quasi-conformal curvature tensor is filled with radiation and extremely hot gases. We also study dust-like fluid spacetime with vanishing quasi-conformal curvature tensor.
Keywords: Quasi-conformal curvature tensor, Einstein space, perfect fluid spacetime, Einstein's field equation, energy momentum tensor.
MSC(2010): Primary: 53B30; Secondary: 53C50.

1. Introduction

The present paper is concerned with certain investigations in general relativity by the coordinate free method of differential geometry. In this method of study spacetime of general relativity is regarded as a connected four dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g of signature $(-, +, +, +)$. The geometry of the Lorentzian manifold begins with the study of the causal character of vectors of the manifold. It is due to this causality that the Lorentzian manifold becomes a convenient choice for the study of general relativity. The Einstein's equations [19] (p. 337), imply that the

Article electronically published on December 18, 2016.
Received: 16 June 2015, Accepted: 27 September 2015.
*Corresponding author.

energy-momentum tensor is of vanishing divergence. This requirement is satisfied if the energy-momentum tensor is covariant-constant [4]. In the paper [4] M. C. Chaki and Sarbari Ray showed that a general relativistic spacetime with covariant-constant energy-momentum tensor is Ricci symmetric, that is, $\nabla S = 0$, where S is the Ricci tensor of the spacetime. Several authors studied spacetimes in several ways such as spacetimes with semisymmetric energy momentum tensor by De and Velimirović [9], m-Projectively flat spacetimes by Zengin [26], pseudo Z symmetric spacetimes by Mantica and Suh [17] and many others.

The notion of quasi-conformal curvature tensor was given by Yano and Sawaki [25] and is defined as follows:

$$\begin{aligned} C^\star(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY] \\ &\quad - \frac{r}{n}[\frac{a}{n-1} + 2b][g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.1}$$

where a and b are constants, R is the Riemann curvature tensor of type (1,3), S is the Ricci tensor of type (0,2), Q is the Ricci operator and r is the scalar curvature of the manifold.
If $a = 1$ and $b = -\frac{1}{n-2}$, then (1.1) reduces to the conformal curvature tensor. A semi-Riemannian manifold is called quasi-conformally flat if $C^\star = 0$ for $n > 3$. The quasi-conformal curvature tensor have been studied by several authors in various ways such as Amur and Maralabhavi [3], De and Sarkar [6], De and Matsuyama [7], De, Jun and Gazi [8], Guha [12], Hosseinzadeh and Taleshian [13], Özgür and Sular [22], Mantica and Suh [16] and many others.
The present paper is organized as follows:
After introduction, in Section 2, we characterize a spacetime with vanishing quasi-conformal curvature tensor and some geometric properties of such a spacetime have been obtained. In the next Section, we study perfect fluid spacetime with vanishing quasi-conformal curvature tensor and investigate some geometric and physical properties of this spacetime under certain condition. Finally, we consider dust-like fluid spacetime admitting vanishing quasi-conformal curvature tensor.

2. Spacetime with vanishing quasi-conformal curvature tensor

Let V_4 be the spacetime of general relativity, then from equation (1.1) we have

$$\begin{aligned} \tilde{C}^\star(X,Y,Z,W) &= a\tilde{R}(X,Y,Z,W) + b[S(Y,Z)g(X,Z) - S(X,Z)g(Y,W) \\ &\quad + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)] \\ &\quad - \frac{r}{4}[\frac{a}{3} + 2b][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \end{aligned} \tag{2.1}$$

where $\tilde{C}^\star(X,Y,Z,W) = g(C^\star(X,Y)Z,W)$ and $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$.

If $\tilde{C}^{\star}(X,Y,Z,W)=0$, then equation (2.1) leads to

$$\begin{aligned} &aR(X,Y,Z,W)+b[S(Y,Z)g(X,Z)-S(X,Z)g(Y,W)\\ &\qquad +g(Y,Z)S(X,W)-g(X,Z)S(Y,W)]\\ &-\frac{r}{4}[\frac{a}{3}+2b][g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]=0. \end{aligned} \tag{2.2}$$

Taking a frame field over X and W, we have from (2.2) that

$$(a+2b)S(Y,Z)=(a+2b)\frac{r}{4}g(Y,Z), \tag{2.3}$$

where S and r denote the Ricci tensor and the scalar curvature of the manifold respectively.
Thus we can state the following theorem.

Theorem 2.1. *A quasi-conformally flat spacetime is an Einstein spacetime, provided $a+2b\neq 0$.*

Again, equations (2.2) and (2.3) give

$$\tilde{R}(X,Y,Z,W)=\frac{r}{12}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]. \tag{2.4}$$

Thus we have the following.

Theorem 2.2. *A quasi-conformally flat spacetime is a spacetime of constant curvature, provided $a+2b\neq 0$.*

Remark 2.3. The spaces of constant curvature play a significant role in cosmology. The simplest cosmological model is obtained by making the assumption that the universe is isotropic and homogeneous. This is known as cosmological principle. This principle, when translated into the language of Riemannian geometry, asserts that the three dimensional position space is a space of maximal symmetry [23], that is, a space of constant curvature whose curvature depends upon time. The cosmological solution of Einstein equations which contain a three dimensional spacelike surface of a constant curvature are the Robertson-Walker metrics, while four dimensional space of constant curvature is the de Sitter model of the universe ([18, 23]).

Let us consider a spacetime satisfying the Einstein's field equation with cosmological constant

$$S(X,Y)-\frac{r}{2}g(X,Y)+\lambda g(X,Y)=\kappa T(X,Y), \tag{2.5}$$

where S, r and κ denote the Ricci tensor, scalar curvature and the gravitational constant respectively. λ is the cosmological constant and $T(X,Y)$ is the energy momentum tensor.
Using (2.3) and (2.5), we obtain

$$T(X,Y)=\frac{1}{\kappa}[\lambda-\frac{r}{4}]g(X,Y). \tag{2.6}$$

Taking covariant derivative of (2.6) we get

$$(\nabla_Z T)(X,Y) = -\frac{1}{4\kappa} dr(Z) g(X,Y). \tag{2.7}$$

Since quasi-conformally flat spacetime is Einstein, therefore the scalar curvature r is constant. Hence

$$dr(X) = 0, \tag{2.8}$$

for all X.
Equations (2.7) and (2.8) together yield

$$(\nabla_Z T)(X,Y) = 0.$$

Thus we can state the following.

Theorem 2.4. *In a quasi-conformally flat spacetime satisfying Einstein's field equation with cosmological constant, the energy momentum tensor is covariant constant.*

Katzin et al. [15] were the pioneers in carring out a detailed study of curvature collineation(CC), in the context of the related particle and field conservation laws that may be admitted in the standard form of general relativity.

The geometrical symmetries of a spacetime are expressed through the equation

$$\pounds_\xi A - 2\Omega A = 0, \tag{2.9}$$

where A represents a geometrical/physical quantity, $\pounds_\xi$ denotes the Lie derivative with respect to the vector field ξ and Ω is a scalar [15].
One of the most simple and widely used example is the metric inheritance symmetry for $A = g$ in (2.9) and for this case, ξ is the Killing vector field if $\Omega = 0$. Therefore,

$$(\pounds_\xi g)(X,Y) = 2\Omega g(X,Y). \tag{2.10}$$

A spacetime M is said to admit a symmetry called a curvature collineation(CC) ([10], [11]) provided there exists a vector field ξ such that

$$(\pounds_\xi R)(X,Y)Z = 0, \tag{2.11}$$

where R is the Riemann curvature tensor.

Now we shall investigate the role of such symmetry inheritance for the spacetime admitting quasi-conformal curvature tensor.

Let us consider a spacetime admitting quasi-conformal curvature tensor with a Killing vector field ξ is a CC. Then we have

$$(\pounds_\xi g)(X,Y) = 0. \tag{2.12}$$

Again, since M admits a CC we have from (2.11)

$$(\pounds_\xi S)(X,Y) = 0, \tag{2.13}$$

where S is the Ricci tensor of the manifold.
Taking Lie derivative of (1.1) and then using (2.11), (2.12) and (2.13) we obtain

$$(\pounds_\xi C^\star)(X,Y)Z = 0.$$

Thus we can state the following:

Theorem 2.5. *If a spacetime M admitting the quasi-conformal curvature tensor with ξ as a Killing vector field is CC, then the Lie derivative of the quasi-conformal curvature tensor vanishes along the vector field ξ.*

The well known symmetry of the energy momentum tensor T is the matter collineation defined by

$$(\pounds_\xi T)(X,Y) = 0,$$

where ξ is the vector field generating the symmetry and $\pounds_\xi$ is the Lie derivative operator along the vector field ξ.
Let ξ be a Killing vector field on the spacetime with vanishing quasi-conformal curvature tensor. Then

$$(\pounds_\xi g)(X,Y) = 0, \tag{2.14}$$

where $\pounds_\xi$ denotes Lie derivative with respect to ξ.
Taking Lie derivatives of both sides of (2.6) with respect to ξ we obtain

$$\frac{1}{\kappa}(\lambda - \frac{r}{4})(\pounds_\xi g)(X,Y) = (\pounds_\xi T)(X,Y). \tag{2.15}$$

In virtue of (2.14), it follows from (2.15) that

$$(\pounds_\xi T)(X,Y) = 0,$$

which implies that the spacetime admits matter collineation.
Conversely, if $(\pounds_\xi T)(X,Y) = 0$, it follows from (2.15) that

$$(\pounds_\xi g)(X,Y) = 0.$$

Hence we can state the following Theorem:

Theorem 2.6. *If a spacetime obeying Einstein's field equation has vanishing quasi-conformal curvature tensor, then the spacetime admits matter collineation with respect to a vector field ξ if and only if ξ is a Killing vector field.*

Next, let us suppose that ξ is a conformal Killing vector field. Then we have

$$(\pounds_\xi g)(X,Y) = 2\phi g(X,Y), \tag{2.16}$$

where ϕ is a scalar.
Then from (2.15) we get

$$(\lambda - \frac{r}{4})2\phi g(X,Y) = \kappa(\pounds_\xi T)(X,Y). \tag{2.17}$$

Using (2.6) in (2.17) we obtain

$$(\pounds_\xi T)(X,Y) = 2\phi T(X,Y). \tag{2.18}$$

From (2.18) we can say that the energy-momentum tensor has Lie inheritance property along ξ.
Conversely, if (2.18) holds, then it follows that (2.16) holds, that is, ξ is a conformal Killing vector field. Thus we state the following:

Theorem 2.7. *If a spacetime obeying Einstein's field equation has vanishing quasi-conformal curvature tensor, then a vector field ξ on the spacetime is a conformal Killing vector field if and only if the energy-momemtum tensor has the Lie inheritance property along ξ.*

3. Perfect fluid spacetime with vanishing quasi-conformal curvature tensor

In this section we consider a perfect fluid spacetime with vanishing quasi-conformal curvature tensor obeying Einstein's field equation without cosmological constant.
The energy momentum tensor T of a perfect fluid is given by [19]

$$T(X,Y) = (\sigma + p)A(X)A(Y) + pg(X,Y), \tag{3.1}$$

where σ is the energy density, p the isotropic pressure and A is a non-zero 1-form such that $g(X,U) = A(X)$, for all X, U being the velocity vector field of the flow, that is, $g(U,U) = -1$.
Einstein's field equation without cosmological constant is given by

$$S(X,Y) - \frac{r}{2}g(X,Y) = kT(X,Y), \tag{3.2}$$

where r is the scalar curvature of the manifold and $\kappa \neq 0$.
In this case Einstein equation can be written as

$$-(\frac{r}{4} + kp)g(X,Y) = k(\sigma + p)A(X)A(Y). \tag{3.3}$$

Taking a frame field after contraction over X and Y we obtain

$$r = \kappa(\sigma - 3p). \tag{3.4}$$

In virtue of (2.3) and (3.4), the Ricci tensor of a quasi-conformally flat spacetime can be written as

$$S(X,Y) = \frac{\kappa(\sigma - 3p)}{4}g(X,Y). \tag{3.5}$$

Let Q be the Ricci operator given by

$$g(QX,Y) = S(X,Y),$$

and

$$S(QX,Y) = S^2(X,Y).$$

Then we have

$$A(QX) = g(QX,U) = S(X,U).$$

Hence we obtain from (3.5) that

$$S(QX,Y) = \frac{\kappa^2(\sigma-3p)^2}{16} g(X,Y). \tag{3.6}$$

Taking a frame field after contraction over X and Y we obtain from (3.6) that

$$\|Q\|^2 = \frac{\kappa^2(\sigma-3p)^2}{4}. \tag{3.7}$$

Hence we obtain the following theorem:

Theorem 3.1. *If a quasi-conformally flat perfect fluid spacetime obeys Einstein's field equation without cosmological constant, then the square of the length of the Ricci operator of the spacetime is* $\frac{\kappa^2(\sigma-3p)^2}{4}$.

Now putting $X = Y = U$ in (3.3) we obtain

$$r = 4\kappa\sigma. \tag{3.8}$$

Equations (3.4) and (3.8) together give $\sigma + p = 0$. Therefore equation (3.1) in this case takes the form

$$T(X,Y) = pg(X,Y). \tag{3.9}$$

Since the scalar curvature r of a quasi-conformally flat spacetime is constant, therefore from (3.8) it follows that $\sigma = constant$ and hence from $\sigma + p = 0$ we get $p = constant$. Now $\sigma + p = 0$ means the fluid behaves as a cosmological constant [24]. This is also termed as Phantom Barrier [5]. Now in a cosmology we know such a choice $\sigma = -p$ leads to rapid expansion of the spacetime which is now termed as inflation [2].
Thus we can state the following:

Theorem 3.2. *If a perfect fluid spacetime with vanishing quasi-conformal curvature tensor obeying Einstein's equation without cosmological constant, then the spacetime has constant energy density and isotropic pressure and the spacetime represents inflation and also the fluid behaves as a cosmological constant.*

We know [20] that if the Ricci tensor S of type (0,2) of the spacetime satisfies the condition

$$S(X,X) > 0, \tag{3.10}$$

for every timelike vector field X, then (3.10) is called the timelike convergence condition.

Equations (3.1) and (3.2) together yield

$$S(X,Y) - \frac{r}{2} g(X,Y) = \kappa[(\sigma+p)A(X)A(Y) + pg(X,Y)]. \tag{3.11}$$

Putting $X = Y = U$ in (3.11) and using (3.8) we obtain

$$S(U,U) = -\kappa\sigma. \tag{3.12}$$

Since the spacetime under consideration satisfies timelike convergence condition and $\kappa > 0$, we have

$$\sigma < 0. \tag{3.13}$$

Hence $p > 0$ and $r < 0$. This implies that the scalar curvature r of the spacetime is negative and the isotropic pressure p of the spacetime is positive. Thus we can state the following theorem:

Theorem 3.3. *If a quasi-conformally flat perfect fluid spacetime satisfying Einstein's field equation without cosmological constant obeys the timelike convergence condition, then such a spacetime has positive isotropic pressure.*

Taking a frame field after contraction over X and Y we get from (3.2) that

$$r = -\kappa t, \tag{3.14}$$

where $t = trace\, T$.
Therefore, equation (3.2) can be written as

$$S(X,Y) = \kappa[T(X,Y) - \frac{t}{2}g(X,Y)]. \tag{3.15}$$

Einstein's field equation without cosmological constant for a purely electromagnetic distribution takes the form [1]

$$S(X,Y) = \kappa T(X,Y). \tag{3.16}$$

Using (3.15) and (3.16) we obtain $t = 0$. So from (3.14) we get $r = 0$. Thus from (2.4) we obtain $\tilde{R}(X,Y,Z,W) = 0$ which means that the spacetime is flat. Thus we can state the following theorem.

Theorem 3.4. *A quasi-conformally flat spacetime satisfying Einstein's equation without cosmological constant for a purely electromagnetic distribution is an Euclidean space.*

Remark 3.5. This theorem points out towards a condition under which a semi-Riemannian space can be reduced to an Euclidean space.

Taking again a frame field after contraction we have from (3.1) that

$$t = 3p - \sigma, \tag{3.17}$$

where $t = trace\, T$. Since for a purely electromagnetic distribution $t = 0$ here, we obtain from (3.17) that $\sigma = 3p$. This means that the spacetime under consideration is filled with radiation and extremely relavistic fluid or extremely hot gases. Thus we can state the following theorem:

Theorem 3.6. *In a purely electromagnetic distribution the spacetime with vanishing quasi-conformal curvature tensor is filled with radiation and extremely hot gases.*

In a quasi-conformally flat perfect fluid spacetime, from (2.4) it follows that the curvature tensor R is given by

$$R(X,Y)Z = \frac{r}{12}[g(Y,Z)X - g(X,Z)Y], \tag{3.18}$$

where r is the scalar curvature of the spacetime.
Since quasi-conformally flat spacetime is Einstein space, it follows that r=constant. Let $U^{\perp}$ denote the 3-dimensional distribution in a quasi-conformally flat perfect fluid spacetime orthogonal to U.
Then

$$R(X,Y)Z = \frac{r}{12}[g(Y,Z)X - g(X,Z)Y], \tag{3.19}$$

for all X,Y,$Z \in U^{\perp}$ and

$$R(X,U)U = -\frac{r}{12}X, \tag{3.20}$$

for every X$\in U^{\perp}$.
According to Karchar [14] a Lorentzian manifold is called infinitesimally spatially isotropic relative to timelike unit vector field U if its curvature tensor R satisfies the relation

$$R(X,Y)Z = l[g(Y,Z)X - g(X,Z)Y], \tag{3.21}$$

for all X,Y,$Z \in U^{\perp}$ and $R(X,U)U = mX$ for all X$\in U^{\perp}$, where l, m are real valued function on the manifold. So by virtue of (3.19) and (3.20) we can state the following theorem.

Theorem 3.7. *A quasi-conformally flat perfect fluid spacetime obeying the Einstein's field equation without cosmological constant and having the vector field U as the velocity vector field is infinitesimally spatially isotropic relative to the unit timelike vector field U.*

Let X,Y$\in U^{\perp}$, K_1 denote the sectional curvature of the spacetime determined by X, Y and K_2 denote the sectional curvature of the spacetime determined by X, U.
Then

$$K_1 = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - \{g(X,Y)\}^2} = \frac{r}{12}$$

$$K_2 = \frac{g(R(X,U)U,X)}{g(X,X)g(U,U) - \{g(X,U)\}^2} = \frac{r}{12}$$

Thus we can state the following:

Theorem 3.8. *In a quasi-conformally flat perfect fluid spacetime the sectional curvature determined by two vectors X,$Y \in U^{\perp}$ and the sectional curvature determined by two vectors X and U are equal.*

4. Dust fluid spacetime with vanishing quasi-conformal curvature tensor

In a dust or pressureless fluid spacetime, the energy momentum tensor is in the form [21]

$$T(X,Y) = \sigma A(X)A(Y), \tag{4.1}$$

where σ is the energy density of the dust-like matter and A is a non-zero 1-form such that $g(X,U) = A(X)$, for all X, U being the velocity vector field of the flow, that is, $g(U,U) = -1$.
Using (2.6) and (4.1) we obtain

$$(\lambda - \frac{r}{4})g(X,Y) = \kappa\sigma A(X)A(Y). \tag{4.2}$$

A frame field after contraction over X and Y leads to

$$\lambda = \frac{r}{4} - \frac{\kappa\sigma}{4}. \tag{4.3}$$

Again, if we put $X = Y = U$ in (4.2), we get

$$\lambda = \frac{r}{4} - \kappa\sigma. \tag{4.4}$$

Thus combining the equations (4.3)and (4.4), we finally obtain that

$$\sigma = 0. \tag{4.5}$$

Thus from (4.1) and (4.5) we conclude that

$$T(X,Y) = 0.$$

This means that the spacetime is devoid of the matter. Thus we can state the following:

Theorem 4.1. *A quasi-conformally flat dust fluid spacetime satisfying Einstein's field equation with cosmological constant is vacuum.*

Acknowledgement

The authors wish to express their sincere thanks and gratitude to the referee for his/her valuable suggestions towards the improvement of the paper.

References

[1] Z. Ahsan and S. A. Siddiqui, Concircular Curvature Tensor and Fluid Spacetimes, *Int. J. Theor. Phys.* **48** (2009), no. 11, 3202–3212.

[2] L. Amendola and S. Tsujikawa, Dark Energy: Theory and Observations, Cambridge Univ. Press, Cambridge, 2010.

[3] K. Amur and Y. B. Maralabhavi, On quasi-conformally flat spaces, *Tensor (N.S.)* **31** (1977), no. 2, 194–198.

[4] M. C. Chaki and S. Roy, Space-times with covariant-constant energy-momentum tensor, *Internat. J. Theoret. Phys.* **35** (1996), no. 5, 1027–1032.

[5] S. Chakraborty, N. Mazumder and R. Biswas, Cosmological evolution across phantom crossing and the nature of the horizon, *Astrophys. Space Sci. Libr.* **334** (2011) 183–186.
[6] U. C. De and A. Sarkar, On the quasi-conformal curvature tensor of a (k, μ)-contact metric manifold, *Math. Rep.* **14** (2012), no. 2, 115–129.
[7] U. C. De and Y. Matsuyama, Quasi-conformally flat manifolds satisfying certain conditions on the Ricci tensor, *SUT J. Math.* **42** (2006), no. 2, 295–303.
[8] U. C. De, J. B. Jun and A. K. Gazi, Sasakian manifolds with quasi-conformal curvature tensor, *Bull. Korean Math. Soc.* **45** (2008), no. 2, 313–319.
[9] U. C. De and L. Velimirović, Spacetimes with semisymmetric energy momentum tensor, *Int. J. Theor. Phys.* **54** (2015), no. 6, 1779–1783.
[10] K. L. Duggal, Curvature Collineations and Conservation Laws of General Relativity, presented at the Canadian conference on general relativity and relativistic Astro-physics, Halifax, Canada, 1985.
[11] K. L. Duggal, Curvature inheritance symmetry in Riemannian spaces with applications to fluid space times, *J. Math. Phys.* **33** (1992), no. 9, 2989–2997.
[12] S. Guha, On a perfect fluid space-time admitting quasi conformal curvature tensor, *Facta Univ. Ser. Mech. Automat. Control Robot.* **3** (2003), no. 14, 843–849.
[13] A. Hosseinzadeh and A. Taleshian, On conformal and quasi-conformal curvature tensors of an $N(k)$-quasi Einstein manifold, *Commun. Korean Math. Soc.* **27** (2012), no. 2, 317–326.
[14] H. Karchar, Infinitesimal characterization of Friedmann universe, *Arch. Math. (Basel)* **38** (1992), no. 1, 58–64.
[15] G. H. Katzin, J. Levine and W. R. Davis, Curvature collineations: A fundamental symmetry property of the spacetime of general relativity defined by the vanishing Lie derivative of the Riemannian curvature tensor, *J. Math. Phys.* **10** (1969) 617–629.
[16] C. A. Mantica and Y. J. Suh, Conformally symmetric manifolds and quasi conformally recurrent Riemannian manifolds, *Balkan J. Geom. Appl.* **16** (2011), no. 1, 66–77.
[17] C. A. Mantica and Y. J. Suh, Pseudo-Z symmetric space-times, *J. Math. Phys.* **55** (2014), no. 4, 12 pages.
[18] J. V. Narlikar, General Relativity and Gravitation, The Macmillan Co. India, 1978.
[19] B. O'Neill, Semi-Riemannian Geometry, Academic Press, NY, 1983.
[20] R. K. Sach and W. Hu, General Relativity for Mathematician, Springer Verlag, New York, 1977.
[21] S. K. Srivastava, General Ralativity and Cosmology, Prentice-Hall of India, Private Limited, New Delhi, 2008.
[22] C. Özgür and S. Sular, On N(k)-quasi-Einstein manifolds satisfying certain conditions, *Balkan J. Geom. Appl.* **13** (2008) 74–79.
[23] H. Stephani, General Relativity-An Introduction to the Theory of Gravitational Field, Cambridge Univ. Press, Cambridge, 1982.
[24] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, Exact Solutions of Einstein's Field Equations, Cambridge Monogr. on Math. Phys. Cambridge Univ. Press, 2nd edition, Cambridge, 2003.
[25] K. Yano and S. Sawaki, Riemannian manifolds admitting a conformal transformation group, *J. Differential Geom.* **2** (1968) 161–184.
[26] F. O. Zengin, M-projectively flat spacetimes, *Math. Rep.* **4** (2012), no. 4, 363–370.

(Sahanous Mallick) Department of Mathematics, Chakdaha College, P. O. Chakdaha, Dist-Nadia, West Bengal, India.
E-mail address: sahanousmallick@gmail.com

(Peibiao Zhao) Department of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094, P. R. China.
E-mail address: pbzhao@njust.edu.cn

(Uday Chand De) Department of Pure Mathematics, University of Calcutta, 35, B. C. Road, Kolkata 700019, West Bengal, India.
E-mail address: uc_de@yahoo.com