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# PARABOLIC MARCINKIEWICZ INTEGRALS ON PRODUCT SPACES

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(Communicated by Madjid Eshaghi Gordji)

Abstract. In this paper, we study the $L^p$ $(1 < p < \infty)$ boundedness for the parabolic Marcinkiewicz integral when the kernel function $\Omega$ belongs to the class $L(\log L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. Our result essentially extend and improve some known results.
**Keywords:** $L^p$ boundedness, parabolic Marcinkiewicz integrals, rough kernels, product spaces.
**MSC(2010):** Primary: 40B20; Secondary: 40B15, 40B25.

## 1. **Introduction and preliminaries**

Let $\mathbf{R}^N$ $(N = n$ or $m)$, $N \geq 2$ be the $N$-dimensional Euclidean space, and let $\mathbf{S}^{N-1}$ be the unit sphere in $\mathbf{R}^N$ which is equipped with the normalized Lebesgue surface measure $d\sigma = d\sigma(\cdot)$. Also, let $p'$ denote to the exponent conjugate to $p$; that is $1/p + 1/p' = 1$.

For $i = 1, 2, \cdots, N$, let $\alpha_i$ be fixed real numbers such that $\alpha_i \geq 1$. For fixed $z \in \mathbf{R}^N$, the function $F(z, \rho) = \sum_{i=1}^{N} \frac{z_i^2}{\rho^{2\alpha_i}}$ is decreasing in $\rho > 0$. The unique solutions of the equations $F(z, \rho) = 1$ is denoted by $\rho(z)$.

For $\lambda > 0$, let $A_\lambda = \begin{bmatrix} \lambda^{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & \lambda^{\alpha_N} \end{bmatrix}$, and let $K_{\Omega,\rho}(z) = \Omega(z)\rho(z)^{1-\alpha}$, where $\alpha = \sum_{i=1}^{N} \alpha_i$ and $\Omega$ is a real valued and measurable function on $\mathbf{R}^N$ with $\Omega \in L^1(\mathbf{S}^{N-1})$ satisfying the conditions

$$\Omega(A_\lambda z) \quad = \quad \Omega(z) \quad \text{and} \quad \int_{\mathbf{S}^{N-1}} \Omega(z')J(z')d\sigma(z') = 0,$$

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where $J(z')$ is defined as in [8]. The parabolic Marcinkiewicz integral $\mu_\Omega$, which was introduced by Ding, Xue and Yabuta in [15], is defined by

$$\mu_\Omega f(z) = \left( \int_0^\infty |F_{\Omega,t}(z)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(z) = \int_{\rho(u)\le t} K_{\Omega,\rho}(u) f(z-u) du.$$

In particular, the authors of [15] proved that the parabolic Littlewood-Paley operator $\mu_\Omega$ is bounded for $p \in (1,\infty)$ provided that $\Omega \in L^q(\mathbf{S}^{N-1})$ for $q > 1$. Subsequently, the study of the $L^p$ boundedness of $\mu_\Omega$ under various conditions on the function $\Omega$ has been studied by many authors (see for example [**?**,8,24]). A particular result that is closely related to our work is the boundedness result of $\mu_\Omega$ obtained by Cheng and Ding in [8]. If fact, they proved that $\mu_\Omega$ is bounded under the condition $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ for $1 < p < \infty$.

We point out that the class of the operators $\mu_\Omega$ is related to the class of the parabolic singular integral operators

$$T_\Omega f(z) = p.v. \int_{\mathbf{R}^N} \frac{\Omega(u)}{\rho(u)^\alpha} f(z-u) du.$$

The class of the operators $T_\Omega$ belongs to the class of singular Radon transforms, which has considered to study by many mathematicians (we refer the readers, in particular, to [18, 21]).

If $\alpha_1 = \cdots = \alpha_N = 1$, then $\rho(z) = |z|$, $\alpha = N$ and $(\mathbf{R}^N, \rho) = (\mathbf{R}^N, |\cdot|)$. In this case, $\mu_\Omega$ is just the classical Marcinkiewicz integral, which were introduced by Stein in [23]. For more information about the importance and the recent advances on the study of such operators, the readers are refereed (for instance to [3, 4, 10, 14, 17, 19], and the references therein).

Although some open problems related to the boundedness of parabolic Marcinkiewicz integral in the one-parameter setting remain open, the investigation of $L^p$ estimates of the Marcinkiewicz integral on product spaces has been started (see for example [1, 2, 5–7, 11–13].)

Our main interest in this paper is to study the $L^p$ boundedness of the parabolic Marcinkiewicz integral with a rough kernel on product spaces. Namely, for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$, let $\alpha_i$, $\beta_j$ be fixed real numbers such that $\alpha_i, \beta_j \ge 1$, and let $K_{\Omega,\rho_1,\rho_2}(x, y) = \Omega(x, y)\rho_1(x)^{1-\alpha}\rho_2(y)^{1-\beta}$, where $\alpha = \sum_{i=1}^{n} \alpha_i$, $\beta = \sum_{j=1}^{m} \beta_j$ and $\Omega$ is a real valued and measurable function on $\mathbf{R}^n \times \mathbf{R}^m$ with

$\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ satisfying the conditions

$$\Omega(A_{\lambda_1}x, A_{\lambda_2}y) = \Omega(x, y) \quad \text{and} \tag{1.1}$$

$$\int_{\mathbf{S}^{n-1}} \Omega(x', .)J(x', .)d\sigma(x') = \int_{\mathbf{S}^{m-1}} \Omega(., y')J(., y')d\sigma(y') = 0, \tag{1.2}$$

where $\lambda_1, \lambda_2 > 0$, and $J(x', y')$ is a function on the unit sphere $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ in $\mathbf{R}^n \times \mathbf{R}^m$, that will be defined later.

The parabolic Marcinkiewicz integral operator $\mathcal{M}_\Omega$ for $f \in \mathcal{S}(\mathbf{R}^n \times \mathbf{R}^m)$ is given by

$$\mathcal{M}_\Omega f(x, y) = \left(\int_0^\infty \int_0^\infty |F_{t,s}(x, y)|^2 \frac{dtds}{t^3 s^3}\right)^{1/2}, \tag{1.3}$$

where

$$F_{t,s}(x, y) = \int_{\rho_1(u) \le t} \int_{\rho_2(v) \le s} K_{\Omega,\rho_1,\rho_2}(u, v) f(x - u, y - v)dudv.$$

When $\alpha_1 = \cdots = \alpha_n = 1$, and $\beta_1 = \cdots = \beta_m = 1$, then $\rho_1(x) = |x|$, $\rho_2(y) = |y|$, $\alpha = n$, and $\beta = m$. In this case, $\mathcal{M}_\Omega$ is just the classical Marcinkiewicz integral on product domains, which was studied by many mathematicians. For instance, the author of [13] gave the $L^2$ boundedness of $\mathcal{M}_\Omega$ if $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. Subsequently, it was verified in [11] that $\mathcal{M}_\Omega$ is bounded for all $1 < p < \infty$ provided that $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. This result was improved (for $p = 2$) in [12] in which the author established that $\mathcal{M}_\Omega$ is bounded on $L^2(\mathbf{R}^n \times \mathbf{R}^m)$ for all $\Omega \in L(\log L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. Recently, Al-Qaseem *et al.* found in [1] that the boundedness of $\mathcal{M}_\Omega$ is obtained under the condition $\Omega \in L(\log L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for $1 < p < \infty$. Furthermore, they proved that the exponent 1 is the best possible.

In this article, we extend and improve the corresponding results in [1,11,12]. Our main result is formulated as follows.

**Theorem 1.1.** *Suppose that $\Omega \in L(\log L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and satisfies (1.2)-(1.3). Then $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbf{R}^n \times \mathbf{R}^m)$ for $p \in (1, \infty)$.*

Throughout this paper, the letter $C$ denotes a bounded positive constant that may vary at each occurrence but independent of the essential variables.

## 2. Some lemmas

In this section, we give some auxiliary lemmas used in the sequel. The following is found in [8, 24]. For $(x, y) \in \mathbf{R}^n \times \mathbf{R}^m$, set

$$\begin{aligned}
&x_1=\rho_1^{\alpha_1}\cos\theta_1\cdots\cos\theta_{n-2}\cos\theta_{n-1}, && y_1=\rho_2^{\beta_1}\cos\vartheta_1\cdots\cos\vartheta_{m-2}\cos\vartheta_{m-1},\\
&x_2=\rho_1^{\alpha_2}\cos\theta_1\cdots\cos\theta_{n-2}\sin\theta_{n-1}, && y_2=\rho_2^{\beta_2}\cos\vartheta_1\cdots\cos\vartheta_{m-2}\sin\vartheta_{m-1},\\
&\vdots && \vdots\\
&x_{n-1}=\rho_1^{\alpha_{n-1}}\cos\theta_1\sin\theta_2, && y_{m-1}=\rho_2^{\beta_{m-1}}\cos\vartheta_1\sin\vartheta_2,\\
&x_n=\rho_1^{\alpha_n}\sin\theta_1, && y_m=\rho_2^{\beta m}\sin\vartheta_1.
\end{aligned}$$

Then $dxdy=\rho_1^{\alpha-1}\rho_2^{\beta-1}J_1(\theta_1,\cdots,\theta_{n-1})J_2(\vartheta_1,\cdots,\vartheta_{m-1})d\rho_1 d\rho_2 d\sigma(x')d\sigma(y')$, where $\rho_1^{\alpha-1}J_1$
$(\theta_1,\cdots,\theta_{n-1})$ and $\rho_2^{\beta-1}J_2(\vartheta_1,\cdots,\vartheta_{m-1})$ are the Jacobians of the above transforms. In [18], it was shown that $J_1=J_1(\theta_1,\cdots,\theta_{n-1})$ is a $C^\infty$ function in the variable $x'\in\mathbf{S}^{n-1}$, also it was proved that $1\le J_1(\theta_1,\cdots,\theta_{n-1})\le L$ for some real number $L\ge 1$, and so for $J_2(\vartheta_1,\cdots,\vartheta_{m-1})$. For simplicity, we denote $J_1(\theta_1,\cdots,\theta_{n-1})J_2(\vartheta_1,\cdots,\vartheta_{m-1})$ by $J(x',y')$.

In order to prove Theorem 1.1, we need the following lemmas.

**Lemma 2.1.** [22] *Suppose that $\lambda_i'$s and $\alpha_i'$s are fixed real numbers, and $\Gamma(t)=(\lambda_1 t^{\alpha_1},\cdots,\lambda_N t^{\alpha_N})$ is a function from $\boldsymbol{R}^+$ to $\boldsymbol{R}^N$. For suitable $f$, let $\mathcal{M}_\Gamma$ be the maximal operator defined on $\boldsymbol{R}^N$ by*

$$\mathcal{M}_\Gamma f(x)=\sup_{h>0}\frac{1}{h}\left|\int_0^h f(x-\Gamma(t))dt\right|$$

*for $x\in\boldsymbol{R}^N$. Then for $1<p\le\infty$, there exists a constant $C_p>0$ such that*

$$\|\mathcal{M}_\Gamma f\|_p\le C_p\|f\|_p.$$

*The constant $C_p$ is independent of $\lambda_i'$s and $f$.*

**Lemma 2.2.** *Suppose that $a_i'$s, $b_i'$s, $\alpha_i'$s, and $\beta_i'$s are fixed real numbers. Let $\Gamma(t)=(a_1t^{\alpha_1},\cdots,a_nt^{\alpha_n})$ and $\Lambda(t)=(b_1t^{\beta_1},\cdots,b_mt^{\beta_m})$; and let $\mathcal{M}_{\Gamma,\Lambda}$ be the maximal operator defined on $\boldsymbol{R}^n\times\boldsymbol{R}^m$ by*

$$\mathcal{M}_{\Gamma,\Lambda}f(x,y)=\sup_{h_1,h_2>0}\frac{1}{h_1h_2}\left|\int_0^{h_1}\int_0^{h_2}f(x-\Gamma(t),y-\Lambda(r))dtdr\right|$$

*for $(x,y)\in\boldsymbol{R}^n\times\boldsymbol{R}^n$. Then for $1<p\le\infty$, there exists a constant $C_p>0$ (independent of $a_i'$s, $b_j'$s, and $f$) such that*

$$\|\mathcal{M}_{\Gamma,\Lambda}f\|_p\le C_p\|f\|_p.$$

The proof of Lemma 2.2 follows immediately by using Lemma 2.1 and the inequality $\mathcal{M}_{\Gamma,\Lambda}f(x,y)\le\mathcal{M}_\Lambda\circ\mathcal{M}_\Gamma f(x,y)$, where $\circ$ denotes the composition of operators.

We shall recall the following lemma due to Madych.

**Lemma 2.3.** [20] *Let $\Phi \in \mathcal{S}(\boldsymbol{R}^n)$ satisfy $\hat{\Phi}(0)=0$. Denote $\Phi_t(x) = t^{-\alpha}\Phi(A_{t^{-1}}x)$ for $t > 0$, the Littlewood-Paley g-function related to the transform A is defined by*

$$g_\Phi(f)(x) = \left(\int_0^\infty |\Phi_t * f(x)|^2 \frac{dt}{t}\right)^{1/2}.$$

*Then there is a positive constant C such that $\|g_\Phi(f)\|_p \le C\|f\|_p$ for any $f \in L^p(\boldsymbol{R}^n)$ and $1 < p < \infty$.*

Similarly, we derive the following lemma.

**Lemma 2.4.** *Let $\Phi \in \mathcal{S}(\boldsymbol{R}^n), \Psi \in \mathcal{S}(\boldsymbol{R}^m)$ with $\hat{\Phi}(0) = \hat{\Psi}(0) = 0$. For $s, t > 0$, let $\Phi_t(x) = t^{-\alpha}\Phi(A_{t^{-1}}x)$, $\Psi_s(y) = s^{-\beta}\Psi(A_{s^{-1}}y)$ and $\Gamma_{t,s}(x, y) = \Phi_t(x)\Psi_s(y)$. Assume that the Littlewood-Paley g-function is defined by*

$$g_{\Phi,\Psi}(f)(x,y) = \left(\int_0^\infty \int_0^\infty |\Gamma_{t,s} * f(x,y)|^2 \frac{dtds}{ts}\right)^{1/2}.$$

*Then there exists $C > 0$ such that $\|g_{\Phi,\Psi}(f)\|_p \le C\|f\|_p$ for any $f \in L^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ and $1 < p < \infty$.*

**Lemma 2.5.** [8] *Let $\gamma \in [0, 1]$ and $u, \xi \in \boldsymbol{R}^n$. Then*

$$\left|\int_1^2 e^{A_\lambda u\cdot\xi} \frac{d\lambda}{\lambda}\right| \le C\,|u\cdot\xi|^{-\frac{\gamma}{\tau}},$$

*where $A_\lambda$ is defined as above and $\tau$ denotes the distinct numbers of $\{\alpha_i\}$.*

For a two-parameter family of measures $\nu = \{\nu_{t,s} : t, s \in \mathbf{R}\}$ on $\mathbf{R}^n \times \mathbf{R}^m$, we define the operator $G_\nu$ and its corresponding maximal operator $\nu^*$ by

$$G_\nu(f)(x,y) = \left(\iint_{\mathbf{R}\times\mathbf{R}} |\nu_{t,s} * f(x,y)|^2\, dtds\right)^{1/2} \tag{2.1}$$

and

$$\nu^*(f) = \sup_{t,s\in\mathbf{R}} ||\nu_{t,s}| * f|. \tag{2.2}$$

We write $t^{\pm\alpha} = \min\{t^{+\alpha}, t^{-\alpha}\}$ and $\|\nu_{t,s}\|$ for the total variation of $\nu_{t,s}$.

The following is the main lemma of this section.

**Lemma 2.6.** *Let $a, b \ge 2$, $\gamma_1, \gamma_2 > 0$, $q > 1$ and $B > 0$. Suppose that the family of measures $\{\nu_{t,s} : t, s \in \boldsymbol{R}\}$ satisfies the following conditions:*
*(i)* $\|\nu_{t,s}\| \le CB$ *for* $t, s \in \mathbf{R}$;
*(ii)* $\|\widehat{\nu}_{t,s}(\xi,\eta)\| \le CB\,|A_{a^t}\xi|^{\pm\gamma_1/\ln a}\,|A_{b^t}\eta|^{\pm\gamma_2/\ln b}$ *for* $(\xi,\eta) \in \mathbf{R}^n \times \mathbf{R}^m$ *and* $t, s \in \mathbf{R}$;
*(iii)* $\|\nu^*(f)\|_q \le CB\,\|f\|_q$ *for* $f \in L^q(\mathbf{R}^n \times \mathbf{R}^m)$.
*Then, for every p satisfying $|1/p - 1/2| < 1/(2q)$, there is a constant $C_p$ (independent of $a, b, B, f$) such that for any $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$,*

$$\|G_\nu(f)\|_p \le C_p B\|f\|_p.$$

*Proof.* We employ some ideas from [1, 8]. For $\kappa > 2$, let $\varphi^{(\kappa)}$ be a $\in C^\infty$ function supported in $[4/(5\kappa), (5\kappa)/4]$ such that

$(i)$ $\varphi^{(\kappa)}(\zeta) = \varphi^{(\kappa)}(\rho(\zeta))$ for $\rho > 0$ and $\zeta \in \mathbf{R}^N$;

$(ii)$ $0 < \varphi^{(\kappa)}(\zeta) \leq 1$;

$(iii)$ $\int_0^\infty \frac{\varphi^{(\kappa)}(t)}{t} dt = 2\ln\kappa$.

For $a, b > 2$, and for $(\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}^m$, let $\Phi \in C^\infty(\mathbf{R}^n)$ and $\Psi \in C^\infty(\mathbf{R}^m)$ be given by $\hat{\Phi}(\xi) = \varphi^{(a)}(\rho_1(\xi)^2)$ and $\hat{\Psi}(\eta) = \varphi^{(b)}(\rho_2(\eta)^2)$. For $x \in \mathbf{R}^n, y \in \mathbf{R}^m$ and $t, s \in \mathbf{R}$, set

$$\Phi_t(x) = t^{-\alpha}\Phi(A_{t^{-1}}x)\,, \Psi_s(y) = s^{-\beta}\Phi(A_{s^{-1}}y) \quad and \quad \Gamma_{t,s}(x,y) = \Phi_t(x)\Psi_s(y).$$

Thus, for any $f \in \mathcal{S}(\mathbf{R}^n \times \mathbf{R}^m)$, we get

$$f(x,y) = \int_{\mathbf{R}\times\mathbf{R}} \Gamma_{a^t,b^s} * f(x,y)dtds. \tag{2.3}$$

By Minkowski's inequality, we reach that

$$\begin{aligned} G_\nu(f)(u,v) &= \left(\int\limits_{\mathbf{R}\times\mathbf{R}} \left| \int\limits_{\mathbf{R}\times\mathbf{R}} \Gamma_{a^{t+u},b^{s+v}} * \nu_{t,s} * f(x,y)dudv\right|^2 dtds\right)^{1/2} \\ &\leq \int\limits_{\mathbf{R}\times\mathbf{R}} (H_{u,v}f)(x,y)dudv, \end{aligned} \tag{2.4}$$

where

$$(H_{u,v}f)(x,y) = \left(\int\limits_{\mathbf{R}\times\mathbf{R}} \left|\Gamma_{a^{t+u},b^{s+v}} * \nu_{t,s} * f(x,y)\right|^2 dtds\right)^{1/2}. \tag{2.5}$$

Let us start estimating $\| H_{u,v}f \|_2$ for the case $u, v \geq 0$; the proof for the other cases are essentially the same and require only minor modifications. By Plancherel's theorem, assumption $(ii)$ and the techniques used in [8], we conclude that

$$\begin{aligned} \| H_{u,v}f \|_2^2 \leq\ & CB\int_{\mathbf{R}\times\mathbf{R}} \left(\int_{\mathbf{R}^n\times\mathbf{R}^m} |\hat{f}(\xi,\eta)|^2 |\varphi^{(a)}(\rho_1(A_{a^{u+t}}\xi)^2)|^2 |A_{a^t}\xi|^{2\gamma_1/\ln a} \right. \\ & \left. \times\, |\varphi^{(b)}(\rho_2(A_{b^{v+s}}\eta)^2)|^2 |A_{b^s}\eta|^{2\gamma_2/\ln b}\right) d\xi d\eta dtds \\ \leq\ & CB\int_{\mathbf{R}\times\mathbf{R}} \left(\int_{E_{u,v,t,s}} |\hat{f}(\xi,\eta)|^2 \{|A_{a^t}\xi|^{2\gamma_1/\ln a}\}\{|A_{b^s}\eta|^{2\gamma_2/\ln b}\} d\xi d\eta\right) dtds \\ \leq\ & CBe^{-2(|u|\gamma_1+|v|\gamma_2)} \| f \|_2^2, \end{aligned} \tag{2.6}$$

where $E_{u,v,t,s} = \{(\xi,\eta) \in \mathbf{R}^n \times \mathbf{R}^m :$

$$\frac{2}{\sqrt{5}}a^{-u-1/2} \leq a^t\rho_1(\xi) \leq \frac{\sqrt{5}}{2}a^{-u+1/2}, \quad \frac{2}{\sqrt{5}}b^{-v-1/2} \leq b^s\rho_2(\eta) \leq \frac{\sqrt{5}}{2}b^{-v+1/2}\}.$$

Now, let us estimate $\| H_{u,v}f \|_{p_0}$ for $p_0$ satisfying $\left|\frac{1}{2}-\frac{1}{p_0}\right| = \frac{1}{2q}$ with $p_0 \neq 2$. First, we consider $1 < p_0 < 2$. By the assumption $(i)$, we get

$$\left\|\int_{\mathbf{R}\times\mathbf{R}} \nu_{t,s} * \Gamma_{a^{u+t},b^{v+s}} * f(x,y)dtds\right\|_1 \leq CB \left\|\int_{\mathbf{R}\times\mathbf{R}} \Gamma_{a^t,b^s} * f(x,y)dtds\right\|_1 . \tag{2.7}$$

Further, by using the assumption $(iii)$, we achieve that

$$\begin{aligned}\left\|\sup_{t,s\in\mathbf{R}} \left|\nu_{t,s} * \Gamma_{a^{u+t},b^{v+s}} * f\right|\right\|_q &\leq \left\|\nu^*(\sup_{t,s\in\mathbf{R}} \left|\Gamma_{a^t,b^s} * f\right|)\right\|_q \\ &\leq CB \left\|\sup_{t,s\in\mathbf{R}} \left|\Gamma_{a^t,b^s} * f\right|\right\|_q . \end{aligned} \tag{2.8}$$

By using the interpolation theorem between (2.7) and (2.8) and the Lemma 2.4, we deduce that

$$\begin{aligned}\|H_{u,v}f\|_{p_0} &\leq CB \left\|\int_{\mathbf{R}\times\mathbf{R}} \left(\left|\Gamma_{a^t,b^s} * f\right|^2 dtds\right)^{1/2}\right\|_{p_0} \\ &\leq CB \|f\|_{p_0} . \end{aligned} \tag{2.9}$$

Next, consider the case $2 < p_0 < \infty$. As $q = (\frac{p_0}{2})'$ and $\left\|(H_{u,v}(f))^{1/2}\right\|_{p_0} = \|H_{u,v}(f)\|_{p_0/2}^{1/2}$, there is a non-negative function $F \in L^q(\mathbf{R}^n \times \mathbf{R}^m)$ with $\|F\|_q \leq 1$ such that

$$\begin{aligned}\|H_{u,v}(f)\|_{p_0}^2 &= \left\|\left(\int_{\mathbf{R}\times\mathbf{R}} \left|\Gamma_{a^{u+t},b^{v+s}} * \nu_{t,s} * f(x,y)\right|^2 dtds\right)^{1/2}\right\|_{p_0}^2 \\ &= \int_{\mathbf{R}^n\times\mathbf{R}^m}\int_{\mathbf{R}\times\mathbf{R}} \left|\Gamma_{a^{u+t},b^{v+s}} * \nu_{t,s} * f(x,y)\right|^2 dtds F(x,y)dxdy. \end{aligned} \tag{2.10}$$

By using Holder's inequality, (2.10), Lemma 2.4 plus the assumptions $(i)$ and $(iii)$, we obtain that

$$\begin{aligned}\|H_{u,v}(f)\|_{p_0}^2 &\leq \| \nu_{t,s} \|_1 \int_{\mathbf{R}^n\times\mathbf{R}^m}\int_{\mathbf{R}\times\mathbf{R}} \mid \nu_{t,s} \mid * \left|\Gamma_{a^{u+t},b^{v+s}} * f\right|^2 dtds \mid F(x,y) \mid dxdy \\ &\leq CB \left(\int_{\mathbf{R}^n\times\mathbf{R}^m}\left(\int_{\mathbf{R}\times\mathbf{R}}\left|\Gamma_{a^{u+t},b^{v+s}} * f(x,y)\right|^2 dtds\right)^{p_0/2} dxdy\right)^{2/p_0} \| \nu^*(\widetilde{F}) \|_q \\ &\leq CB\|f\|_{p_0}^2, \end{aligned} \tag{2.11}$$

where $\widetilde{F}(x,y) = F(-x,-y)$. Thus, by (2.9) and (2.11) we reach that

$$\|H_{u,v}(f)\|_{p_0} \leq CB \left|f\right|_{p_0} \tag{2.12}$$

for any $p_0$ satisfying $\mid \frac{1}{2}-\frac{1}{p_0} \mid = \frac{1}{2q}$ with $p_0 \neq 2$. Hence, interpolation between (2.6) and (2.12) gives that

$$\|H_{u,v}(f)\|_{p_0} \leq CBe^{-(|u|\gamma_1+|v|\gamma_2)} \|f\|_{p_0} . \tag{2.13}$$

Therefore, by this and (2.2), we deduce that

$$\|G_\nu(f)\|_p \leq \int_{\mathbf{R}\times\mathbf{R}} \|H_{u,v}(f)\|_p \, dudv \leq C_p \|f\|_p \, . \tag{2.14}$$

□

## 3. Proof of Theorem 1.1

We prove Theorem 1.1 by applying the same approaches found in [1,8], which have their roots in [16, 17]. Let us assume that $\Omega \in L(\log L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and satisfies (1.2)-(1.3). For $k \in \mathbf{N}$, let $E_k = \{(x, y) \in \mathbf{S}^{n-1} \times \mathbf{S}^{m-1} : 2^{k-1} \leq |\Omega(x, y)| < 2^k\}$, $D = \{k \in \mathbf{N} : \sigma(E_k) > 2^{-4k}\}$. Denote $\varrho_n = \int_{\mathbf{S}^{n-1}} J(x, \cdot)d\sigma(x)$ and $\varrho_m = \int_{\mathbf{S}^{m-1}} J(\cdot, y)d\sigma(y)$. For $k \in \mathbf{N}$, define $\Omega_k$ by

(3.1)

$$\Omega_k(x,y) = \Omega(x,y)\chi_{E_k}(x,y) - \frac{1}{\varrho_n}\int\limits_{\mathbf{S}^{n-1}} \Omega(x,y)J(x,y)\chi_{E_k}(x,y)d\sigma(x)$$
$$-\frac{1}{\varrho_m}\int\limits_{\mathbf{S}^{m-1}}\Omega(x,y)J(x,y)\chi_{E_k}(x,y)d\sigma(y)+\frac{1}{\varrho_n\varrho_m}\int\limits_{E_k}(x,y)\Omega(x,y)J(x,y)d\sigma(x)d\sigma(y),$$

and

$$\Omega_0(x,y) \quad = \quad \Omega(x,y) - \sum_{k\in D}\Omega_k(x,y). \tag{3.2}$$

As in [8], it is easy to verify that $\|\Omega_0\|_2 \leq C$; and for $k \in \mathbf{N} \cup \{0\}$, $\Omega_k$ satisfies (1.2)-(1.3). For $k \in D$, we define the family of measures $\nu^{(k)} = \{\nu_{k,t,s} : t, s \in \mathbf{R}\}$ on $\mathbf{R}^n \times \mathbf{R}^m$ by

$$\int_{\mathbf{R}^n\times\mathbf{R}^m} f d\nu_{k,t,s} = \frac{1}{2^{k(t+s)}}\int_{\rho_1(u)\leq 2^{kt}}\int_{\rho_2(v)\leq 2^{ks}} \frac{\Omega_k(u,v)}{\rho_1(u)^{\alpha-1}\rho_2(v)^{\beta-1}} f(u,v)dudv.$$

Set $a_k = b_k = 2^k$, $B_k = 2^k\sigma(E_k)$, $\gamma_1 = \frac{2\delta\ln 2}{N_1^*}$, and $\gamma_2 = \frac{2\delta\ln 2}{N_2^*}$, where $N_1^*, N_2^*$ denote the distinct numbers $\{\alpha_i\}$, $\{\beta_j\}$, respectively; and $0 < \delta < \min\{1, \frac{N_1^*}{2}, \frac{N_2^*}{2}, \frac{N_1^*}{\alpha}, \frac{N_2^*}{\beta}\}$. Thus,

$$\|\nu_{k,t,s}\|_1 \leq \frac{1}{2^{k(t+s)}}\int_0^{2^{kt}}\int_0^{2^{ks}}\iint_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega_k(u,v)|\, J(u,v)d\sigma(u)d\sigma(v)d\rho_1 d\rho_2$$
$$\leq C2^k\sigma(B_k) = CB_k. \tag{3.3}$$

By the cancelation properties of $\Omega_k$, and a simple change of variables, we derive that

$$|\hat{\nu}_{k,t,s}| \leq \frac{1}{2^{k(t+s)}}\int_0^{2^{kt}}\int_0^{2^{ks}}\iint_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} J(u,v)\,|\Omega_k(u,v)|\left|e^{-iA_{\rho_1}u\cdot\xi} - 1\right| d\sigma(u)d\sigma(v)d\rho_2 d\rho_1$$
$$\leq \frac{1}{2^{kt}}\int_0^{2^{kt}}\iint_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega_k(u,v)|\,|A_{\rho_1}u\cdot\xi|\, d\sigma(u)d\sigma(v)d\rho_1$$

$$\leq C\,|A_{2^{kt}}\xi|\int_0^1\iint_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}|\Omega_k(u,v)|\,|A_{\rho_1}u|\,d\sigma(u)d\sigma(v)d\rho_1$$
$$\leq CB_k\left|A_{a_k^t}\xi\right|,$$

which when combined with the trivial estimate $|\hat{\nu}_{k,t,s}|\leq CB_k$ gives that

$$|\hat{\nu}_{k,t,s}|\quad\leq\quad CB_k\left|A_{a_k^t}\xi\right|^{\frac{\gamma_1}{\ln a_k}}. \tag{3.4}$$

In the same manner, we attain

$$|\hat{\nu}_{k,t,s}|\quad\leq\quad CB_k\left|A_{b_k^s}\eta\right|^{\frac{\gamma_2}{\ln b_k}}. \tag{3.5}$$

On the other hand, Lemma 2.5 and Hölder's inequality lead to

$$|\hat{\nu}_{k,t,s}|^2\leq\sum_{i,j=0}^{\infty}\frac{1}{2^{i+j}}\int\limits_{2^{kt-j-1}}^{2^{kt-j}}\int\limits_{2^{ks-i-1}}^{2^{ks-i}}\left|\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}J(u,v)\Omega_k(u,v)\right.$$
$$\times\left.e^{-i(A_{\rho_1}u\cdot\xi+A_{\rho_2}v\cdot\eta)}d\sigma(u)d\sigma(v)\right|^2\frac{d\rho_1d\rho_2}{\rho_1\rho_2}$$
$$\leq C\int_{\mathbf{S}^{m-1}}\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}}\Omega_k(u,v)\overline{\Omega_k(x,v)}$$
$$\times\sum_{j=0}^{\infty}\frac{1}{2^j}\left|\int\limits_1^2 e^{-iA_{2^{kt-j-1}\rho_1}(u-x)\cdot\xi}\frac{d\rho_1}{\rho_1}\right|d\sigma(u)d\sigma(x)d\sigma(v)$$
$$\leq C\int_{\mathbf{S}^{m-1}}\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}}\Omega_k(u,v)\overline{\Omega_k(x,v)}$$
$$\times\sum_{j=0}^{\infty}\frac{1}{2^j}\left(|A_{2^{kt-j-1}}(u-x)\cdot\xi|\right)^{\frac{-\delta}{N_1^*}}d\sigma(u)d\sigma(x)d\sigma(v)$$
$$\leq C\sum_{j=0}^{\infty}\frac{1}{2^j}2^{(\alpha\delta/N_1^*)(j+1)}\,|A_{2^{kt}}\xi|^{\frac{-\delta}{N_1^*}}$$
$$\times\int_{\mathbf{S}^{m-1}}\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}}\Omega_k(u,v)\overline{\Omega_k(x,v)}\left|(u-x)\cdot\frac{A_{2^{kt-j-1}}\xi}{|\,A_{2^{kt-j-1}}\xi\,|}\right|^{\frac{-\delta}{N_1^*}}d\sigma(u)d\sigma(x)d\sigma(v)$$
$$\leq C\sum_{j=0}^{\infty}\frac{1}{2^j}\,|A_{2^{kt}}\xi|^{\frac{-\delta}{N_1^*}}\,\|\Omega_k\|_2^2.$$

Thus,

$$|\hat{\nu}_{k,t,s}|\leq Ca_kB_k\,|A_{2^{kt}}\xi|^{\frac{-\delta}{2N_1^*}}.$$

Combining this estimate with the trivial estimate $|\hat{\nu}_{k,t,s}|\leq CB_k$ yields

$$|\hat{\nu}_{k,t,s}|\leq CB_k\,|A_{2^{kt}}\xi|^{-\frac{\gamma_1}{\ln a_k}}. \tag{3.6}$$

Similarly, we derive

$$|\hat{\nu}_{k,t,s}|\leq CB_k\,|A_{2^{ks}}\xi|^{-\frac{\gamma_2}{\ln b_k}}. \tag{3.7}$$

Finally, by definition of $(\nu^{(k)})^*(f)$, we have that

$$\begin{aligned}
&(\nu^{(k)})^*(f)(x,y) = \sup_{t,s\in\mathbf{R}} ||\nu_{k,t,s}| * f| \leq C \iint_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega_k(u,v)| \\
\times \quad & \left(\sup_{t,s\in\mathbf{R}} \frac{1}{2^{k(t+s)}} \int_0^{2^{kt}} \int_0^{2^{ks}} |f(x-A_{\rho_1}u, y-A_{\rho_2}v)|\, d\rho_2 d\rho_1\right) d\sigma(u)d\sigma(v) \\
\leq \quad & C \iint_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega_k(u,v)|\, M_{\Gamma,\Lambda}(f)(x,y) d\sigma(u)d\sigma(v).
\end{aligned}$$

Therefore, by Lemma 2.2, we obtain that

$$\begin{aligned}
\left\|(\nu^{(k)})^*(f)\right\|_q &\leq C\|f\|_p \|\Omega_k(u,v)\|_1 \\
&\leq CB_k \|f\|_q \qquad (3.8)
\end{aligned}$$

Lemma 2.6 and (3.3)-(3.8) give that

$$\|G_{\nu^{(k)}}(f)\|_p \leq C_p \|f\|_p . \qquad (3.9)$$

By this and Minkowski's inequality, we conclude that

$$\begin{aligned}
\|\mathcal{M}_\Omega(f)\|_p &\leq C_p \|\mathcal{M}_{\Omega_0}(f)\|_p + \sum_{k\in D} (\ln 2^k) B_k \|G_{\nu^{(k)}}(f)\|_p \\
&\leq C_p \left(1 + \sum_{k\in D}(k)B_k\right) \|f\|_p \\
&\leq C_p \left(1 + \|\Omega\|_{L(\log L)}\right) \|f\|_p \leq C_p \|f\|_p
\end{aligned}$$

for $1 < p < \infty$ and $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$. Thus, we finish the proof of Theorem 1.1.


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