Title:
Parabolic Marcinkiewicz integrals on product spaces

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PARABOLIC MARCINKIEWICZ INTEGRALS ON PRODUCT SPACES

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Abstract. In this paper, we study the $L^p$ ($1 < p < \infty$) boundedness for the parabolic Marcinkiewicz integral when the kernel function $\Omega$ belongs to the class $L(\log L)(S^{n-1} \times S^{m-1})$. Our result essentially extend and improve some known results.

Keywords: $L^p$ boundedness, parabolic Marcinkiewicz integrals, rough kernels, product spaces.


1. Introduction and preliminaries

Let $\mathbb{R}^N$ ($N = n$ or $m$), $N \geq 2$ be the $N$-dimensional Euclidean space, and let $S^{N-1}$ be the unit sphere in $\mathbb{R}^N$ which is equipped with the normalized Lebesgue surface measure $d\sigma = d\sigma(z)$. Also, let $p'$ denote to the exponent conjugate to $p$; that is $1/p + 1/p' = 1$.

For $i = 1, 2, \ldots, N$, let $\alpha_i$ be fixed real numbers such that $\alpha_i \geq 1$. For fixed $z \in \mathbb{R}^N$, the function $F(z, \rho) = \sum_{i=1}^{N} \frac{z_i^2}{\rho^{\alpha_i}}$ is decreasing in $\rho > 0$. The unique solutions of the equations $F(z, \rho) = 1$ is denoted by $\rho(z)$.

For $\lambda > 0$, let $A_\lambda = \begin{bmatrix} \lambda^{\alpha_1} & 0 & \cdots & 0 \\ 0 & \lambda^{\alpha_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda^{\alpha_N} \end{bmatrix}$, and let $K_{\Omega, \rho}(z) = \Omega(z)\rho(z)^{1-\alpha}$,

where $\alpha = \sum_{i=1}^{N} \alpha_i$ and $\Omega$ is a real valued and measurable function on $\mathbb{R}^N$ with $\Omega \in L^1(S^{N-1})$ satisfying the conditions

$\Omega(A_\lambda z) = \Omega(z)$ and $\int_{S^{N-1}} \Omega(z')J(z')d\sigma(z') = 0$,

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where \( J(z') \) is defined as in [8]. The parabolic Marcinkiewicz integral \( \mu_\Omega \), which was introduced by Ding, Xue and Yabuta in [15], is defined by

\[
\mu_\Omega f(z) = \left( \int_0^\infty |F_{\Omega,t}(z)|^2 \frac{dt}{t^3} \right)^{1/2},
\]

where

\[
F_{\Omega,t}(z) = \int_{|\rho(u)|\leq t} K_{\Omega,\rho}(u)f(z-u)du.
\]

In particular, the authors of [15] proved that the parabolic Littlewood-Paley operator \( \mu_\Omega \) is bounded for \( p \in (1, \infty) \) provided that \( \Omega \in L^q(S^{N-1}) \) for \( q > 1 \). Subsequently, the study of the \( L^p \) boundedness of \( \mu_\Omega \) under various conditions on the function \( \Omega \) has been studied by many authors (see for example [7, 8, 24]). A particular result that is closely related to our work is the boundedness result of \( \mu_\Omega \) obtained by Cheng and Ding in [8]. In fact, they proved that \( \mu_\Omega \) is bounded under the condition \( \Omega \in L(\log L)^{1/2}(S^{n-1}) \) for \( 1 < p < \infty \).

We point out that the class of the operators \( \mu_\Omega \) is related to the class of the parabolic singular integral operators

\[
T_\Omega f(z) = p.v. \int_{\mathbb{R}^N} \frac{\Omega(u)}{\rho(u)^\alpha} f(z-u)du.
\]

The class of the operators \( T_\Omega \) belongs to the class of singular Radon transforms, which has considered to study by many mathematicians (we refer the readers, in particular, to [18, 21]).

If \( \alpha_1 = \cdots = \alpha_N = 1 \), then \( \rho(z) = |z| \), \( \alpha = N \) and \((\mathbb{R}^N, \rho) = (\mathbb{R}^N, |\cdot|)\). In this case, \( \mu_\Omega \) is just the classical Marcinkiewicz integral, which were introduced by Stein in [23]. For more information about the importance and the recent advances on the study of such operators, the readers are refereed (for instance to [3, 4, 10, 14, 17, 19], and the references therein).

Although some open problems related to the boundedness of parabolic Marcinkiewicz integral in the one-parameter setting remain open, the investigation of \( L^p \) estimates of the Marcinkiewicz integral on product spaces has been started (see for example [1, 2, 5–7, 11–13].)

Our main interest in this paper is to study the \( L^p \) boundedness of the parabolic Marcinkiewicz integral with a rough kernel on product spaces. Namely, for \( i = 1, 2, \cdots, n \) and \( j = 1, 2, \cdots, m \), let \( \alpha_i, \beta_j \) be fixed real numbers such that \( \alpha_i, \beta_j \geq 1 \), and let \( K_{\Omega,\rho_1,\rho_2}(x,y) = \Omega(x,y)\rho_1(x)^{1-\alpha}\rho_2(y)^{1-\beta} \), where \( \alpha = \sum_{i=1}^n \alpha_i \), \( \beta = \sum_{j=1}^m \beta_j \) and \( \Omega \) is a real valued and measurable function on \( \mathbb{R}^n \times \mathbb{R}^m \) with
\( \Omega \in L^1(S^{n-1} \times S^{m-1}) \) satisfying the conditions

\[
\begin{align*}
(1.1) & & \Omega(A_1, x, A_2 y) & = \Omega(x, y) & \text{and} \\
(1.2) & & \int_{S^{n-1}} \Omega(x', .) J(x', .) d\sigma(x') & = \int_{S^{m-1}} \Omega(., y') J(., y') d\sigma(y') = 0,
\end{align*}
\]

where \( \lambda_1, \lambda_2 > 0 \), and \( J(x', y') \) is a function on the unit sphere \( S^{n-1} \times S^{m-1} \) in \( \mathbb{R}^n \times \mathbb{R}^m \), that will be defined later.

The parabolic Marcinkiewicz integral operator \( \mathcal{M}_\Omega \) for \( f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m) \) is given by

\[
(1.3) \quad \mathcal{M}_\Omega f(x, y) = \left( \int_0^\infty \int_0^\infty |F_{t,s}(x, y)|^2 \frac{dt ds}{t s^2} \right)^{1/2},
\]

where

\[
F_{t,s}(x, y) = \int_{\rho_1(t) \leq t} \int_{\rho_2(s) \leq s} K_{\Omega, \rho_1, \rho_2}(u, v) f(x - u, y - v) du dv.
\]

When \( \alpha_1 = \cdots = \alpha_n = 1 \), and \( \beta_1 = \cdots = \beta_m = 1 \), then \( \rho_1(x) = |x| \), \( \rho_2(y) = |y| \), \( \alpha = n \), and \( \beta = m \). In this case, \( \mathcal{M}_\Omega \) is just the classical Marcinkiewicz integral on product domains, which was studied by many mathematicians. For instance, the author of [13] gave the \( L^2 \) boundedness of \( \mathcal{M}_\Omega \) if \( \Omega \in L(\log L)^2(S^{n-1} \times S^{m-1}) \). Subsequently, it was verified in [11] that \( \mathcal{M}_\Omega \) is bounded for all \( 1 < p < \infty \) provided that \( \Omega \in L(\log L)^2(S^{n-1} \times S^{m-1}) \). This result was improved (for \( p = 2 \)) in [12] in which the author established that \( \mathcal{M}_\Omega \) is bounded on \( L^2(\mathbb{R}^n \times \mathbb{R}^m) \) for all \( \Omega \in L(\log L)(S^{n-1} \times S^{m-1}) \). Recently, Al-Qaseem et al. found in [1] that the boundedness of \( \mathcal{M}_\Omega \) is obtained under the condition \( \Omega \in L(\log L)(S^{n-1} \times S^{m-1}) \) for \( 1 < p < \infty \). Furthermore, they proved that the exponent 1 is the best possible.

In this article, we extend and improve the corresponding results in [1, 11, 12]. Our main result is formulated as follows.

**Theorem 1.1.** Suppose that \( \Omega \in L(\log L)(S^{n-1} \times S^{m-1}) \) and satisfies (1.2)-(1.3). Then \( \mathcal{M}_\Omega \) is bounded on \( L^p(\mathbb{R}^n \times \mathbb{R}^m) \) for \( p \in (1, \infty) \).

Throughout this paper, the letter \( C \) denotes a bounded positive constant that may vary at each occurrence but independent of the essential variables.

### 2. Some lemmas

In this section, we give some auxiliary lemmas used in the sequel. The following is found in [8, 24]. For \((x, y) \in \mathbb{R}^n \times \mathbb{R}^m\), set
\[ x_1 = \rho_1^\alpha_1 \cos \theta_1 \cdots \cos \theta_{n-2} \cos \theta_{n-1}, \quad y_1 = \rho_2^\beta_1 \cos \theta_1 \cdots \cos \theta_{m-2} \cos \theta_{m-1}, \]
\[ x_2 = \rho_1^\alpha_2 \cos \theta_1 \cdots \cos \theta_{n-2} \sin \theta_{n-1}, \quad y_2 = \rho_2^\beta_2 \cos \theta_1 \cdots \cos \theta_{m-2} \sin \theta_{m-1}, \]
\[ \vdots \]
\[ x_{n-1} = \rho_1^\alpha_{n-1} \cos \theta_1 \sin \theta_2, \quad y_{m-1} = \rho_2^\beta_{m-1} \cos \theta_1 \sin \theta_2, \]
\[ x_n = \rho_1^\alpha_n \sin \theta_1, \quad y_m = \rho_2^\beta_m \sin \theta_1. \]

Then \( \rho_1^\alpha \rho_2^\beta = \rho_1^\alpha \rho_2^\beta \), \( J_1(\theta_1, \cdots, \theta_{n-1}) J_2(\theta_1, \cdots, \theta_{m-1}) \) is a function from \( \mathbb{R}^n \) to \( \mathbb{R}^m \). For suitable \( f \), let \( M_{f} \) be the maximal operator defined on \( \mathbb{R}^n \) by

\[ M_{f}(x) = \sup_{h>0} \frac{1}{h} \left| \int_0^h f(x - \Gamma(t)) dt \right| \]

for \( x \in \mathbb{R}^n \). Then for \( 1 < p \leq \infty \), there exists a constant \( C_p > 0 \) such that

\[ \| M_{f} \|_p \leq C_p \| f \|_p. \]

The constant \( C_p \) is independent of \( \alpha_i \) and \( f \).

**Lemma 2.2.** Suppose that \( \alpha_i \), \( \beta_i \), \( \alpha'_i \), and \( \beta'_i \) are fixed real numbers. Let \( \Gamma(t) = (a_1 t^{\alpha_1}, \cdots, a_n t^{\alpha_n}) \) and \( \Lambda(t) = (b_1 t^{\beta_1}, \cdots, b_m t^{\beta_m}) \), and let \( M_{f, \Lambda} \) be the maximal operator defined on \( \mathbb{R}^n \times \mathbb{R}^m \) by

\[ M_{f, \Lambda}(x, y) = \sup_{h_1, h_2 > 0} \frac{1}{h_1 h_2} \left| \int_0^{h_2} \int_0^{h_1} f(x - \Gamma(t), y - \Lambda(r)) dr dt \right| \]

for \((x, y) \in \mathbb{R}^n \times \mathbb{R}^m \). Then for \( 1 < p \leq \infty \), there exists a constant \( C_p > 0 \) (independent of \( \alpha_i \), \( \beta_i \), \( \alpha'_i \), \( \beta'_i \), and \( f \)) such that

\[ \| M_{f, \Lambda} \|_p \leq C_p \| f \|_p. \]

The proof of Lemma 2.2 follows immediately by using Lemma 2.1 and the inequality \( M_{f, \Lambda}(x, y) \leq M_\Lambda \circ M_f(x, y) \), where \( \circ \) denotes the composition of operators.

We shall recall the following lemma due to Madych.

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Lemma 2.3. [20] Let \( \Phi \in \mathcal{S}(\mathbb{R}^n) \) satisfy \( \hat{\Phi}(0) = 0 \). Denote \( \Phi_t(x) = t^{-\alpha} \Phi(A_{t^{-1}}x) \) for \( t > 0 \), the Littlewood-Paley \( g \)-function related to the transform \( A \) is defined by

\[
g_{\Phi}(f)(x) = \left( \int_0^\infty |\Phi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2}.
\]

Then there is a positive constant \( C \) such that \( \|g_{\Phi}(f)\|_p \leq C\|f\|_p \) for any \( f \in L^p(\mathbb{R}^n) \) and \( 1 < p < \infty \).

Similarly, we derive the following lemma.

Lemma 2.4. Let \( \Phi \in \mathcal{S}(\mathbb{R}^n), \Psi \in \mathcal{S}(\mathbb{R}^m) \) with \( \hat{\Phi}(0) = \hat{\Psi}(0) = 0 \). For \( s, t > 0 \), let \( \Phi_t(x) = t^{-\alpha} \Phi(A_{t^{-1}}x) \), \( \Psi_s(y) = s^{-\beta} \Psi(A_{s^{-1}}y) \) and \( \Gamma_{t,s}(x,y) = \Phi_t(x)\Psi_s(y) \).

Assume that the Littlewood-Paley \( g \)-function is defined by

\[
g_{\Phi,\Psi}(f)(x,y) = \left( \int_0^\infty \int_0^\infty |\Gamma_{t,s} * f(x,y)|^2 \frac{dt}{ts} \right)^{1/2}.
\]

Then there exists \( C > 0 \) such that \( \|g_{\Phi,\Psi}(f)\|_p \leq C\|f\|_p \) for any \( f \in L^p(\mathbb{R}^n \times \mathbb{R}^m) \) and \( 1 < p < \infty \).

Lemma 2.5. [8] Let \( \gamma \in [0,1] \) and \( u, \xi \in \mathbb{R}^n \). Then

\[
\left| \int_1^2 e^{A_\lambda u \cdot \xi} \frac{d\lambda}{\lambda} \right| \leq C |u \cdot \xi|^{-\gamma},
\]

where \( A_\lambda \) is defined as above and \( \tau \) denotes the distinct numbers of \( \{\alpha_i\} \).

For a two-parameter family of measures \( \nu = \{\nu_{t,s} : t, s \in \mathbb{R}\} \) on \( \mathbb{R}^n \times \mathbb{R}^m \), we define the operator \( G_\nu \) and its corresponding maximal operator \( \nu^* \) by

\[
G_\nu(f)(x,y) = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |\nu_{t,s} * f(x,y)|^2 \, dt \, ds \right)^{1/2}
\]

and

\[
\nu^*(f) = \sup_{t,s \in \mathbb{R}} \|\nu_{t,s} * f\|.
\]

We write \( t^\pm = \min\{t^+, t^-\} \) and \( \|\nu_{t,s}\|_\nu \) for the total variation of \( \nu_{t,s} \).

The following is the main lemma of this section.

Lemma 2.6. Let \( a, b \geq 2, \gamma_1, \gamma_2 > 0, q > 1 \) and \( B > 0 \). Suppose that the family of measures \( \nu_{t,s} : t, s \in \mathbb{R} \) satisfies the following conditions:

(i) \( \|\nu_{t,s}\|_\nu \leq CB \) for \( t, s \in \mathbb{R} \);

(ii) \( \|\nu_{t,s}(\xi,\eta)\|_\nu \leq CB |A_{a^\gamma_1/\ln a} \xi|^{\pm\gamma_1/\ln a} |A_{b^\gamma_2/\ln b} \eta|^{\pm\gamma_2/\ln b} \) for \( (\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^m \) and \( t, s \in \mathbb{R} \);

(iii) \( \|\nu^*(f)\|_q \leq CB \|f\|_q \) for \( f \in L^q(\mathbb{R}^n \times \mathbb{R}^m) \).

Then, for every \( p \) satisfying \( |1/p - 1/2| < 1/(2q) \), there is a constant \( C_p \) (independent of \( a, b, B, f \)) such that for any \( f \in L^p(\mathbb{R}^n \times \mathbb{R}^m) \),

\[
\|G_\nu(f)\|_p \leq C_p B \|f\|_p.
\]
Proof. We employ some ideas from [1, 8]. For $\kappa > 2$, let $\varphi^{(\kappa)}$ be a $C^{\infty}$ function supported in $[4/(5\kappa), (5\kappa)/4]$ such that

(i) $\varphi^{(\kappa)}(\xi) = \varphi^{(\kappa)}(\rho(\xi))$ for $\rho > 0$ and $\xi \in \mathbb{R}^N$;

(ii) $0 < \varphi^{(\kappa)}(\xi) \leq 1$;

(iii) $\int_{0}^{\infty} \frac{\varphi^{(\kappa)}(t)}{t} dt = 2 \ln \kappa$.

For $a, b > 2$, and for $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m$, let $\Phi \in C^{\infty}(\mathbb{R}^n)$ and $\Psi \in C^{\infty}(\mathbb{R}^m)$ be given by $\Phi(\xi) = \varphi^{(a)}(\rho_1(\xi)^2)$ and $\Psi(\eta) = \varphi^{(b)}(\rho_2(\eta)^2)$. For $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ and $t, s \in \mathbb{R}$, set

$$\Phi_t(x) = t^{-\alpha} \Phi(A_{t^{-1}}x), \Psi_s(y) = s^{-\beta} \Phi(A_{s^{-1}}y) \quad \text{and} \quad \Gamma_{t,s}(x,y) = \Phi_t(x)\Psi_s(y).$$

Thus, for any $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, we get

$$f(x,y) = \int_{\mathbb{R} \times \mathbb{R}} \Gamma_{a^t,b^s} * f(x,y)dtds. \quad (2.3)$$

By Minkowski’s inequality, we reach that

$$G_{u,v}(f)(u,v) = \left( \int_{\mathbb{R} \times \mathbb{R}} \left| \int_{\mathbb{R} \times \mathbb{R}} \Gamma_{a^t+u,b^s+v} * \nu_{t,s} * f(x,y)dudv \right|^2 dtds \right)^{1/2} \quad (2.4)$$

where

$$H_{u,v}(f)(x,y) = \left( \int_{\mathbb{R} \times \mathbb{R}} \left| \int_{\mathbb{R} \times \mathbb{R}} \Gamma_{a^t+u,b^s+v} * \nu_{t,s} * f(x,y) \right|^2 dtds \right)^{1/2}. \quad (2.5)$$

Let us start estimating $\|H_{u,v}f\|_2$ for the case $u, v \geq 0$; the proof for the other cases are essentially the same and require only minor modifications. By Plancherel’s theorem, assumption $(ii)$ and the techniques used in [8], we conclude that

$$\|H_{u,v}f\|_2^2 \leq CB \int_{\mathbb{R} \times \mathbb{R}} \left( \int_{\mathbb{R}^n \times \mathbb{R}^m} \left| \hat{f}(\xi, \eta) \right|^2 \left| \varphi^{(a)}(\rho_1(A_{u^{-1}}\xi) ) \right|^2 |A_{u} \xi|^{2\gamma_1/lna} \right. \times \left| \varphi^{(b)}(\rho_2(A_{v^{-1}}\eta) ) \right|^2 |A_{v} \eta|^{2\gamma_2/lnb} d\xi d\eta dt ds \\leq CB \int_{\mathbb{R} \times \mathbb{R}} \left( \int_{E_{u,v,t,s}} \left| \hat{f}(\xi, \eta) \right|^2 \left| A_{u} \xi \right|^{2\gamma_1/lna} \left| A_{v} \eta \right|^{2\gamma_2/lnb} d\xi d\eta \right) dt ds \quad (2.6)$$

where $E_{u,v,t,s} = \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m : \}$

$$\frac{2}{\sqrt{5}} a^{-u - 1/2} \leq a^t \rho_1(\xi) \leq \frac{\sqrt{5}}{2} a^{-u+1/2}, \quad \frac{2}{\sqrt{5}} b^{-v - 1/2} \leq b^s \rho_2(\eta) \leq \frac{\sqrt{5}}{2} b^{-v+1/2}. \}$$
Now, let us estimate \( \| H_{u,v} f \|_{p_0} \) for \( p_0 \) satisfying \( \frac{1}{2} - \frac{1}{p_0} = \frac{1}{2q} \) with \( p_0 \neq 2 \).

First, we consider \( 1 < p_0 < 2 \). By the assumption \((i)\), we get

\[
(2.7) \quad \left\| \int_{\mathbb{R} \times \mathbb{R}} \nu_{t,s} \ast \Gamma_{u^{n+1},b^{n+1}} f(x,y) dt ds \right\|_1 \leq CB \left\| \int_{\mathbb{R} \times \mathbb{R}} \Gamma_{a^{n},b^{n}} f(x,y) dt ds \right\|_1.
\]

Further, by using the assumption \((iii)\), we achieve that

\[
(2.8) \quad \| \sup_{t,s \in \mathbb{R}} \| \nu_{t,s} \ast \Gamma_{u^{n+1},b^{n+1}} f \|_q \| \leq CB \| \sup_{t,s \in \mathbb{R}} \| \Gamma_{a^{n},b^{n}} f \|_q.
\]

By using the interpolation theorem between (2.7) and (2.8) and the Lemma 2.4, we deduce that

\[
(2.9) \quad \| H_{u,v} f \|_{p_0} \leq CB \left\| \int_{\mathbb{R} \times \mathbb{R}} \left( \| \Gamma_{a^{n},b^{n}} f \|_p \right)^2 dt ds \right\|_{p_0}^{1/2} \leq CB \| f \|_{p_0}.
\]

Next, consider the case \( 2 < p_0 < \infty \). As \( q = \left( \frac{p_0}{2} \right)' \) and \( \left\| (H_{u,v}(f))^{1/2} \right\|_{p_0} = \| H_{u,v}(f) \|_{1/2} \), there is a non-negative function \( F \in L^q(\mathbb{R}^n \times \mathbb{R}^m) \) with \( \| F \|_q \leq 1 \) such that

\[
(2.10) \quad \| H_{u,v}(f) \|_{p_0}^2 = \left\| \left( \int_{\mathbb{R} \times \mathbb{R}} \left| \Gamma_{u^{n+1},b^{n+1}} \ast \nu_{t,s} f(x,y) \right|^2 dt ds \right)^{1/2} \right\|_{p_0}^2 \leq \int_{\mathbb{R}^n \times \mathbb{R}^m} \int_{\mathbb{R} \times \mathbb{R}} \left| \Gamma_{a^{n+1},b^{n+1}} \ast \nu_{t,s} f(x,y) \right|^2 dt ds f(x,y) dx dy.
\]

By using Holder’s inequality, (2.10), Lemma 2.4 plus the assumptions \((i)\) and \((iii)\), we obtain that

\[
(2.11) \quad \| H_{u,v}(f) \|_{p_0}^2 \leq \| \nu_{t,s} \|_1 \left\| \int_{\mathbb{R}^n \times \mathbb{R}^m} \int_{\mathbb{R} \times \mathbb{R}} \left| \Gamma_{a^{n+1},b^{n+1}} \ast f(x,y) \right|^2 dt ds \right\|_{p_0/2}^{p_0/2} \| F \|_{q} \leq CB \left( \int_{\mathbb{R}^n \times \mathbb{R}^m} \int_{\mathbb{R} \times \mathbb{R}} \left| \Gamma_{a^{n+1},b^{n+1}} \ast f(x,y) \right|^2 dt ds \right)^{p_0/2} dx dy \left\| \nu^{*}(\tilde{F}) \right\|_q.
\]

where \( \tilde{F}(x,y) = F(-x,-y) \). Thus, by (2.9) and (2.11) we reach that

\[
(2.12) \quad \| H_{u,v}(f) \|_{p_0} \leq CB \| f \|_{p_0}
\]

for any \( p_0 \) satisfying \( \frac{1}{2} - \frac{1}{p_0} = \frac{1}{2q} \) with \( p_0 \neq 2 \). Hence, interpolation between (2.6) and (2.12) gives that

\[
(2.13) \quad \| H_{u,v}(f) \|_{p_0} \leq CBE^{-\langle |u|^\gamma_1 + |v|^\gamma_2 \rangle} \| f \|_{p_0}.
\]
Therefore, by this and (2.2), we deduce that
\[
\|G_\nu(f)\|_p \leq \int_{\mathbb{R} \times \mathbb{R}} \|H_{u,v}(f)\|_p \, dudv \leq C_p \|f\|_p.
\]

3. Proof of Theorem 1.1

We prove Theorem 1.1 by applying the same approaches found in [1, 8], which have their roots in [16, 17]. Let us assume that $\Omega \in L(\log L)\left(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}\right)$ and satisfies (1.2)-(1.3). For $k \in \mathbb{N}$, let $E_k = \{(x, y) \in \mathbb{S}^{n-1} \times \mathbb{S}^{m-1} : 2^{k-1} \leq |\Omega(x, y)| < 2^k\}$, $D = \{k \in \mathbb{N} : \sigma(E_k) > 2^{-2k}\}$. Denote $\varrho_n = \int_{\mathbb{S}^{n-1}} J(x, \cdot) \sigma(x)$ and $\varrho_m = \int_{\mathbb{S}^{m-1}} J(\cdot, y) \sigma(y)$. For $k \in \mathbb{N}$, define $\Omega_k$ by
\[
\Omega_k(x, y) = \Omega(x, y) \chi_{E_k}(x, y) - \frac{1}{\varrho_n} \int_{\mathbb{S}^{n-1}} \Omega(x, y) J(x, y) \chi_{E_k}(x, y) d\sigma(x)
\]
\[-\frac{1}{\varrho_m} \int_{\mathbb{S}^{m-1}} \Omega(x, y) J(x, y) \chi_{E_k}(x, y) d\sigma(y) + \frac{1}{\varrho_n \varrho_m} \int_{E_k} (x, y) \Omega(x, y) J(x, y) d\sigma(x) d\sigma(y),
\]
and
\[
\Omega_0(x, y) = \Omega(x, y) - \sum_{k \in D} \Omega_k(x, y).
\]

As in [8], it is easy to verify that $\|\Omega_0\|_2 \leq C$; and for $k \in \mathbb{N} \cup \{0\}$, $\Omega_k$ satisfies (1.2)-(1.3). For $k \in D$, we define the family of measures $\nu^{(k)} = \{\nu_{k,t,s} : t, s \in \mathbb{R}\}$ on $\mathbb{R}^n \times \mathbb{R}^m$ by
\[
\int_{\mathbb{R}^n \times \mathbb{R}^m} f \, d\nu_{k,t,s} = \frac{1}{2^{k(t+s)}} \int_{\rho_1(u) \leq 2^{kt}} \int_{\rho_2(v) \leq 2^{ks}} \frac{\Omega_k(u, v)}{\rho_1(u)^\alpha \rho_2(v)^\beta} f(u, v) \, dudv.
\]

Set $a_k = b_k = 2^k$, $B_k = 2^k \sigma(E_k)$, $\gamma_1 = \frac{2\ln 2}{N_1^2}$, and $\gamma_2 = \frac{2\ln 2}{N_2^2}$, where $N_1^2, N_2^2$ denote the distinct numbers $\{\alpha_i\}, \{\beta_j\}$, respectively; $0 < \delta < \min\{1, \frac{N_1^2}{2}, \frac{N_2}{2}, \frac{N_1^2}{\alpha}, \frac{N_2}{\beta}\}$. Thus,
\[
\|\nu_{k,t,s}\|_1 \leq \frac{1}{2^{k(t+s)}} \int_0^{2^{kt}} \int_{2^{ks}}^{2^{2ks}} \int_{\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}} |\Omega_k(u, v)| J(u, v) \sigma(u) \sigma(v) \, d\rho_1 \, d\rho_2 \, dJ(u, v)
\]
\[
\leq C 2^{k} \sigma(B_k) = CB_k.
\]

By the cancelation properties of $\Omega_k$, and a simple change of variables, we derive that
\[
|\nu_{k,t,s}| \leq \frac{1}{2^{k(t+s)}} \int_0^{2^{kt}} \int_{2^{ks}}^{2^{2ks}} \int_{\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}} J(u, v) \Omega_k(u, v) \left| e^{-iA \mu_i u \cdot \xi} - 1 \right| \sigma(u) \sigma(v) \, d\rho_1 \, d\rho_2 \, dJ(u, v)
\]
\[
\leq \frac{1}{2^{kt}} \int_0^{2^{kt}} \int_{\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}} |\Omega_k(u, v)| |A \mu_i u \cdot \xi| \sigma(u) \sigma(v) \, d\rho_1 \, d\rho_2 \, dJ(u, v)
\]
Combining this estimate with the trivial estimate $|\hat{\nu}_{k,t,s}| \leq CB_k$ gives that
\begin{equation}
|\hat{\nu}_{k,t,s}| \leq CB_k \left| A_{\eta_k} \xi \right|^\frac{m}{m+k} \tag{3.4}
\end{equation}

In the same manner, we attain
\begin{equation}
|\hat{\nu}_{k,t,s}| \leq CB_k \left| A_{\eta_k} \eta \right|^\frac{m}{m+k} \tag{3.5}
\end{equation}

On the other hand, Lemma 2.5 and Hölder’s inequality lead to
\begin{align*}
|\hat{\nu}_{k,t,s}|^2 &\leq \sum_{i,j=0}^{\infty} \frac{1}{2^{i+j}} \int_{2^{i+j-1}}^{2^{i+j}} \int_{2^{i+j-1}}^{2^{i+j}} \left| \int_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} J(u,v) \Omega_k(u,v) ight| \left| e^{-i(A_{\rho_1} u \xi + A_{\rho_2} v \eta)} d\sigma(u)d\sigma(v) \right|^2 \frac{d\rho_1 d\rho_2}{\rho_1 \rho_2} \\
&\leq C \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \Omega_k(u,v) \Omega_k(x,v) \\
&\times \sum_{j=0}^{\infty} \frac{1}{2^j} \int_{2^{j-1}}^{2^j} e^{-iA_{2^{j-1} \rho_1} (u-x) \cdot \xi} d\sigma(u) d\sigma(x) d\sigma(v) \\
&\leq C \sum_{j=0}^{\infty} \frac{1}{2^j} 2^{(\alpha^N)^{j+1}} \left| A_{2^j \xi} \xi \right|^{\frac{m}{m+k}} \\
&\times \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \Omega_k(u,v) \Omega_k(x,v) \left| (u-x) \right| A_{2^j \rho_1} \left| \frac{2^j \xi}{A_{2^j \rho_1}} \right|^{\frac{m}{m+k}} d\sigma(u) d\sigma(x) d\sigma(v) \\
&\leq C \sum_{j=0}^{\infty} \frac{1}{2^j} \left| A_{2^j \xi} \xi \right|^{\frac{m}{m+k}} \left\| \Omega_k \right\|_2^2.
\end{align*}

Thus,
\begin{equation}
|\hat{\nu}_{k,t,s}| \leq C a_k B_k \left| A_{2^j \xi} \xi \right|^{\frac{m}{m+k}} \tag{3.6}
\end{equation}

Combining this estimate with the trivial estimate $|\hat{\nu}_{k,t,s}| \leq CB_k$ yields
\begin{equation}
|\hat{\nu}_{k,t,s}| \leq CB_k \left| A_{2^j \xi} \xi \right|^{-\frac{m}{m+k}} \tag{3.7}
\end{equation}

Similarly, we derive
\begin{equation}
|\hat{\nu}_{k,t,s}| \leq CB_k \left| A_{2^j \xi} \xi \right|^{-\frac{m}{m+k}} \tag{3.7}
\end{equation}
Finally, by definition of \( (\nu^{(k)})^* (f) \), we have that
\[
(\nu^{(k)})^*(f)(x,y) = \sup_{t,s \in \mathbb{R}} |[\nu_{k,t,s}] * f| \leq C \int_{S^{n-1} \times S^{m-1}} |\Omega_k(u,v)|
\]
\[
\times \left( \sup_{t,s \in \mathbb{R}} \frac{1}{2^{k(t+s)}} \int_0^{2^{k(t+s)}} \int_0^{2^{k(t+s)}} |f(x - A_{\rho_1} u, y - A_{\rho_2} v)| d\rho_1 d\rho_2 \right) d\sigma(u) d\sigma(v)
\]
\[
\leq C \int_{S^{n-1} \times S^{m-1}} |\Omega_k(u,v)| M_{\Gamma, A}(f)(x,y) d\sigma(u) d\sigma(v).
\]
Therefore, by Lemma 2.2, we obtain that
\[
\left\| (\nu^{(k)})^*(f) \right\|_q \leq C \| f \|_p \|\Omega_k(u,v)\|_1
\]
(3.8)

Lemma 2.6 and (3.3)-(3.8) give that
\[
\| G_{\nu^{(k)}}(f) \|_p \leq C_p \| f \|_p.
\]
(3.9)

By this and Minkowski's inequality, we conclude that
\[
\| M_{\Gamma}(f) \|_p \leq C_p \| M_{\Omega}_k(f) \|_p + \sum_{k \in \mathcal{D}} (\ln 2^k) B_k \| G_{\nu^{(k)}}(f) \|_p
\]
\[
\leq C_p \left( 1 + \sum_{k \in \mathcal{D}} (k) B_k \right) \| f \|_p
\]
\[
\leq C_p \left( 1 + \| \Omega \|_{L(\log L)} \right) \| f \|_p \leq C_p \| f \|_p
\]
for \( 1 < p < \infty \) and \( f \in L^p(\mathbb{R}^n \times \mathbb{R}^m) \). Thus, we finish the proof of Theorem 1.1.

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References


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