Bulletin of the Iranian Mathematical Society Vol. 32 *No. 1 (2006), pp 45-63.*

ON PRE-θ-OPEN SETS AND TWO CLASSES OF FUNCTIONS

M. CALDAS, S. JAFARI*, G. NAVALAGI AND T. NOIRI

Abstract. In this paper we introduce the notions of pre-θ-derived, pre-θ-border, pre-θ-frontier and pre-θ-exterior of a set and study some of their basic properties. We also introduce two classes of functions called θ-preopen functions and θ-preclosed functions. We obtain their characterizations, their basic properties and their relationships with other types of functions between topological spaces.

1. Introduction and preliminaries

Mashhour et al. [14] defined a function $f : (X, \tau) \to (Y, \sigma)$ from a topological space (X, τ) into a topological space (Y, σ) to be precontinuous if $f^{-1}(U)$ is preopen in X for every open set U in Y. Since then, these functions have been extensively investigated. Precontinuity was called near continuity by Pták [21] and also called almost continuity by Husain [11]. Recently Noiri ([17], [18]) introduced the notions of strongly θ-precontinuous and θ-precontinuous functions which are stronger than those of precontinuous and weakly precontinuous functions respectively. In 2003, Cho [8] continued the work of Noiri and gave some other characterizations of strongly θ-precontinuous functions including a characterization using nets. Baker [5] also introduced a weak form of strong θ-precontinuity which he called weak θ-precontinuity. However, it is

MSC(2000): Primary: 54A05, 54C10, 54D10; Secondary: 54C08
Keywords: Pre-θ-open sets, pre-θ-closed sets, θ-preopen functions, θ-preclosed functions, quasi H-closed spaces
Received: 7 May 2006 , Revised: 18 August 2006
*Corresponding author

shown in Theorem 3.2 of [18] that weak θ-precontinuity in the sense of Baker is equivalent to θ-precontinuity. In this paper, we introduce the notions of pre-θ-derived, pre-θ-border, pre-θ-frontier and pre-θ-exterior of a set and show that some of their properties are analogous to those for open sets. Also, we give some additional properties of pre-θ-closure and pre-θ-interior of a set due to ([17], [19]). Moreover, we define the concepts of θ-preopenness and θ-preclosedness as a natural dual to the θ-precontinuity due to Noiri. We obtain several characterizations and properties of these functions. Moreover, we also study these functions comparing with other types of already known functions. It turns out that strong θ-preopenness implies θ-preopenness but not conversely. We show that under a certain condition the converse is also true.

Throughout this paper, (X,τ) and (Y,σ) (or simply, X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. If S is any subset of a space X, then Cl(S) and Int(S) denote the closure and the interior of S, respectively. Recall that a set S is called regular open (resp. regular closed) if $S = \mathrm{Int}(\mathrm{Cl}(S))$ (resp. $S = \mathrm{Cl}(\mathrm{Int}(S))$).

A subset S of X is called preopen [14] (resp. α-open [15] , β-open [1] or semipreopen [2]) if $S \subset \mathrm{Int}(\mathrm{Cl}(S))$ (resp. $S \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(S)))$, $S \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(S)))$). The complement of a preopen set is called preclosed. The intersection of all preclosed sets containing S is called the preclosure [10] of S and is denoted by $\mathrm{pCl}(S)$. The preinterior of S is defined by the union of all preopen sets contained in S and is denoted by $\mathrm{pInt}(S)$. The family of all preopen sets of X is denoted by $PO(X)$.

A point $x \in X$ is called a θ-cluster [26] (resp. pre-θ-cluster [17]) point of S if $S \cap \mathrm{Cl}(U) \neq \emptyset$ (resp. $S \cap \mathrm{pCl}(U) \neq \emptyset$) for each open (rep. preopen) set U containing x. The set of all θ-cluster (resp. pre-θ-cluster) points of S is called the θ-closure (resp. pre-θ-closure) of S and is denoted by $\mathrm{Cl}_\theta(S)$ (resp. $\mathrm{pCl}_\theta(S)$). A subset S is called θ-closed (resp. pre-θ-closed) if $\mathrm{Cl}_\theta(S) = S$ (resp. $\mathrm{pCl}_\theta(S) = S$). The complement of a θ-closed (resp. pre-θ-closed) set is called θ-open (resp. pre-θ-open). The family of all pre-θ-open (resp. pre-θ-closed) sets of a space X is denoted by $P\theta O(X,\tau)$(resp. $P\theta C(X,\tau)$).

Definition 1.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be

(i) weakly open ([23], [24]) if $f(U) \subset Int(f(Cl(U)))$ for each open subset U of X,
(ii) weakly closed [24] if $Cl(f(Int(F))) \subset f(F)$ for each closed subset F of X,
(iii) strongly continuous ([13],[3]) if for every subset A of X, $f(Cl(A)) \subset f(A)$,
(iv) almost open in the sense of Singal and Singal , written as a.o.S., [25] if the image of each regular open set U of X is an open set of Y, equivalently if $f(U) \subset Int(f(Int(Cl(U))))$ for each open subset U of X,
(v) preopen [14](resp. preclosed [14]) if for each open set U (resp. closed set F) of X, $f(U)$ is pre-open (resp. $f(F)$ is pre-closed) set in Y,
(vi) contra θ-preopen (resp. contra-closed [4]) if $f(U)$ is pre-θ-closed (resp. open) in Y for each open (resp. closed) subset U of X,

2. Some properties of pre-θ-open sets

The pre-θ-interior and the θ-interior of S, denoted by $\mathrm{pInt}_\theta(S)$ and $\mathrm{Int}_\theta(S)$, are defined as follows:
$\mathrm{pInt}_\theta(S) = \{x \in X :$ for some preopen subset U of X, $x \in U \subset \mathrm{pCl}(U) \subset S\}$ and
$\mathrm{Int}_\theta(S) = \{x \in X :$ for some open subset U of X, $x \in U \subset \mathrm{Cl}(U) \subset S\}$.

Recall that a space X is said to be pre-regular [19] if for each preclosed set F and each point $x \in X - F$, there exist disjoint preopen sets U and V such that $x \in U$ and $F \subset V$, equivalently if for each $U \in \mathrm{PO}(X)$ and each point $x \in U$, there exists $V \in \mathrm{PO}(X, x)$ such that $x \in V \subset \mathrm{pCl}(V) \subset U$.

Theorem 2.1. *Let A be a subset of a space X.*

(i) *A is a pre-θ-open set if and only if $A = \mathrm{pInt}_\theta(A)$.*
(ii) *$X - \mathrm{pInt}_\theta(A) = \mathrm{pCl}_\theta(X - A)$ and $\mathrm{pInt}_\theta(X - A) = X - \mathrm{pCl}_\theta(A)$.*
(iii) *$\mathrm{pCl}_\theta(A)$ is pre-closed, but it is not in general pre-θ-closed.*

Proof. (iii) One can check that $\mathrm{pCl}_\theta(A) \subset \mathrm{pCl}(\mathrm{pCl}_\theta(A))$. On the other hand, let $x \in \mathrm{pCl}(\mathrm{pCl}_\theta(A))$. Then for each $U \in PO(X, x)$, $U \cap$

$\mathrm{pCl}_\theta(A) \neq \emptyset$. Therefore, there exists $z \in U \cap \mathrm{pCl}_\theta(A)$. Hence $z \in U$ and $z \in \mathrm{pCl}_\theta(A)$. Then $U \in \mathrm{PO}(X, z)$ and $A \cap \mathrm{pCl}(U) \neq \emptyset$. Therefore, for each $U \in \mathrm{PO}(X, x)$, $A \cap \mathrm{pCl}(U) \neq \emptyset$, i.e., $x \in \mathrm{pCl}_\theta(A)$. Thus $\mathrm{pCl}(\mathrm{pCl}_\theta(A)) \subset \mathrm{pCl}_\theta(A)$. □

Example 2.2. Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$. Let $A = \{a\}$. Then $\mathrm{pCl}_\theta(A) = \{a, b\}$ and $\mathrm{pCl}_\theta(\mathrm{pCl}_\theta(A)) = X$. Therefore $\mathrm{pCl}_\theta(\mathrm{pCl}_\theta(A)) \neq \mathrm{pCl}_\theta(A)$. Hence $\mathrm{pCl}_\theta(A)$ is not pre-θ-closed.

Theorem 2.3. ([17], [8]). *For a subset A of a space X, the following hold*
(i) $A \subset \mathrm{pCl}(A) \subset \mathrm{pCl}_\theta(A)$ *and* $\mathrm{pInt}_\theta(A) \subset \mathrm{pInt}(A) \subset A$,
(ii) *If A is preopen (resp. preclosed),* $\mathrm{pCl}(A) = \mathrm{pCl}_\theta(A)$ *(resp.* $\mathrm{pInt}_\theta(A) = \mathrm{pInt}(A)$*).*

Example 2.4. Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Then it can be easily verified that for $A = \{b\}$, we obtain $\mathrm{pCl}_\theta(A) = X$, $\mathrm{pCl}(A) = \{b\}$. Therefore $\mathrm{pCl}_\theta(A) \not\subset \mathrm{pCl}(A)$, i.e., in general the converse of Theorem 2.3 (i) may not be true.

Theorem 2.5. *For an open (resp. closed) subset A of a space X,* $\mathrm{Cl}(A) = \mathrm{pCl}(A) = \mathrm{pCl}_\theta(A) = \mathrm{Cl}_\theta(A)$ *(resp.* $\mathrm{Int}_\theta(A) = \mathrm{pInt}_\theta(A) = \mathrm{pInt}(A) = \mathrm{Int}(A)$*).*

Theorem 2.6. [5]. *A space X is pre-regular if and only if* $\mathrm{pCl}(A) = \mathrm{pCl}_\theta(A)$ *for any subset A of X.*

Theorem 2.7. *Let A be a subset of a pre-regular space X.*

(i) *Every pre-closed subset A of X is pre-θ-closed.*

(ii) *$pCl_\theta(A)$ (resp. $pInt_\theta(A)$) is a pre-θ-closed (resp. pre-θ-open) set.*

Recall that a space X is said to be pre-Hausdorff [12], if for each pair of distinct points x and y of X, there exists a pair of disjoint preopen sets, one containing x and the other containing y.

Theorem 2.8. *A space X is pre-Hausdorff if and only if for each $x \in X$, the singleton $\{x\}$ is pre-θ-closed.*

Proof. *Necessity.* It is clear that $\{x\} \subset pCl_\theta(\{x\})$. Let $y \notin \{x\}$. Then $x \neq y$. Since X is pre-Hausdorff, there exist $U \in PO(X,x)$ and $V \in PO(X,y)$ such that $U \cap V = \emptyset$; hence $pCl(V) \cap U = \emptyset$. Thus, we have $pCl(V) \cap \{x\} = \emptyset$. Then $y \notin pCl_\theta(\{x\})$. Hence $pCl_\theta(\{x\}) \subset \{x\}$. Therefore $pCl_\theta(\{x\}) = \{x\}$, i.e., $\{x\}$ is pre-θ-closed.
Sufficiency. Let x and y be two distinct points of X. Then $y \notin \{x\} = \mathrm{pCl}_\theta(\{x\})$ and there exists $V \in \mathrm{PO}(X,y)$ such that $\mathrm{pCl}(V) \cap \{x\} = \emptyset$. Put $U = X \backslash \mathrm{pCl}(V)$. Then $U \in \mathrm{PO}(X,x)$ and $U \cap V = \emptyset$. Therefore X is pre-Hausdorff. □

Definition 2.9. Let A be a subset of a space X. A point $x \in X$ is called a pre-θ-limit point of A if for each pre-θ-open set U containing x, $U \cap (A \backslash \{x\}) \neq \emptyset$. The set of all pre-$\theta$-limit points of A is called the pre-θ-derived set of A and is denoted by $pD_\theta(A)$.

Theorem 2.10. *For subsets A, B of a space X, the following statements hold*

(i) $pD(A) \subset pD_\theta(A)$, *where $pD(A)$ is the pre-derived set of A,*
(ii) *If $A \subset B$, then $pD_\theta(A) \subset pD_\theta(B)$,*
(iii) $pD_\theta(A) \cup pD_\theta(B) = pD_\theta(A \cup B)$ *and* $pD_\theta(A \cap B) \subset pD_\theta(A) \cap pD_\theta(B)$,
(iv) $pD_\theta(pD_\theta(A)) \backslash A \subset pD_\theta(A)$,
(v) $pD_\theta(A \cup pD_\theta(A)) \subset A \cup pD_\theta(A) \subset pCl_\theta(A)$.

Proof. (i) It suffices to observe that every pre-θ-open set is preopen.
(iii) $pD_\theta(A \cup B) = pD_\theta(A) \cup pD_\theta(B)$ is a modification of the standard proof for D, where open sets are replaced by pre-θ-open sets.
(iv) If $x \in pD_\theta(pD_\theta(A)) \backslash A$ and U is a pre-θ-open set containing x, then $U \cap (pD_\theta(A) \backslash \{x\}) \neq \emptyset$. Let $y \in U \cap (pD_\theta(A) \backslash \{x\})$. Since $y \in pD_\theta(A)$ and $y \in U$, $U \cap (A \backslash \{y\}) \neq \emptyset$. Let $z \in U \cap (A \backslash \{y\})$. Then $z \neq x$ for $z \in A$ and $x \notin A$. Hence $U \cap (A \backslash \{x\}) \neq \emptyset$. Therefore $x \in pD_\theta(A)$.
(v) Let $x \in pD_\theta(A \cup pD_\theta(A))$. If $x \in A$, the result is obvious. So let $x \in pD_\theta(A \cup pD_\theta(A)) \backslash A$. Then for pre-$\theta$-open set U containing x, $U \cap (A \cup pD_\theta(A) \backslash \{x\}) \neq \emptyset$. Thus $U \cap (A \backslash \{x\}) \neq \emptyset$ or $U \cap (pD_\theta(A) \backslash \{x\}) \neq \emptyset$. Now it follows similarly from (4) that $U \cap (A \backslash \{x\}) \neq \emptyset$. Hence $x \in pD_\theta(A)$. Therefore, in any case $pD_\theta(A \cup pD_\theta(A)) \subset A \cup pD_\theta(A)$. Since $pD_\theta(A) \subset pCl_\theta(A)$, $A \cup pD_\theta(A) \subset pCl_\theta(A)$. □

Theorem 2.11. *For subsets A, B of a space X, the following statements are true*

(i) $pInt_\theta(pInt_\theta(A)) \subset pInt_\theta(A)$,
(ii) $A \subset B$, *then* $pInt_\theta(A) \subset pInt_\theta(B)$,
(iii) $pInt_\theta(A) \cup pInt_\theta(B) \subset pInt_\theta(A \cup B)$,
(iv) $pInt_\theta(A \cap B) \subset pInt_\theta(A) \cap pInt_\theta(B)$.

Definition 2.12. For a subset A of a space X, $pb_\theta(A) = A \backslash pInt_\theta(A)$ is called the pre-θ-border of A.

Theorem 2.13. *For a subset A of a space X, the following statements hold*

(i) $pb(A) \subset pb_\theta(A)$, *where $pb(A)$ denotes the pre-border of A*,
(ii) $A = pInt_\theta(A) \cup pb_\theta(A)$,
(iii) $pInt_\theta(A) \cap pb_\theta(A) = \emptyset$,
(iv) *A is a pre-θ-open set if and only if* $pb_\theta(A) = \emptyset$,
(v) $pInt_\theta(pb_\theta(A)) = \emptyset$,
(vi) $pb_\theta(pb_\theta(A)) = pb_\theta(A)$,
(vii) $pb_\theta(A) = A \cap pCl_\theta(X \backslash A)$.

Proof. (v) If $x \in \mathrm{pInt}_\theta(pb_\theta(A))$, then $x \in pb_\theta(A)$. On the other hand, since $pb_\theta(A) \subset A$, $x \in \mathrm{pInt}_\theta(pb_\theta(A)) \subset \mathrm{pInt}_\theta(A)$. Hence $x \in \mathrm{pInt}_\theta(A) \cap pb_\theta(A)$ which contradicts (iii). Thus $\mathrm{pInt}_\theta(pb_\theta(A)) = \emptyset$.
(vii) $pb_\theta(A) = A \backslash \mathrm{pInt}_\theta(A) = A \backslash (X \backslash \mathrm{pCl}_\theta(X \backslash A) = A \cap \mathrm{pCl}_\theta(X \backslash A)$. □

Example 2.14. Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Let $A = \{a, c\}$. Then $\mathrm{pCl}_\theta(X \backslash A) = \mathrm{pCl}_\theta(\{b\}) = X$, $\mathrm{pCl}(X \backslash A) = \mathrm{pCl}(\{b\}) = \{b\}$. Therefore we obtain $pb_\theta(A) \not\subset pb(A)$, i.e., in general the converse of Theorem 2.13(i) may not be true.

Definition 2.15. For a subset A of a space X, $pFr_\theta(A) = pCl_\theta(A) \backslash pInt_\theta(A)$ is called the pre-θ-frontier of A.

Theorem 2.16. *For a subset A of a space X, the following statements hold*

(i) $pFr(A) \subset pFr_\theta(A)$, *where $pFr(A)$ denotes the pre-frontier of A*,
(ii) $pCl_\theta(A) = pInt_\theta(A) \cup pFr_\theta(A)$,
(iii) $pInt_\theta(A) \cap pFr_\theta(A) = \emptyset$,

(iv) $pb_\theta(A) \subset pFr_\theta(A)$,
(v) $pFr_\theta(A) = pCl_\theta(A) \cap pCl_\theta(X\backslash A)$,
(vi) $pFr_\theta(A) = pFr_\theta(X\backslash A)$,
(vii) $pInt_\theta(A) = A\backslash pFr_\theta(A)$.

Proof. (ii) $pInt_\theta(A) \cup pFr_\theta(A) = pInt_\theta(A) \cup (pCl_\theta(A)\backslash pInt_\theta(A)) = pCl_\theta(A)$.
(iii) $pInt_\theta(A) \cap pFr_\theta(A) = pInt_\theta(A) \cap (pCl_\theta(A)\backslash pInt_\theta(A)) = \emptyset$.
(iv) $pFr_\theta(A) = pCl_\theta(A)\backslash pInt_\theta(A) = pCl_\theta(A) \cap pCl_\theta(X\backslash A)$.
(vii) $A\backslash pFr_\theta(A) = A\backslash(pCl_\theta(A)\backslash pInt_\theta(A)) = pInt_\theta(A)$. □

The converses of (i) and (iv) of the Theorem 2.16 are not true in general, as are shown by the following example.

Example 2.17. Consider the topological space (X, τ) given in Example 2.8. If $A = \{a, c\}$, then $pFr_\theta(A) = X \not\subset \{b\} = pFr(A)$ and also $pFr_\theta(A) = X \not\subset \{a, c\} = pb_\theta(A)$.

Definition 2.18. For a subset A of a space X, $pExt_\theta(A) = \mathrm{pInt}_\theta(X\backslash A)$ is called the pre-θ-exterior of A.

Theorem 2.19. *For a subset A of a space X, the following statements hold*

(i) $pExt_\theta(A) \subset pExt(A)$, *where* $pExt(A)$ *denotes the pre-exterior of* A,
(ii) $pExt_\theta(A) = pInt_\theta(X\backslash A) = X\backslash pCl_\theta(A)$,
(iii) $pExt_\theta(pExt_\theta(A)) = pInt_\theta(pCl_\theta(A))$,
(iv) *If* $A \subset B$, *then* $pExt_\theta(A) \supset pExt_\theta(B)$,
(v) $pExt_\theta(X) = \emptyset$,
(vi) $pExt_\theta(\emptyset) = X$,
(vii) $pExt_\theta(X\backslash pExt_\theta(A)) \subset pExt_\theta(A)$,
(viii) $pInt_\theta(A) \subset pExt_\theta(pExt_\theta(A))$.

Proof. (iii) $pExt_\theta(pExt_\theta(A)) = pExt_\theta(X\backslash \mathrm{pCl}_\theta(A)) = \mathrm{pInt}_\theta(X\backslash(X\backslash \mathrm{pCl}_\theta(A)))$
$= \mathrm{pInt}_\theta(\mathrm{pCl}_\theta(A))$.
(vii) $pExt_\theta(X\backslash pExt_\theta(A)) = pExt_\theta(X\backslash pInt_\theta(X\backslash A)) = \mathrm{pInt}_\theta(X\backslash(X\backslash \mathrm{pInt}_\theta(X\backslash A)))$
$= \mathrm{pInt}_\theta(\mathrm{pInt}_\theta(X\backslash A)) \subset \mathrm{pInt}_\theta(X\backslash A) = pExt_\theta(A)$.
(viii) $\mathrm{pInt}_\theta(A) \subset .\mathrm{pInt}_\theta(\mathrm{pCl}_\theta(A)) = \mathrm{pInt}_\theta(X\backslash \mathrm{pInt}_\theta(X\backslash A)))$
$= \mathrm{pInt}_\theta(X\backslash pExt_\theta(A)) = pExt_\theta(pExt_\theta(A))$. □

3. θ-preopen functions

In [18], Noiri defined a function $f : X \to Y$ to be θ-precontinuity if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a preopen set U of X containing x such that $f(pCl(U)) \subset Cl(V)$. He also showed that a function $f : X \to Y$ is θ-precontinuous if and only if $f^{-1}(V) \subset pInt_\theta(f^{-1}(Cl(V)))$ for every open set V of Y. We shall define θ-preopen functions as a natural by dual of θ-precontinuity.

Definition 3.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be

(i) θ-preopen if $f(U) \subset pInt_\theta(f(Cl(U)))$ for each open set U of X,
(ii) weakly preopen [6] if $f(U) \subset pInt(f(Cl(U)))$ for each open set U of X,
(iii) strongly θ-preopen if $f(U)$ is pre-θ-open in Y for each open set U of X.

Now we have the following diagram in which none of the implications reverses as shown the following examples.

$$\begin{array}{ccc} \textit{strongly } \theta\textit{-preopen} & \Rightarrow & \textit{preopen} \\ \Downarrow & & \Downarrow \\ \theta\textit{-preopen} & \Rightarrow & \textit{weakly preopen} \end{array}$$

Example 3.2. (i) A θ-preopen function need not be strongly θ-preopen. Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{a, b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then f is θ-preopen and preopen since $pInt_\theta(f(Cl(\{a\}))) = pInt_\theta(f(Cl(\{a, b\}))) = pInt_\theta(f(Cl(\{a, c\}))) = Y$ but f is not strongly θ-preopen since $f(\{a\}) \neq pInt_\theta(f(\{a\}))$

(ii) A weakly open function need not be θ-preopen. Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma = \{X, \emptyset, \{b\}, \{a, b\}, \{b, c\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is weakly open and hence weakly preopen since $Int(f(Cl(\{a\}))) = \{a, b\}$, $Int(f(Cl(\{c\}))) = \{b, c\}$, $Int(f(Cl(\{a, c\}))) = X$ but f is neither θ-preopen nor preopen since $f(\{a\}) \not\subset pInt_\theta(f(Cl(\{a\})))$.

(iii) A θ-preopen function need not be weakly open. Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{c\}, \{a, b\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{b, c\}\}$. The identity function $f : (X, \tau) \to (X, \sigma)$ is θ-preopen and preopen but not weakly open.

Since a topological space Y is submaximal if and only if $PO(Y,\sigma) = \sigma$ [22], then $f : (X,\tau) \to (Y,\sigma)$ is open whenever f is preopen and (Y,σ) a submaximal space. A function $f : (X,\tau) \to (Y,\sigma)$ is θ-preopen if f is weakly preopen and Y is pre-regular.

Example 3.2 shows that (1) θ-preopenness and weakly openness are independent notions and (2) preopenness and weak openness are independent of each other.

Theorem 3.3. *Let X be a regular space. A function $f : (X,\tau) \to (Y,\sigma)$ is θ-preopen if and only if f is strongly θ-preopen.*

Proof. The sufficiency is clear.
For the necessity, Let W be a nonempty open subset of X. For each x in W, let U_x be an open set such that $x \in U_x \subset Cl(U_x) \subset W$. Hence we obtain that $W = \cup\{U_x : x \in W\} = \cup\{Cl(U_x) : x \in W\}$ and, $f(W) = \cup\{f(U_x) : x \in W\} \subset \cup\{pInt_\theta(f(Cl(U_x))) : x \in W\} \subset pInt_\theta(f(\cup\{Cl(U_x) : x \in W\}) = pInt_\theta(f(W))$. Thus f is strongly θ-preopen. □

The following result gives several characterizations of θ-preopen functions.

Theorem 3.4. *For a function $f : (X,\tau) \to (Y,\sigma)$, the following conditions are equivalent.*

(i) *f is θ-preopen,*
(ii) *$f(Int_\theta(A)) \subset pInt_\theta(f(A))$ for every subset A of X,*
(iii) *$Int_\theta(f^{-1}(B)) \subset f^{-1}(pInt_\theta(B))$ for every subset B of Y,*
(iv) *$f^{-1}(pCl_\theta(B)) \subset Cl_\theta(f^{-1}(B))$ for every subset B of Y,*
(v) *$f(Int(F)) \subset pInt_\theta(f(F))$ for each closed subset F of X,*
(vi) *$f(Int(Cl(U))) \subset pInt_\theta(f(Cl(U)))$ for each open subset U of X,*
(vii) *$f(U) \subset pInt_\theta(f(Cl(U)))$ for every regular open subset U of X,*
(viii) *$f(U) \subset pInt_\theta(f(Cl(U)))$ for every α-open subset U of X.*

Proof. The proofs of (v) → (vi) → (vii) → (viii) → (i) are straightforward and are omitted.
(i) → (ii). Let A be any subset of X and $x \in \mathrm{Int}_\theta(A)$. Then , there exists an open set U such that $x \in U \subset \mathrm{Cl}(U) \subset A$. Hence $f(x) \in f(U) \subset$

$f(\mathrm{Cl}(U)) \subset f(A)$. Since f is θ-preopen, $f(U) \subset \mathrm{pInt}_\theta(f(\mathrm{Cl}(U))) \subset \mathrm{pInt}_\theta(f(A))$. It implies that $f(x) \in \mathrm{pInt}_\theta(f(A))$. Therefore $x \in f^{-1}(\mathrm{pInt}_\theta(f(A)))$. Thus $\mathrm{Int}_\theta(A) \subset f^{-1}(\mathrm{pInt}_\theta(f(A)))$, and so $f(\mathrm{Int}_\theta(A)) \subset \mathrm{pInt}_\theta(f(A))$.
(ii) $\rightarrow$ (iii). Let B be any subset of Y. Then by (ii),
$f(\mathrm{Int}_\theta(f^{-1}(B))) \subset \mathrm{pInt}_\theta(B)$. Therefore $\mathrm{Int}_\theta(f^{-1}(B)) \subset f^{-1}(\mathrm{pInt}_\theta(B))$.
(iii) $\rightarrow$ (iv). Let B be any subset of Y. Using (iii), we have $X - \mathrm{Cl}_\theta(f^{-1}(B)) = \mathrm{Int}_\theta(X - f^{-1}(B)) = \mathrm{Int}_\theta(f^{-1}(Y - B)) \subset f^{-1}(\mathrm{pInt}_\theta(Y - B)) = f^{-1}(Y - \mathrm{pCl}_\theta(B)) = X - (f^{-1}(\mathrm{pCl}_\theta(B))$. Therefore, we obtain $f^{-1}(\mathrm{pCl}_\theta(B)) \subset \mathrm{Cl}_\theta(f^{-1}(B))$.
(iv) $\rightarrow$ (v). Let F be any closed set of X. Then by (iv) $f^{-1}(\mathrm{pCl}_\theta(Y - f(F))) \subset \mathrm{Cl}_\theta(f^{-1}(Y - f(F)))$. We have $f^{-1}(\mathrm{pCl}_\theta(Y - f(F))) = f^{-1}(Y - \mathrm{pInt}_\theta(f(F))) = X - f^{-1}(\mathrm{pInt}_\theta(f(F)))$. On the other hand $\mathrm{Cl}_\theta(f^{-1}(Y - f(F))) = \mathrm{Cl}_\theta(X - f^{-1}(f(F))) \subset \mathrm{Cl}_\theta(X - F) = X - \mathrm{Int}_\theta(F) = X - \mathrm{Int}(F)$, since F is closed. Therefore, $\mathrm{Int}(F) \subset f^{-1}(\mathrm{pInt}_\theta(f(F)))$ and hence $f(\mathrm{Int}(F)) \subset \mathrm{pInt}_\theta(f(F))$. □

Theorem 3.5. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a bijective function. Then the following statements are equivalent.*

(i) *f is is θ-preopen.*
(ii) *$pCl_\theta(f(U)) \subset f(Cl(U))$ for each open set U in X.*
(iii) *$pCl_\theta(f(Int(F))) \subset f(F)$ for each closed set F in X.*

Proof. (i) $\rightarrow$ (iii). Let F be a closed set in X. Then we have $f(X - F) = Y - f(F) \subset \mathrm{pInt}_\theta(f(\mathrm{Cl}(X - F)))$ and so $Y - f(F) \subset Y - \mathrm{pCl}_\theta(f(\mathrm{Int}(F)))$. Hence $\mathrm{pCl}_\theta(f(\mathrm{Int}(F))) \subset f(F)$.
(iii) $\rightarrow$ (ii). Let U be an open set in X. Since $\mathrm{Cl}(U)$ is a closed set and $U \subset \mathrm{Int}(\mathrm{Cl}(U))$, by (iii) we have $\mathrm{pCl}_\theta(f(U)) \subset \mathrm{pCl}_\theta(f(\mathrm{Int}(\mathrm{Cl}(U))) \subset f(\mathrm{Cl}(U))$.
(ii) $\rightarrow$ (i). Let U be any open set of X. By (ii), we have $\mathrm{pCl}_\theta(f(X - \mathrm{Cl}(U))) \subset f(\mathrm{Cl}(X - \mathrm{Cl}(U)))$. Since f is bijective, $\mathrm{pCl}_\theta(f(X - \mathrm{Cl}(U))) = Y - \mathrm{pInt}_\theta(f(\mathrm{Cl}(U)))$ and $f(\mathrm{Cl}(X - \mathrm{Cl}(U))) = f(X - \mathrm{Int}(\mathrm{Cl}(U))) \subset f(X - U) = Y - f(U)$. Therefore, we obtain $f(U) \subset \mathrm{pInt}_\theta(f(\mathrm{Cl}(U)))$ and hence f is θ-preopen. □

Theorem 3.6. *Let X be a regular space. Then for a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following conditions are equivalent.*

(i) *f is θ-preopen.*
(ii) *For each θ-open set A in X, $f(A)$ is pre-θ-open in Y.*

(iii) *For any set B of Y and any θ-closed set A in X containing $f^{-1}(B)$. there exists a pre-θ-closed set F in Y containing B such that $f^{-1}(F) \subset A$.*

Proof. (i) $\to$ (ii) : Let A be a θ-open set in X. Since X is regular, by Theorem 3.3 f is strongly θ-preopen and A is open. Therefore $f(A)$ is pre- θ-open in Y.
(ii) $\to$ (iii) : Let B be any set in Y and A be a θ-closed set in X such that $f^{-1}(B) \subset A$. Since $X - A$ is θ-open in X, by (ii), $f(X - A)$ is pre-θ-open in Y. Let $F = Y - f(X - A)$. Then F is pre-θ-closed and $B \subset F$. Now, $f^{-1}(F) = f^{-1}(Y - f(X - A)) = X - f^{-1}(f(X - A)) \subset A$.
(iii) $\to$ (i) : Let B be any set in Y. Let $A = \mathrm{Cl}_\theta(f^{-1}(B))$. Since X is regular, A is a θ-closed set in X and $f^{-1}(B) \subset A$. Then there exists a pre-θ-closed set F in Y containing B such that $f^{-1}(F) \subset A$. Since F is pre-θ-closed $f^{-1}(\mathrm{pCl}_\theta(B)) \subset f^{-1}(F) \subset \mathrm{Cl}_\theta(f^{-1}(B))$. Therefore by Theorem 3.4, f is a θ-preopen function. □

Theorem 3.7. *If $f : (X, \tau) \to (Y, \sigma)$ is θ-preopen and strongly continuous, then f is strongly θ-preopen.*

Proof. Let U be an open subset of X. Since f is θ-preopen, $f(U) \subset \mathrm{pInt}_\theta(f(\mathrm{Cl}(U)))$. However, because f is strongly continuous, $f(U) \subset \mathrm{pInt}_\theta(f(U))$ and therefore $f(U)$ is pre-θ-open. Hence, f is strongly θ-preopen. □

Example 3.8. A strongly θ-preopen function need not be strongly continuous.
Let $X = \{a, b, c\}$, and let τ be the indiscrete topology for X. Then the identity function $f : (X, \tau) \to (X, \tau)$ is a strongly θ-preopen function which is not strongly continuous.

Theorem 3.9. *If $f : (X, \tau) \to (Y, \sigma)$ is closed and a.o.S., then f is a θ-preopen function.*

Proof. Let U be an open set in X. Since f is a.o.S. and $\mathrm{Int}(\mathrm{Cl}(U))$ is regular open, $f(\mathrm{Int}(\mathrm{Cl}(U)))$ is open in Y and hence $f(U) \subset f(\mathrm{Int}(\mathrm{Cl}(U))) \subset \mathrm{Int}(f(\mathrm{Cl}(U)))$. Since f is closed, $f(U) \subset \mathrm{pInt}_\theta(f(\mathrm{Cl}(U)))$ by Theorem 2.5. This shows that f is θ-preopen. □

The converse of Theorem 3.9 is not true in general.

Example 3.10. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{b, c\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is θ-preopen but it is not a.o.S., since $\{c\}$ is a regular open set in (X, τ) and $f(\{c\}) = \{c\}$ is not open in (X, σ). It is easy to verify that f is not a closed function.

Lemma 3.11. [14] *If $f : (X, \tau) \to (Y, \sigma)$ is a precontinuous function, then for any open set U of X , $f(Cl(U)) \subset Cl(f(U))$.*

Theorem 3.12. *If $f : (X, \tau) \to (Y, \sigma)$ is a weakly preopen and precontinuous function, then f is preopen.*

Proof. Let U be an open set in X. Then by weak preopenness of f, $f(U) \subset pInt(f(Cl(U)))$. Since f is precontinuous, $f(Cl(U)) \subset Cl(f(U))$. Hence we obtain that $f(U) \subset pInt(f(Cl(U))) \subset pInt(Cl(f(U))) \subset Int(Cl(f(U)))$. Therefore, $f(U) \subset Int(Cl(f(U)))$ which shows that $f(U)$ is a preopen set in Y. Thus, f is a preopen function. □

Definition 3.13. A space X is said to be hyperconnected [16] if every nonempty open subset of X is dense in X.

Theorem 3.14. *Let X be a hyperconnected space, then $f : (X, \tau) \to (Y, \sigma)$ is θ-preopen if and only if $f(X)$ is pre-θ-open in Y.*

Proof. The sufficiency is clear. For the necessity observe that for any open subset U of $X, f(U) \subset f(X) = pInt_\theta(f(X)) = pInt_\theta(f(Cl(U)))$. □

4. θ-preclosed functions

Definition 4.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be

(i) θ-preclosed if $pCl_\theta(f(Int(F))) \subset f(F)$ for each closed set F in X,
(ii) strongly θ-preclosed if every closed set F of X, $f(X)$ is pre-θ-closed in Y.

Clearly, every strongly θ-preclosed function is θ-preclosed, since $\mathrm{pCl}_\theta(f(\mathrm{Int}(A))) \subset \mathrm{pCl}_\theta(f(A)) = f(A)$ for every closed subset A of X. But this is not true conversely.

Example 4.2. Let $f : (X,\tau) \to (Y,\sigma)$ be the function defined in Example 3.2(i) (resp.Example 3.2(ii)). Then it is shown that f is a θ-preclosed function which is not strongly θ-preclosed (resp. weakly closed function need not be θ-preclosed).

Theorem 4.3. *For a function $f : (X,\tau) \to (Y,\sigma)$, the following conditions are equivalent*

(i) *f is θ-preclosed.*
(ii) *$pCl_\theta(f(U)) \subset f(Cl(U))$ for every open subset U of X.*
(iii) *$pCl_\theta(f(U)) \subset f(Cl(U))$ for each preopen subset U of X.*
(iv) *$pCl_\theta(f(Int(F))) \subset f(F)$ for each preclosed subset F of X.*
(v) *$pCl_\theta(f(Int(F))) \subset f(F)$ for every α-closed subset F of X.*
(vi) *$pCl_\theta(f(Int(Cl(U)))) \subset f(Cl(U))$ for each subset U of X.*
(vii) *$pCl_\theta(f(U)) \subset f(Cl(U))$ for each preopen subset U of X.*

Proof. (i) $\to$ (ii). Let U be any open subset of X. Then $\mathrm{pCl}_\theta(f(U)) = \mathrm{pCl}_\theta(f(\mathrm{Int}(U))) \subset \mathrm{pCl}_\theta(f(\mathrm{Int}(\mathrm{Cl}(U)))) \subset f(\mathrm{Cl}(U))$.
(ii) $\to$ (iii). Let U be any preopen set of X. Then $\mathrm{pCl}_\theta(f(U)) \subset \mathrm{pCl}_\theta(f(\mathrm{Int}(\mathrm{Cl}(U)))) \subset f(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(U)))) \subset f(\mathrm{Cl}(U))$.
(iii) $\to$ (iv). Let F be any preclosed set of X. Then, we have $\mathrm{pCl}_\theta(f(\mathrm{Int}(F))) \subset f(\mathrm{Cl}(\mathrm{Int}(F))) \subset f(F)$.
It is clear that (iv) $\to$ (v) $\to$ (vi) $\to$ (vii) $\to$ (i). □

Theorem 4.4. *Let Y be a pre-regular space. Then for a function $f : (X,\tau) \to (Y,\sigma)$, the following conditions are equivalent.*

(i) *f is θ-preclosed.*
(ii) *$pCl_\theta(f(U)) \subset f(Cl(U))$ for each regular open subset U of X.*
(iii) *For each subset F in Y and each open set U in X with $f^{-1}(F) \subset U$, there exists a pre-θ-open set A in Y with $F \subset A$ and $f^{-1}(A) \subset Cl(U)$.*
(iv) *For each point y in Y and each open set U in X with $f^{-1}(y) \subset U$, there exists a pre-θ-open set A in Y containing y and $f^{-1}(A) \subset Cl(U)$.*

Proof. It is clear that (i) $\to$ (ii) and (iii) $\to$ (iv).
(ii) $\to$ (iii). Let F be a subset of Y and U an open set in X with $f^{-1}(F) \subset U$. Then $f^{-1}(F) \cap \mathrm{Cl}(X - \mathrm{Cl}(U)) = \phi$ and consequently, $F \cap f(\mathrm{Cl}(X - \mathrm{Cl}(U))) = \phi$. Since $X - \mathrm{Cl}(U)$ is regular open, $F \cap \mathrm{pCl}_\theta(f(X - \mathrm{Cl}(U))) = \phi$ by (ii). Let $A = Y - \mathrm{pCl}_\theta(f(X - \mathrm{Cl}(U)))$. Then

A is a pre-θ-open set with $F \subset A$ and $f^{-1}(A) \subset X - f^{-1}(\mathrm{pCl}_\theta(f(X - \mathrm{Cl}(U)))) \subset X - f^{-1}f(X - \mathrm{Cl}(U)) \subset \mathrm{Cl}(U)$.
(iv) $\rightarrow$ (i). Let F be closed in X and let $y \in Y - f(F)$. Since $f^{-1}(y) \subset X - F$, there exists a pre-θ-open set A in Y with $y \in A$ and $f^{-1}(A) \subset \mathrm{Cl}(X - F) = X - \mathrm{Int}(F)$ by (iv). Therefore $A \cap f(\mathrm{Int}(F)) = \phi$, so that $y \in Y - \mathrm{pCl}_\theta(f(\mathrm{Int}(F)))$. Thus $\mathrm{pCl}_\theta(f(\mathrm{Int}(F))) \subset f(F)$.

Theorem 4.5. *A bijection $f : (X, \tau) \rightarrow (Y, \sigma)$ is θ-preopen if and only if f is θ-preclosed.*

Proof. This is an immediate consequence of Theorem 3.5.

Next we investigate conditions under which θ-preclosed functions are strongly θ-preclosed.

Theorem 4.6. (i) *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is preclosed and contra-closed, then f is strongly θ-preclosed and closed.*
(ii) *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra θ-preopen, then f is θ-preclosed.*

Proof. (i) Let F be a closed subset of X. Since f is preclosed, $pCl_\theta(Int(f(F))) = Cl(Int(f(F))) \subset f(F)$ and since f is contra-closed, $f(F)$ is open. Therefore by Theorem 2.5 $Cl(f(F)) = pCl_\theta(f(F)) = pCl_\theta(Int(f(F))) \subset f(F)$ and hence $f(F)$ is pre-θ-closed and closed in Y. Therefore, f is strongly θ-preclosed and closed.

(ii) Let F be a closed subset of X. Then, $pCl_\theta(f(Int(F))) = f(Int(F)) \subset f(F)$.

Example 4.7. Example 3.2(i) shows that θ-preclosedness does not imply contra θ-preopenness.

Example 4.8. Contra-closedness and θ-preclosedness are independent notions. Example 3.2(i) shows that θ-preclosedness does not imply contra-closedness while the reverse is shown in the Example 3.2(ii).

Theorem 4.9. *If Y is a pre-regular space and if $f : (X, \tau) \rightarrow (Y, \sigma)$ is one-to-one and θ-preclosed, then for every subset F of Y and every open set U in X with $f^{-1}(F) \subset U$, there exists a pre-θ-closed set B in Y such that $F \subset B$ and $f^{-1}(B) \subset Cl(U)$.*

Proof. Let F be a subset of Y and U an open subset of X with $f^{-1}(F) \subset U$. Put $B = pCl_\theta(f(Int(Cl(U))))$, then by Theorem 2.7 B is a pre-θ-closed subset of Y such that $F \subset B$, since $F \subset f(U) \subset f(Int(Cl(U))) \subset pCl_\theta(f(Int(Cl(U)))) = B$. Now since f is θ-preclosed, $f^{-1}(B) \subset Cl(U)$.

Taking the set F in Theorem 4.9 to be $\{y\}$ for $y \in Y$ we obtain the following result.

Corollary 4.10. *If Y is a pre-regular space and if $f : (X, \tau) \to (Y, \sigma)$ is one-to-one and θ-preclosed, then for every point y in Y and every open set U in X with $f^{-1}(y) \subset U$, there exists a pre-θ-closed set B in Y containing y such that $f^{-1}(B) \subset \mathrm{Cl}(U)$.*

Recall that a set F in X is θ-compact [24] if for each cover Ω of F by open U in X, there is a finite family $U_1, ..., U_n$ in Ω such that $F \subset \mathrm{Int}(\bigcup\{\mathrm{Cl}(U_i) : i = 1, 2, ..., n\})$.

Theorem 4.11. *Let (Y, σ) be a pre-regular space. If $f : (X, \tau) \to (Y, \sigma)$ is a θ-preclosed function with θ-closed fibers, then $f(F)$ is pre-θ-closed for each θ-compact F in X.*

Proof. Let F be θ-compact and $y \in Y - f(F)$. Then $f^{-1}(y) \cap F = \phi$ and for each $x \in F$ there is an open $U_x \subset X$ with $x \in U_x$ such that $Cl(U_x) \cap f^{-1}(y) = \phi$. Clearly, $\Omega = \{U_x : x \in F\}$ is an open cover of F and since F is θ-compact, there is a finite family $\{U_{x_1}, ..., U_{x_n}\} \subset \Omega$ such that $F \subset \mathrm{Int}(A)$, where $A = \bigcup\{Cl(U_{x_i}) : i = 1, ..., n\}$. Since f is θ-preclosed, by Theorem 4.4 there exists a pre-θ-open $B \subset Y$ with $f^{-1}(y) \subset f^{-1}(B) \subset Cl(X - A) = X - Int(A) \subset X - F$. Therefore $y \in B$ and $B \cap f(F) = \phi$. By Theorem 2.1 (i), there exists a preopen set W with $y \in W$ such that $pCl(W) \subset B$. Therefore $pCl(W) \cap f(F) = \phi$. Thus $y \in Y - pCl_\theta(f(F))$. This shows that $f(F)$ is pre-θ-closed.

Two nonempty subsets A and B in X are said to be strongly separated [24] if there exist open sets U and V in X with $A \subset U$ and $B \subset V$ and $Cl(U) \cap Cl(V) = \phi$. If A and B are singletons we may speak of points being strongly separated. We will use the fact that in a normal space, disjoint closed sets are strongly separated.

A topological space (X, τ) is said to be θ-pre-T_2 if for $x, y \in X$ with $x \neq y$ there exist disjoint pre-θ-open sets U and V such that $x \in U$ and $y \in V$.

Theorem 4.12. *Let (Y, σ) be a pre-regular space. If $f : (X, \tau) \to (Y, \sigma)$ is a θ-preclosed surjection and all pairs of disjoint fibers are strongly separated, then Y is θ-pre-T_2 (hence pre-T_2).*

Proof. Let y and z be two points in Y. Let U and V be open sets in X such that $f^{-1}(y) \in U$ and $f^{-1}(z) \in V$ with $Cl(U) \cap Cl(V) = \phi$. By θ-preclosedness (Theorem 4.4) there are pre-θ-open sets F and B in Y such that $y \in F$ and $z \in B$, $f^{-1}(F) \subset Cl(U)$ and $f^{-1}(B) \subset Cl(V)$. Therefore $F \cap B = \phi$, because $Cl(U) \cap Cl(V) = \phi$ and f is surjective. Then Y is θ-pre-T_2.

Corollary 4.13. *If Y is a pre-regular space and if $f : (X, \tau) \to (Y, \sigma)$ is a θ-preclosed surjection with closed fibers and X is normal, then Y is θ-pre-T_2 (hence pre-T_2).*

Definition 4.14. A subset S of a topological space (X, τ) is said to be *quasi H-closed relative to* (X, τ) [20] (resp. *p-closed relative to* (X, τ) [9]) if for every cover $\{U_\alpha \mid \alpha \in A\}$ of S by open (resp. preopen) sets of X, there exists a finite subset A_0 of A such that $S \subset \cup\{Cl(U_\alpha) \mid \alpha \in A_0\}$ (resp. $S \subset \cup\{pCl(U_\alpha) \mid \alpha \in A_0\}$). A topological space (X, τ) is said to be *quasi H-closed* [7] (resp. *p-closed*) if the subset X is quasi H-closed relative to (X, τ) (resp. p-closed relative to (X, τ)).

Every p-closed space is quasi H-closed since $pCl(U) = Cl(U)$ for every open set U. It is shown in Theorem 2.5 of [9] that a T_0 space X is p-closed if and only if it is quasi H-closed and strongly irresolvable. It is also shown in Theorem 5.3 of [18] that if $f : X \to Y$ is a θ-precontinuous function and K is p-closed relative to X then $f(K)$ is quasi H-closed relative to Y.

Theorem 4.15. *Let Y be a pre-regular space. If $f : X \to Y$ is a θ-preclosed surjection with compact point inverses and K is p-closed relative to Y, then $f^{-1}(K)$ is quasi H-closed relative to X.*

Proof. Let $\{U_\alpha \mid \alpha \in A\}$ be any cover of $f^{-1}(K)$ by open sets of X. For each $y \in K$, $f^{-1}(y)$ is compact and $f^{-1}(y) \subset \bigcup_{\alpha \in A} U_\alpha$. There exists a finite subset $A(y)$ of A such that $f^{-1}(y) \subset \bigcup_{\alpha \in A(y)} U_\alpha$. Put $U(y) = \bigcup_{\alpha \in A(y)} U_\alpha$. Since f is θ-preclosed, by Theorem 4.4 there exists a pre-θ-open set $V(y)$ containing y such that $f^{-1}(V(y)) \subset Cl(U(y))$. Since $V(y)$ is pre-θ-open, there exists a preopen set $V_0(y)$ such that $y \in V_0(y) \subset pCl(V_0(y)) \subset V(y)$. Since the family $\{V_0(y) \mid y \in K\}$ is a cover of K by preopen sets of Y, there exist a finite number of points, say, y_1, $y_2, \ldots, y_n$ of K such that $K \subset \bigcup_{i=1}^n pCl(V_0(y_i))$; hence $K \subset \bigcup_{i=1}^n V(y_i)$. Therefore, we obtain $f^{-1}(K) \subset \bigcup_{i=1}^n f^{-1}(V(y_i)) \subset \bigcup_{i=1}^n Cl(U(y_i)) = \bigcup_{i=1}^n \bigcup_{\alpha \in A(y_i)} Cl(U_\alpha)$. This shows that $f^{-1}(K)$ is quasi H-closed relative to X.

Corollary 4.16. *Let Y be a pre-regular space and $f : X \to Y$ a θ-preclosed surjection with compact point inverses. If Y is a p-closed space, then X is quasi H-closed.*

Acknowledgment

This work was initiated during the second visit of the first author to the Institute of Mathematics of the Universidad Autónoma de México under the TWAS-UNESCO.

References

1. M. E Abd-El-Monsef, S. N. El-Deeb and R. A. Mahmoud, β-open and β-continuous mappings, *Bull. Fac. Sci. Assiut Univ.* **12** (1983), 70-90.
2. D. Andrijević, Semi-preopen sets, *Mat. Vesnik* **38** (1986), 24-32 .
3. S. P. Arya and R. Gupta, Strongly continuous mappings, *Kyungpook Math. J.* **14** (1974), 131-143.
4. C. W. Baker, Contra-open functions and contra-closed functions, *Math. Today* **15** (1997), 19-24.
5. C. W. Baker, Weakly θ-precontinuous functions, *Acta Math. Hungar.* **100** (2003), 343-351.
6. M. Caldas and G. Navalagi, On Weak forms of preopen and preclosed functions, *Archiv. Math.* **40** (2004), 119-128.
7. D. E. Cameron, Some maximal topologies which are *Q.H.C.*, *Proc. Amer. Math. Soc.* **75** (1979), 149-156.
8. S. H. Cho, A note on strongly θ-precontinuous functions, *Acta Math. Hungar.* **101** (2003), 173-178.
9. J. Dontchev, M. Ganster and T. Noiri, On p-closed spaces, *Internat. J. Math. Math. Sci.* **24** (2000), 203-212.
10. J. N. El-Deeb, I. A. Hasanein, A. S. Mashhour and T. Noiri, On p-regular spaces, *Bull. Math. Soc. Sci. Math. R. S. Roum.* **27** (1983), 311-315.
11. T. Husain, Almost continuous mappings, *Prace Math.* **10** (2003), 173-178.

12. A. Kar and P. Bhattacharyya, Some weak separation axioms, *Bull. Calcutta Math. Soc.* **82** (1990), 415-422.
13. N. Levine, Strong continuity in topological spaces, *Amer. Math. Monthly* **67** (1960), 269.
14. A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, Precontinuous and weak precontinuous mappings, *Proc. Math. Phys. Soc. Egypt.* **53** (1982), 47-53.
15. O. Njåstad, On some classes of nearly open sets, *Pacific J. Math.* **15** (1965), 961-970.
16. T. Noiri, Properties of hyperconnected spaces , *Acta Math. Hungar.* **66** (1995), 147-154.
17. T. Noiri, Strongly θ-precontinuous functions, *Acta Math. Hungar.* **90** (2001), 307-316.
18. T. Noiri, On θ-precontinuous functions, *Internat. J. Math. Math. Sci.* **28** (2001), 285-292.
19. M. C. Pal and P. Bhattacharyya, Feeble and strong forms of preirresolute functions, *Bull. Malaysian Math. Soc.* **19** (1996), 63-75.
20. J. Porter and J. Thomas, On H-closed and minimal Hausdorff spaces, *Trans. Amer. Math. Soc.* **138** (1969), 159-170.
21. V. Pták, Completeness and the open mapping Theorem, *Bull. Soc. Math. France* **86** (1958), 41-78.
22. I. L. Reilly and M. K. Vamanamurthy, On some questions concerning preopen sets, *Kyungpook Math. J.* **30** (1990), 87-93.
23. D. A. Rose, On weak openness and almost openness, *Internat. J. Math. Math. Sci.* **7** (1984), 35-40.
24. D. A. Rose and D. S. Janković,. Weakly closed functions and Hausdorff spaces, *Math. Nachr.* **130** (1987),105-110.
25. M. K. Singal and A. R. Singal, Almost continuous mappings, *Yokohama Math. J.* **16** (1968), 63-73.
26. N. V. Veličko, H-closed topological spaces, *Amer. Math. Soc. Transl.* **78** (1968), 103-118.

M. Caldas
Departamento de Matematica Aplicada
Universidade Federal Fluminense
Rua Mario Santos Braga ,s/n,
CEP: 24020-140,
Niteroi , RJ Brasil
e-mail: gmamccs@vm.uff.br

S. Jafari
College of Vestsjaelland South
Herrestraede 11,
4200 Slagelse, Denmark
e-mail: jafari@stofanet.dk

G. B. Navalagi
Department of Mathematics
KLE Societys
G.H.College, Haveri-581110,
Karnataka, India
e-mail: gnavalagi@hotmail.com

T. Noiri
2949-1 Shiokita-cho, Hinagu,
Yatsushiro-shi, Kumamoto-ken,
869-5142 Japan
e-mail: t.noiri@nifty.com