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MULTIPLICITY RESULT TO SOME KIRCHHOFF-TYPE BIHARMONIC EQUATION INVOLVING EXPONENTIAL GROWTH CONDITIONS

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Abstract. In this paper, we prove a multiplicity result for some biharmonic elliptic equation of Kirchhoff type and involving nonlinearities with critical exponential growth at infinity. Using some variational arguments and exploiting the symmetries of the problem, we establish a multiplicity result giving two nontrivial solutions.
Key words: Biharmonic equation, Kirchhoff-type, radial solution, nonradial solution, Adams inequality.
MSC(2010):Primary:35J35; Secondary: 35A15, 35D30, 35J62.

1. Introduction and statement of main results

In the present work, we consider the equation

$$(P) \quad \left(a+b\int_{\mathbb{R}^4} |\Delta u|^2\,dx\right)\Delta^2 u + V(x)u = f(x,u)+h(x), \text{ in } \mathbb{R}^4,$$

where $a>0$ and $b>0$. We assume

(H_1) $V:\mathbb{R}^4\to[0,+\infty[$ is some continuous function such that

$$V_0=\inf_{x\in\mathbb{R}^4} V(x)>0.$$

Moreover, V is spherically symmetric (radial), that is

$$\forall\ x,y\in\mathbb{R}^4,\ |x|=|y|\Rightarrow V(x)=V(y).$$

(H_2) $f:\mathbb{R}^4\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function which is spherically symmetric with respect to $x\in\mathbb{R}^4$, that is

$$\forall\ (x,y,s)\in\mathbb{R}^4\times\mathbb{R}^4\times\mathbb{R},\ |x|=|y|\Rightarrow f(x,s)=f(y,s).$$

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(H_3) There exist $\alpha > 1,\ \beta > 1,\ p > 0$ and $C_0 > 0$ such that

$$|f(x,s)| \leq C_0 \left(|s|^{\alpha} + |s|^{\beta} \left(e^{ps^2} - 1 \right) \right),\ \text{a.e } x \in \mathbb{R}^4,\ \forall\ s \in \mathbb{R}.$$

(H_4) There exist $A > 0$ and $q > 4$ such that

$$F(x,s) = \int_0^s f(x,t)dt \geq A\,|s|^q\,,\ \text{a.e } x \in \mathbb{R}^4,\ \forall\ s \in \mathbb{R}.$$

(H_5) There exists $\nu > 4$ such that

$$\nu F(x,s) \leq f(x,s)s,\ \text{a.e } x \in \mathbb{R}^4,\ \forall\ s \in \mathbb{R}.$$

(H_6) $h : \mathbb{R}^4 \to \mathbb{R}$ is spherically symmetric such that $h \in L^2(\mathbb{R}^4)$.

Higher order nonlinear equations and especially those involving a biharmonic operator arise in many physical applications such as deformations of an elastic beam in equilibrium state, travelling waves in suspension bridges, thin film theory, surface diffusion on solids, interface dynamics, and phase field models of multiphase systems. The interested reader can be referred to [7, 11, 14, 22] and references therein. On the other hand, nowadays, it is becoming clear and even obvious the importance of studying Kirchhoff-type equations (and in general nonlinear equations containing nonlocal terms) in view of their various possibilities to modelize several physical and biological phenomena. Among these equations, a special attention has recently been given to fourth-order ones. We can, for example, cite [4, 20, 21, 24]. In these cited references and others, the authors are mainly concerned with equation of the form

$$\Delta^2 u - M\left(\int_{\Omega} |\Delta u|^2\,dx \right) \Delta u = g(x,u),\ x \in \Omega,$$

where Ω is some subset of $\mathbb{R}^N,\ N \geq 3$, and g is some Carathéodory function whose growth at infinity is controlled by some polynomial. A special case with great interest is when $M(s) = a + bs$, with a and b two positive constants. In our case we study the situation when the nonlinearities enjoy a critical exponential growth at infinity. This kind of problems has known a very great interest in last few decades but the number of papers dealing with Kirchhoff type equation involving this kind of growth condition is very limited. We can quote [1–3,8,12]. All these works investigated existence and multiplicity of solutions to some second-order equations and they are governed by the well known Trudinger-Moser inequality. Concerning higher order, up to the best knowledge of the author, the present paper is the first attempt to study Kirchhoff-type equation involving exponential growth condition. Another aspect of novelty in this article is that we prove the existence of at least two nontrivial solutions for both cases $h \neq 0$ and $h = 0$. We look for solutions to the problem (P) in the Hilbert space

$$E = \left\{ u \in H^2(\mathbb{R}^4),\ \int_{\mathbb{R}^4} V(x)u^2 dx < +\infty \right\}$$

equipped with the norm

$$\|u\| = \left(\int_{\mathbb{R}^4} \left(|\Delta u|^2 + V(x)u^2 \right) dx \right)^{\frac{1}{2}}.$$

The following results concerning the space $H^2(\mathbb{R}^4)$ are needed. See [16–18] for proofs and more details. For $u \in H^2(\mathbb{R}^4)$, we denote

$$\|u\|_{H^2(\mathbb{R}^4)} = |(-\Delta + I)u|_2 = \left(|\Delta u|_2^2 + 2\,|\nabla u|_2^2 + |u|_2^2 \right)^{\frac{1}{2}},$$

where $|\cdot|_2$ denotes the norm in $L^2(\mathbb{R}^4)$, i.e.

$$|v|_2 = \left(\int_{\mathbb{R}^4} v^2 dx \right)^{\frac{1}{2}},\ v \in L^2(\mathbb{R}^4).$$

Here, we state the Adams inequality for the whole space $\mathbb{R}^4$,

$$\sup_{u \in S} \int_{\mathbb{R}^4} \left(e^{\alpha u^2} - 1 \right) dx \begin{cases} < +\infty & \text{if} \quad \alpha \leq 32\pi^2, \\ = +\infty & \text{if} \quad \alpha > 32\pi^2, \end{cases}$$

where $S = \left\{ u \in H^2(\mathbb{R}^4),\ \|u\|_{H^2(\mathbb{R}^4)} \leq 1 \right\}$. Moreover, if $\alpha > 0,\ q \geq 2$ and $M > 0$ such that $\alpha M^2 < 32\pi^2$, then there exists a constant $C = C(\alpha, q, M) > 0$ such that

$$\int_{\mathbb{R}^4} \left(e^{\alpha u^2} - 1 \right) |u|^q \, dx \leq C \,\|u\|_{H^2(\mathbb{R}^4)}^q,\ \forall\ u \in H^2(\mathbb{R}^4),\ \|u\|_{H^2(\mathbb{R}^4)} \leq M.$$

Now, by (H_1), there exists a constant $\chi_0 > 0$ such that

$$\|u\|_{H^2(\mathbb{R}^4)} \leq \chi_0 \,\|u\|,\ \forall\ u \in H^2(\mathbb{R}^4).$$

It follows that there exists a positive constant $C' = C'(\alpha, q, M) > 0$ such that

$$\int_{\mathbb{R}^4} \left(e^{\alpha u^2} - 1 \right) |u|^q \, dx \leq C' \,\|u\|^q, \tag{1.1}$$

provided that $\|u\| \leq M < \frac{1}{\chi_0} \left(\frac{32\pi^2}{\alpha} \right)^{\frac{1}{2}}$.

Definition 1.1. A function $u \in E$ is said to be a weak solution of the problem (P) if it satisfies

$$\begin{aligned} &a \int_{\mathbb{R}^4} \Delta u \Delta v dx + b \left(\int_{\mathbb{R}^4} |\Delta u|^2 \, dx \right) \int_{\mathbb{R}^4} \Delta u \Delta v dx + \int_{\mathbb{R}^4} V(x) u v dx \\ &= \int_{\mathbb{R}^4} f(x,u) v dx + \int_{\mathbb{R}^4} h v dx,\ \forall\ v \in E. \end{aligned}$$

The main result of this paper is given by the following theorem.

Theorem 1.2. *Assume that* $(H_1)-(H_6)$ *hold true. Then, there exist* $A_0>0$ *and* $h_0>0$ *such that the problem* (P) *admits at least two nontrivial weak solutions provided that* $A>A_0$ *and* $0<|h|_2<h_0$. *Moreover, if* $h=0$ *and* $F(x,s)$ *is even with respect to* $s\in\mathbb{R}$, *then* (P) *has also at least two nontrivial weak solutions.*

2. **Proof of Theorem 1.2**

We will proceed by steps. First, we introduce the energy functional which corresponds to (P),

$$I(u)=\frac{a}{2}\int_{\mathbb{R}^4}|\Delta u|^2\,dx+\frac{b}{4}\left(\int_{\mathbb{R}^4}|\Delta u|^2\,dx\right)^2+\frac{1}{2}\int_{\mathbb{R}^4}V(x)u^2dx$$
$$-\int_{\mathbb{R}^4}F(x,u)dx-\int_{\mathbb{R}^4}hudx,\ u\in E.$$

Lemma 2.1. *Assume that* (H_1), (H_3) *and* (H_6) *hold true. Then, there exist* $\mu>0,\ \rho>0$ *and* $h_0>0$ *such that*

$$I(u)\geq\mu,\ \forall\ u\in E,\ \|u\|=\rho,\ \textit{provided that}\ \ 0\leq|h|_2<h_0.$$

Proof. By (H_3), it yields

$$\int_{\mathbb{R}^4}F(x,u)dx\leq c_1\left(\int_{\mathbb{R}^4}|u|^{\alpha+1}\,dx+\int_{\mathbb{R}^4}|u|^{\beta+1}\left(e^{pu^2}-1\right)dx\right).$$

Then, if $\|u\|\leq\inf\left(1,\ \frac{1}{2\chi_0}\left(\frac{32\pi^2}{p}\right)^{\frac{1}{2}}\right)$, by (1.1) we obtain

$$\int_{\mathbb{R}^4}F(x,u)dx\leq c_1\left(\|u\|^{\alpha+1}+\|u\|^{\beta+1}\right)\leq(2c_1)\,\|u\|^{1+\inf(\alpha,\beta)}.$$

Thus,

$$I(u)\geq\frac{\inf(1,a)}{2}\|u\|^2-(2c_1)\,\|u\|^{1+\inf(\alpha,\beta)}-|h|_2\,\|u\|,$$

for $\|u\|\leq\inf\left(1,\ \frac{1}{2\chi_0}\left(\frac{32\pi^2}{p}\right)^{\frac{1}{2}}\right)$. Since $\inf(\alpha,\beta)>1$, one can find $\rho>0$ small enough such that $\rho<\inf\left(1,\ \frac{1}{2\chi_0}\left(\frac{32\pi^2}{p}\right)^{\frac{1}{2}}\right)$ and $\frac{\inf(1,a)}{2}>2c_1\rho^{\inf(\alpha,\beta)-1}$. Hence, Lemma 2.1 can be concluded by taking $h_0=\frac{\inf(1,a)}{2}\rho-2c_1\rho^{\inf(\alpha,\beta)}$ and $\mu=\frac{\inf(1,a)}{2}\rho^2-2c_1\rho^{1+\inf(\alpha,\beta)}-h_0\rho$. □

Lemma 2.2. *Assume that* (H_1), (H_2), (H_3) *and* (H_6) *hold true. Then, the problem* (P) *admits a nontrivial radial weak solution* U_1 *such that* $I(U_1)<0$ *provided that* $0<|h|_2<h_0$.

Proof. Denote by $I_r = I|_{E_r}$ the restriction of I on the subspace E_r consisting of all radial functions in E. Since $h \neq 0$, there is $\varphi \in E_r$ such that $\varphi \neq 0$ and $\int_{\mathbb{R}^4} h\varphi dx > 0$. Let $0 < t < 1$. We have

$$\begin{aligned}\frac{d}{dt} I_r(t\varphi) = at \int_{\mathbb{R}^4} |\Delta\varphi|^2 dx + bt^3 \left(\int_{\mathbb{R}^4} |\Delta\varphi|^2 dx\right)^2 + t\int_{\mathbb{R}^4} V(x)\varphi^2 dx \\ - \int_{\mathbb{R}^4} \varphi f(x, t\varphi) dx - \int_{\mathbb{R}^4} h\varphi dx.\end{aligned}$$

By the Lebesgue dominated convergence Theorem, it yields

$$\lim_{t\to 0^+} \int_{\mathbb{R}^4} \varphi f(x, t\varphi) dx = 0.$$

Hence, one can find $0 < t_0 < 1$ small enough such that $\frac{d}{dt} I_r(t\varphi) < 0,\ \forall\ 0 < t < t_0$. Since $I_r(0) = 0$, it must exists $0 < t_1 < \inf\left(t_0,\ \frac{\rho}{\|\varphi\|}\right)$ (where ρ is given by Lemma 2.1) such that $I_r(t_1\varphi) < 0$. From the fact that $\|t_1\varphi\| < \rho$, we infer

$$d_\rho = \inf\{I_r(u),\ u \in E_r,\ \|u\| \leq \rho\} \leq I_r(t_1\varphi) < 0.$$

Now, by the virtue of the Ekeland's variational principle (see [6]), there exists a sequence $(u_n) \subset E_r,\ \|u_n\| \leq \rho,\ \forall\, n \geq 0$ such that $I_r(u_n) \to d_\rho$ and $I_r'(u_n) \to 0$. Then, there exists $U_1 \in E_r$ such that $u_n \rightharpoonup U_1$ weakly in E_r. We claim that, up to a subsequence, (u_n) is strongly convergent to U_1 in E_r. Let $0 < \epsilon < 1$; by (H_3) there exists a constant $c_\epsilon > 0$ such that

$$|f(x,s)| \leq \epsilon |s| + c_\epsilon |s|^\beta \left(e^{ps^2} - 1\right),\ \text{a.e } x \in \mathbb{R}^4,\ \forall\ s \in \mathbb{R}.$$

It follows

$$\begin{aligned}\int_{\mathbb{R}^4} |f(x,u_n)(u_n - U_1)| dx \leq \epsilon \int_{\mathbb{R}^4} |u_n(u_n - U_1)| dx \\ + c_\epsilon \int_{\mathbb{R}^4} |u_n|^\beta \left(e^{pu_n^2} - 1\right) |u_n - U_1| dx.\end{aligned} \tag{2.1}$$

We have

$$\begin{aligned}\int_{\mathbb{R}^4} |u_n(u_n - U_1)| dx &\leq \int_{\mathbb{R}^4} \frac{|u_n|^2}{2} dx + \int_{\mathbb{R}^4} \frac{|u_n - U_1|^2}{2} dx \\ &\leq c_2,\ \forall\ n \geq 0.\end{aligned} \tag{2.2}$$

On the other hand, by Hölder's inequality

$$\begin{aligned}&\int_{\mathbb{R}^4} |u_n|^\beta \left(e^{pu_n^2} - 1\right) |u_n - U_1| dx \\ &\leq \left(\int_{\mathbb{R}^4} |u_n - U_1|^3 dx\right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^4} |u_n|^{\frac{3\beta}{2}} \left(e^{\frac{3pu_n^2}{2}} - 1\right) dx\right)^{\frac{2}{3}}.\end{aligned} \tag{2.3}$$

Obviously, one could choose ρ small enough such that

$$\rho < \frac{1}{\chi_0}\left(\frac{32\pi^2}{\frac{3\beta}{2}}\right)^{\frac{1}{2}} = \frac{1}{\chi_0}\left(\frac{64\pi^2}{3\beta}\right)^{\frac{1}{2}}.$$

By (1.1), it yields

$$\int_{\mathbb{R}^4} |u_n|^{\frac{3\beta}{2}} \left(e^{\frac{3pu_n^2}{2}} - 1\right) dx \le c_3 \|u_n\|^{\frac{3\beta}{2}} \le c_4,\ \forall\ n \ge 0.$$

Taking into account that the embeddings $E_r \hookrightarrow\hookrightarrow L^t(\mathbb{R}^4)$ are compact for all $2 < t < +\infty$ (see [13]), we deduce that, up to a subsequence,

$$\int_{\mathbb{R}^4} |u_n - U_1|^3 \, dx \to 0,\ n \to +\infty.$$

By (2.3) , we infer

$$\int_{\mathbb{R}^4} |u_n|^{\beta} \left(e^{pu_n^2} - 1\right) |u_n - U_1| \, dx \to 0,\ n \to +\infty. \tag{2.4}$$

Using (2.4) , (2.2) and (2.1) , we obtain

$$\lim_{n\to+\infty} \int_{\mathbb{R}^4} |f(x,u_n)(u_n - U_1)| \, dx \le c_2\epsilon.$$

Since $0 < \epsilon < 1$ is arbitrary, we deduce that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^4} f(x,u_n)(u_n - U_1) dx = 0. \tag{2.5}$$

Taking the weak convergence of (u_n) to U_1 in E_r into account and using (2.5), it follows

$$\left(a + b\int_{\mathbb{R}^4} |\Delta u_n|^2 \, dx\right) \int_{\mathbb{R}^4} |\Delta(u_n - U_1)|^2 \, dx + \int_{\mathbb{R}^4} V(x)(u_n - U_1)^2 dx \to 0.$$

Consequently, $u_n \to U_1$ strongly in E_r. Hence, $I_r'(U_1) = 0$ and $I_r(U_1) = I(U_1) = d_\rho < 0$. According to the principle of symmetric criticality (see [15,23]), the function U_1 is in fact a critical point of the functional I. ☐

Lemma 2.3. *Assume that $(H_1) - (H_6)$ hold true. Then, there exists $A_0 > 0$ such that the problem (P) admits a nontrivial weak radial solution U_2 such that $I(U_2) > 0$ provided that $A > A_0$.*

Proof. Let $t > 0$ and $\varphi \in E_r$ be such that $\varphi \neq 0$ and $\int_{\mathbb{R}^4} h\varphi dx > 0$. By (H_4), it yields

$$\begin{aligned} I_r(t\varphi) \le{}& \frac{at^2}{2}\int_{\mathbb{R}^4} |\Delta\varphi|^2 \, dx + \frac{b}{4}t^4 \left(\int_{\mathbb{R}^4} |\Delta\varphi|^2 \, dx\right)^2 \\ &+ \frac{t^2}{2}\int_{\mathbb{R}^4} V(x)\varphi^2 dx - At^q \int_{\mathbb{R}^4} |\varphi|^q \, dx. \end{aligned}$$

Since $q > 4$, there exists $\tau_0 > \frac{\rho}{\|\varphi\|}$ large enough such that $I_r(\tau_0\varphi) < 0$. Now, by the Mountain-Pass Theorem without the Palais-Smale condition (see [23, Chapter 2]), there exists a sequence $(u_n) \subset E_r$ such that $I_r(u_n) \to c$ and $I_r'(u_n) \to 0$, where $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} I_r(\gamma(t)) > 0$, and

$$\Gamma = \{\gamma : [0,1] \to E_r,\ \gamma(0) = 0,\ \gamma(1) = \tau_0\varphi\}.$$

Observe that

$$I_r(u_n) - \frac{1}{\nu}\langle I_r'(u_n), u_n\rangle \leq c + o_n(1)\left(1 + \|u_n\|\right),\ \forall\ n \geq 0.$$

Using (H_5), we get

$$\begin{aligned} & a\left(\frac{1}{2} - \frac{1}{\nu}\right)\int_{\mathbb{R}^4} |\Delta u_n|^2\,dx + b\left(\frac{1}{4} - \frac{1}{\nu}\right)\left(\int_{\mathbb{R}^4} |\Delta u_n|^2\,dx\right)^2 \\ & + \left(\frac{1}{2} - \frac{1}{\nu}\right)\int_{\mathbb{R}^4} V(x)u_n^2\,dx \\ & \leq c + o_n(1)\left(1 + \|u_n\|\right) + |h|_2\,\|u_n\|,\ \forall\ n \geq 0. \end{aligned}$$

Thus,

$$\left(\frac{1}{2} - \frac{1}{\nu}\right)\inf(1,a)\,\|u_n\|^2 \leq c + o_n(1)\left(1 + \|u_n\|\right) + |h|_2\,\|u_n\|,\ \forall\ n \geq 0. \tag{2.6}$$

Then, (u_n) is bounded in E_r. By Young's inequality and (2.6), one can easily find a positive constant $c_5 > 0$ such that

$$\left(\frac{1}{2} - \frac{1}{\nu}\right)\frac{\inf(1,a)}{2}\,\|u_n\|^2 \leq c + o_n(1)\left(1 + \|u_n\|\right) + c_5\,|h|_2^2,\ \forall\ n \geq 0. \tag{2.7}$$

Passing to the upper limit in (2.7), we obtain

$$\limsup_{n\to+\infty} \|u_n\|^2 \leq \frac{2c}{\left(\frac{1}{2} - \frac{1}{\nu}\right)\inf(1,a)} + c_6\,|h|_2^2. \tag{2.8}$$

Now, by the even definition of c and (H_4), we have

$$c \leq \max_{t\geq 0} I_r(t\varphi) \leq \max_{t\geq 0}\left(k_1t^2 + k_2t^4 - k_3t^q\right), \tag{2.9}$$

where

$$k_1 = \frac{1+a}{2}\,\|\varphi\|^2,\ k_2 = \frac{b}{4}\left(\int_{\mathbb{R}^4} |\Delta\varphi|^2\,dx\right)^2,\ k_3 = A\,|\varphi|_q^q.$$

For $t \geq 0$, define $\Psi(t) = k_1t^2 + k_2t^4 - k_3t^q$. Clearly, one can choose A large enough such that

$$\frac{4(k_1 + k_2)}{qk_3} < 1. \tag{2.10}$$

That last inequality together with a direct computation lead to

$$\begin{aligned}\max_{0\le t\le 1}\Psi(t) &\le \max_{0\le t\le 1}\left((k_1+k_2)t^2-k_3t^q\right)\\ &= \left(1-\frac{2}{q}\right)(k_1+k_2)\left(\frac{2(k_1+k_2)}{qk_3}\right)^{\frac{2}{q-2}}.\end{aligned}\tag{2.11}$$

On the other hand, again by (2.10) we have

$$\max_{t\ge 1}\Psi(t)\le \max_{t\ge 1}\left((k_1+k_2)t^4-k_3t^q\right)=k_1+k_2-k_3.\tag{2.12}$$

Combining (2.10), (2.11) and (2.12), one can easily find $A_0>0$ large enough such that

$$\max_{t\ge 0}\Psi(t)\le\left(\frac{1}{2}-\frac{1}{\nu}\right)\inf(1,a)\left(\frac{1}{4\chi_0}\right)^2\left(\frac{64\pi^2}{3\beta}\right),\ \forall\ A>A_0.$$

By (2.9), it follows

$$\frac{2c}{\left(\frac{1}{2}-\frac{1}{\nu}\right)\inf(1,a)}\le\frac{1}{8}\left(\frac{1}{\chi_0}\right)^2\left(\frac{64\pi^2}{3\beta}\right),\ \forall\ A>A_0.\tag{2.13}$$

Clearly, one could choose $h_0>0$ small enough such that

$$c_6h_0^2\le\frac{1}{8}\left(\frac{1}{\chi_0}\right)^2\left(\frac{64\pi^2}{3\beta}\right).\tag{2.14}$$

By (2.14) ,(2.13) and (2.8) , we deduce that

$$\limsup_{n\to+\infty}\|u_n\|^2\le\frac{1}{4}\left(\frac{1}{\chi_0}\right)^2\left(\frac{64\pi^2}{3\beta}\right),\ \forall\ A>A_0,\ \forall\ |h|_2<h_0.$$

Consequently, there exists $n_0>1$ large enough such that

$$\|u_n\|\le\frac{1}{2\chi_0}\left(\frac{64\pi^2}{3\beta}\right)^{\frac{1}{2}},\ \forall\ n\ge n_0,\tag{2.15}$$

provided that $A>A_0$ and $|h|_2<h_0$. Denote by $U_2\in E_r$ the weak limit of (u_n). By (2.15) , it is immediate that (2.2) and (2.4) hold and by consequence we get

$$\lim_{n\to+\infty}\int_{\mathbb{R}^4}f(x,u_n)(u_n-U_2)dx=0,$$

which implies that, always up to a subsequence, $u_n\to U_2$ strongly in E_r. Therefore, $I_r'(U_2)=0$ and $I_r(U_2)=I(U_2)=c>0$. Again by the virtue of the principle of symmetric criticality, U_2 is a critical point of I. □

Now, we will treat the case when $h=0$ and $F(x,s)$ is even with respect to $s\in\mathbb{R}$. In this case, we try to prove the existence of at least one nonradial (and sign-changing) solution to (P). For that aim, we will try to adapt some arguments found in [5]. More precisely, set

$$H = O(2) \times O(2) = \left\{ \begin{pmatrix} (a) & (0) \\ (0) & (b) \end{pmatrix},\ (a),\ (b) \in O(2) \right\} \subset O(4).$$

Furthermore, let $\tau = \begin{pmatrix} (0) & i_2 \\ i_2 & (0) \end{pmatrix} \in O(4)$ where i_2 denotes the identity of $\mathbb{R}^2$. Now, consider the subgroup $G = \langle H \cup \{\tau\} \rangle$ of $O(4)$. Observe that $\tau^{-1} = \tau$ and $\tau H = H\tau$. Consequently, $G = H \cup \{h\tau,\ h \in H\}$. We define the action $* : G \times E \to E$ of G on E by

$$h * u(x) = u(h^{-1}x),\ \forall\ x \in \mathbb{R}^4,\ \forall\ u \in E,\ \forall\ h \in H,$$

$$(h\tau) * u(x) = -u(\tau h^{-1}x),\ \forall\ x \in \mathbb{R}^4,\ \forall\ u \in E,\ \forall\ h \in H.$$

Clearly, $\|g * u\| = \|u\|,\ \forall\ u \in E,\ \forall\ g \in G$. Thus, the action of G on E is isometric. Since $V(x)$ and $f(x,s)$ are spherically symmetric with respect to x, then

$$V(gx) = V(x),\ f(gx,s) = f(x,s),\ \forall\ x \in \mathbb{R}^4,\ \forall\ g \in G,\ \forall\ s \in \mathbb{R}.$$

Since the function $u \longmapsto \int_{\mathbb{R}^4} F(x,u)dx$ is even, then $I(g * u) = I(u),\ \forall\ u \in E,\ \forall\ g \in G$ which means that I is G-invariant. Set

$$E_G = Fix_E(G) = \{u \in E,\ g * u = u,\ \forall\ g \in G\}.$$

This space is nothing else than the space of the fixed points in E with respect to the action of G on E. Moreover, we introduce the space

$$W_H^{1,4}(\mathbb{R}^4) = \left\{u \in W^{1,4}(\mathbb{R}^4),\ u(x) = u(hx),\ \forall\ x \in \mathbb{R}^4,\ \forall\ h \in H\right\}.$$

By the virtue of [9, Corollary 4], the embedding $W_H^{1,4}(\mathbb{R}^4) \hookrightarrow L^r(\mathbb{R}^4)$ is compact, for all $4 < r < +\infty$. Observing that $E_G \hookrightarrow W_H^{1,4}(\mathbb{R}^4)$ with continuous embedding, it follows

$$E_G \hookrightarrow\hookrightarrow L^r(\mathbb{R}^4),\ \forall\ 4 < r < +\infty \tag{2.16}$$

with compact embedding.

Remark 2.4. It is clear that every point $u \in E_G \backslash \{0\}$ is non spherically symmetric (i.e. nonradial) and is sign-changing.

Proof of Theorem 1.2 completed in view of Lemma 2.2 and Lemma 2.3, we easily see that, for $0 < |h|_2 < h_0$, the problem (P) has at least two nontrivial radial solutions. Now, we assume that $h = 0$ and $F(x,s)$ is even with respect to $s \in \mathbb{R}$. By Lemma 2.3, the problem (P) admits at least one weak radial solution denoted by U_1 such that $I(U_1) > 0$. In order to conclude the proof of Theorem 1.1, it remains to show that the problem (P) admits a nontrivial nonradial weak solution. But this result can be reached by adapting the arguments used in the proof of Lemma 2.3 to the functional $I_G = I|_{E_G}$, which is the restriction of I on the subspace E_G. It suffices to prove that a similar identity to (2.5) holds. Let $(u_n) \subset E_G$ and $u \in E_G$ be such that $u_n \rightharpoonup u$ weakly in E_G. We claim that

$$\int_{\mathbb{R}^4} f(x,u_n)(u_n-u)dx \to 0,\ n \to +\infty. \tag{2.17}$$

By Hölder's inequality, we have

$$\begin{aligned} &\int_{\mathbb{R}^4} |u_n|^\beta \left(e^{pu_n^2}-1\right)|u_n-u|\,dx \\ &\le \left(\int_{\mathbb{R}^4} |u_n-u|^5\,dx\right)^{\frac{1}{5}} \left(\int_{\mathbb{R}^4} |u_n|^{\frac{5\beta}{4}} \left(e^{\frac{5pu_n^2}{4}}-1\right)dx\right)^{\frac{4}{5}}. \end{aligned} \tag{2.18}$$

As in (2.15), one can easily show that, for $A > A_0$ and n large enough, we have

$$\|u_n\| \le \frac{1}{2\chi_0}\left(\frac{64\pi^2}{3\beta}\right)^{\frac{1}{2}} < \frac{1}{2\chi_0}\left(\frac{128\pi^2}{5\beta}\right)^{\frac{1}{2}}.$$

By (1.1), it yields

$$\int_{\mathbb{R}^4} |u_n|^{\frac{5\beta}{4}} \left(e^{\frac{5pu_n^2}{4}}-1\right)dx \le c_7\,\|u_n\|^{\frac{5\beta}{4}} \le c_8,\ \forall\, n \ge 0.$$

By (2.16), $E_G \hookrightarrow\hookrightarrow L^5(\mathbb{R}^4)$. Then, $|u_n-u|_5 \to 0$, and by consequence (2.17) follows from (2.18) and (2.1). We deduce that I_G admits at least one nontrivial critical point $U_2 \in E_G$ such that $I_G(U_2) = I(U_2) > 0$. Taking into account that the principle of symmetric criticality still holds (see [10]), we conclude that U_2 is a critical point of I. Finally, since $U_2 \neq 0$, Remark 2.4 implies that U_2 is nonradial and sign-changing. This ends the proof of 1.2.

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