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PPF DEPENDENT FIXED POINT THEOREMS FOR MULTI-VALUED MAPPINGS IN BANACH SPACES

A. FARAJZADEH, A. KAEWCHAROEN* AND S. PLUBTIENG

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Abstract. We prove the existence of PPF dependent coincidence points for a pair of single-valued and multi-valued mappings satisfying generalized contractive conditions in Banach spaces. Furthermore, the PPF dependent fixed point and PPF dependent common fixed point theorems for multi-valued mappings are proved.
Keywords: PPF dependent common fixed points, PPF dependent coincidence points, Razumikhin classes, generalized ψ-contractions.
MSC(2010): Primary: 54H25; Secondary: 55M20.

1. Introduction

The common fixed point theorems for single-valued mappings satisfying some contractive conditions have been studied by many authors (see [1, 6–9, 11, 12] and the references contained therein). The existence of PPF dependent fixed points in the Razumikhin class of mappings for single-valued mappings that have different domains and ranges has been proved by Bernfeld et al. [2]. Since then the researchers have extended the existence of PPF dependent fixed points to PPF common dependent fixed points for single-valued mappings satisfying the weaker contractive conditions in Banach spaces (see [3–5]).

In this paper, we prove the existence of PPF dependent coincidence points for a pair of single-valued and multi-valued mappings satisfying generalized contractive conditions in Banach spaces. Furthermore the PPF dependent fixed point and PPF dependent common fixed point theorems for multi-valued mappings are proved.

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2. Preliminaries

Suppose that E is a real Banach space with the norm $\|\cdot\|_E$ and I is a closed interval $[a,b]$ in $\mathbb{R}$. Let $E_0 = C(I,E)$ be the set of all continuous E-valued functions on I equipped with the supremum norm $\|\cdot\|_{E_0}$ defined by

$$\|\phi\|_{E_0} = \sup_{t\in I} \|\phi(t)\|_E = \max_{t\in I} \|\phi(t)\|_E, \tag{2.1}$$

for all $\phi \in E_0$. For a fixed element $c \in I$, the Razumikhin class of mappings in E_0 is defined by

$$\mathcal{R}_c = \{\phi \in E_0 : \|\phi\|_{E_0} = \|\phi(c)\|_E\}. \tag{2.2}$$

Definition 2.1. Let A be a subset of E. Then

(i) A is said to be topologically closed with respect to the norm topology if for each sequence $\{x_n\}$ in A with $x_n \to x$ as $n \to \infty$ implies $x \in A$.
(ii) A is said to be algebraically closed with respect to the difference if $x - y \in A$ for all $x, y \in A$.

Farajzadeh and Kaewcharoen [4] proved that the Razumikhin class $\mathcal{R}_c$ is topologically closed with respect to the norm topology.

Proposition 2.2. ([4]) *The Razumikhin class $\mathcal{R}_c$ is topologically closed with respect to the norm topology.*

The following example shows that the algebraic closedness with respect to the difference of Razumikhin class $\mathcal{R}_c$ may fail. Hence, by using the previous argument, the Razumikhin class $\mathcal{R}_c$ is not convex in general.

Example 2.3. ([4]) Let $E_0 = C([0,1],\mathbb{R})$ and $c = 1$. If we take $\phi(t) = t^2$ and $\alpha(t) = t$ for all $t \in [0,1]$, then $\phi, \alpha \in \mathcal{R}_c$ while $\phi - \alpha \notin \mathcal{R}_c$.

Generally, if ϕ and α are elements of $\mathcal{R}_c$ with $\phi \neq \alpha$ and $\phi(c) = \alpha(c)$ then $\phi - \alpha \notin \mathcal{R}_c$. Because if $\|\phi - \alpha\|_{E_0} = \|(\phi - \alpha)(c)\|_E$, then it follows from $\phi(c) = \alpha(c)$ that $\|\phi - \alpha\|_{E_0} = 0$ and so $\phi = \alpha$ which is a contradiction.

Remark that from the above example and the fact $\lambda\mathcal{R}_c \subseteq \mathcal{R}_c$, for $|\lambda| \le 1$, we get $\frac{1}{2}\phi + \frac{1}{2}(-\alpha) \notin \mathcal{R}_c$ and so the Razumikhin class $\mathcal{R}_c$ may fail to be midconvex.

It is clear from the definition of the Razumikhin class $\mathcal{R}_c$ that

$$\lambda\mathcal{R}_c \subseteq \mathcal{R}_c \quad \text{for each real number } \lambda.$$

Hence the Razumikhin class $\mathcal{R}_c$ is balance, that is $\lambda\mathcal{R}_c \subseteq \mathcal{R}_c \quad$ for $|\lambda| \le 1$.

Therefore we can deduce the next result.

Proposition 2.4. *If the Razumikhin class $\mathcal{R}_c$ is algebraically closed with respect to the difference, then it is a closed linear subspace of $E_0 = C(I,E)$.*

The following example indicates that the converse of Proposition 2.4 is not true in general.

Example 2.5. Let $E_0 = C([0,1],\mathbb{R})$ and $c = 1$. If we take $\phi(t) = t^2$ and $\alpha(t) = t$ for all $t \in [0,1]$, then the closed linear subspace M of E_0 generated by the set $\{\phi, \alpha\}$ is not equal to the Razumikhin class $\mathcal{R}_c$ because $\phi - \alpha \in M$ while $\phi - \alpha \notin \mathcal{R}_c$.

Recall that a point $\phi \in E_0$ is said to be a PPF dependent fixed point or a fixed point with PPF dependence of $T : E_0 \to E$ if $T\phi = \phi(c)$ for some $c \in I$.

Example 2.6. ([4]) Let $T : C([0,1],\mathbb{R}) \to \mathbb{R}$ be defined by

$$T\phi = \frac{1}{2}\Big(\sup_{t\in[0,1]} |\phi(t)|\Big) \text{ for all } \phi \in C([0,1],\mathbb{R}).$$

Therefore T is a contraction with a constant $\frac{1}{2}$. Suppose that $\phi(t) = t^2 + 1$ for all $t \in [0,1]$. Since $T\phi = \frac{1}{2}\Big(\sup_{t\in[0,1]} |\phi(t)|\Big) = 1 = \phi(0)$, we obtain that ϕ is a PPF fixed point with dependence of T.

Definition 2.7. Let $S, T : E_0 \to E$ be two mappings. A point $\phi \in E_0$ is said to be a PPF dependent common fixed point or a common fixed point with PPF dependence of S and T if $S\phi = \phi(c) = T\phi$ for some $c \in I$.

Note that if we take $S = T$, then a PPF dependent common fixed point of S and T collapses to a PPF dependent fixed point.

Definition 2.8. Let $A : E_0 \to E$ and $S : E_0 \to E_0$. A point $\phi \in E_0$ is said to be a PPF dependent coincidence point or coincident point with PPF dependence of A and S if $A\phi = S\phi(c)$ for some $c \in I$.

Let $CB(E)$ be the collection of all nonempty closed bounded subsets of E. Suppose that H is the Hausdorff metric induced by $\|\cdot\|_E$. Therefore, for each $A, B \in CB(E)$,

$$H_E(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\},$$

where $d(a,B) = \inf_{b\in B} \|a - b\|$.

The following result is obtained by the definition of the Hausdorff metric induced by $\|\cdot\|_E$ and the property of the infimum.

Lemma 2.9. *Let $A, B \in CB(E)$ and $a \in A$. Then for all $\varepsilon > 0$, there exists a point $b \in B$ such that $\|a - b\| \leq H_E(A,B) + \varepsilon$.*

Proof. Suppose that there exists $\varepsilon > 0$ such that

$$\|a - b\| > H_E(A,B) + \varepsilon \text{ for all } b \in B.$$

This implies that $d(a,B) \geq H_E(A,B) + \varepsilon > H_E(A,B)$ which is a contradiction. □

In 1989, Mizoguchi and Takahashi [10] obtained a generalization of the Banach contraction principle in a complete metric space as follows:

Theorem 2.10. ([10, Theorem 5]) *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multi-valued mapping satisfying*

$$H_E(Tx, Ty) \leq \varphi(d(x, y))d(x, y), \quad \text{for all } x, y \in X,$$

where $\varphi : [0, \infty) \to [0, 1)$ is a mapping such that

$$\limsup_{r \to t^+} \varphi(r) < 1, \quad \text{for all } t \in [0, \infty).$$

Then T has a fixed point in X.

In this paper, we prove the existence of PPF dependent coincidence points for a pair of single-valued and multi-valued mappings by applying a generalized form of the contractive condition appeared in Theorem 2.10 in the setting of Banach spaces. Furthermore, the PPF dependent fixed point and PPF dependent common fixed point theorems for multi-valued mappings are established.

3. PPF Dependent common fixed points

We begin this section by introducing the concept of a PPF dependent fixed point for a multi-valued mapping which is an extension of the definition of a PPF dependent fixed point for a single-valued mapping in the setting of Banach spaces.

Definition 3.1. Let $T : E_0 \to CB(E)$ be a multi-valued mapping. A point $\phi \in E_0$ is said to be a PPF dependent fixed point or a fixed point with PPF dependence of T if $\phi(c) \in T\phi$ for some $c \in I$.

Note that if $T_1 : E_0 \to E$ is a single-valued mapping, then we can define the multi-valued mapping $T : E_0 \to CB(E)$ by $T(\phi) = \{T_1(\phi)\}$, for all $\phi \in E_0$. Hence, the set of PPF dependent fixed points of T_1 is coincide to the set of PPF dependent fixed point of T. Therefore, the former definition of a PPF dependent fixed point for a multi-valued mapping is a generalization of the corresponding definition for a single-valued mapping.

Definition 3.2. Let $f : E_0 \to E_0$ be a single-valued mapping and $T : E_0 \to CB(E)$ be a multi-valued mapping. A point $\phi \in E_0$ is said to be a PPF dependent coincidence point of f and T if $f\phi(c) \in T\phi$ for some $c \in I$.

Notice that if f is equal to the identity map then Definition 3.2 collapses to Definition 3.1.

The following lemma will be useful for our main theorems.

Lemma 3.3. *Let $A, B \in CB(E)$. Suppose that $\varepsilon > 0$ and $H_E(A, B) < \varepsilon$. Then for all $a \in A$ there exists $b \in B$ such that $\|a - b\| < \varepsilon$*

Proof. Assume that there exists $a \in A$ such that $\|a - b\| \geq \varepsilon$ for all $b \in B$. This implies that $d(a, B) \geq \varepsilon$. Therefore $H_E(A, B) \geq \varepsilon$ which contradicts to the assumption. This completes the proof. $\square$

We now prove the existence of PPF dependent coincidence points for a pair of single-valued and multi-valued mappings satisfying generalized contractive conditions appeared in Theorem 2.10 in Banach spaces.

Theorem 3.4. *Suppose that $T : E_0 \to CB(E)$ is a multi-valued mapping and $f : \mathcal{R}_c \to \mathcal{R}_c$ is a single-valued mapping satisfying the following conditions:*

(i) $T(E_0) \subseteq f(\mathcal{R}_c)(c)$,
(ii) $f(\mathcal{R}_c)$ *is complete,*
(iii) *there exists a function $\varphi : [0, \infty) \to [0, 1)$ such that*

$$\limsup_{r \to t^+} \varphi(r) < 1 \text{ for all } t \in [0, \infty), \tag{3.1}$$

and for all $\phi, \alpha \in \mathcal{R}_c$,

$$H_E(T\phi, T\alpha) \leq \varphi(\|f\phi - f\alpha\|_{E_0})\|f\phi - f\alpha\|_{E_0}. \tag{3.2}$$

If $\mathcal{R}_c$ is algebraically closed with respect to the difference, then T and f have a PPF dependent coincidence point in $\mathcal{R}_c$.

Proof. Define a function $\omega : [0, \infty) \to [0, 1)$ such that $\omega(t) = \frac{\varphi(t)+1}{2}$, for all $t \in [0, \infty)$. This implies that

$$\limsup_{r \to t^+} \omega(r) < 1, \ \varphi(t) < \omega(t) \text{ and } 0 < \omega(t) < 1, \text{ for all } t \in [0, \infty).$$

Let $\phi_0 \in \mathcal{R}_c$. Since $T\phi_0 \subseteq E$, there exists $x_1 \in E$ such that $x_1 \in T\phi_0$. Using the fact that $T(\phi_0) \subseteq f(\mathcal{R}_c)(c)$, we can choose $\phi_1 \in \mathcal{R}_c$ such that

$$f\phi_1(c) = x_1 \in T\phi_0.$$

Setting $\alpha_1 = f\phi_1$, we have $\alpha_1 \in \mathcal{R}_c$. By applying (3.2), we obtain that

$$\begin{aligned} H_E(T\phi_0, T\phi_1) &\leq \varphi(\|f\phi_0 - f\phi_1\|_{E_0})\|f\phi_0 - f\phi_1\|_{E_0} \\ &< \omega(\|f\phi_0 - f\phi_1\|_{E_0})\|f\phi_0 - f\phi_1\|_{E_0}. \end{aligned}$$

If $f\phi_0 = f\phi_1$, then T and f have a PPF dependent coincidence point. Assume that $f\phi_0 \neq f\phi_1$. By Lemma 3.3, there exists $x_2 \in T\phi_1$ such that

$$\|x_1 - x_2\|_E \quad < \quad \omega(\|f\phi_0 - f\phi_1\|_{E_0})\|f\phi_0 - f\phi_1\|_{E_0}.$$

Since $T(\phi_1) \subseteq f(\mathcal{R}_c)(c)$, we can choose $\phi_2 \in \mathcal{R}_c$ such that

$$f\phi_2(c) = x_2 \in T\phi_1.$$

Setting $\alpha_2 = f\phi_2$, we obtain that $\alpha_2 \in \mathcal{R}_c$. Therefore

$$\|\alpha_1(c) - \alpha_2(c)\|_E \quad < \quad \omega(\|\alpha_0 - \alpha_1\|_{E_0})\|\alpha_0 - \alpha_1\|_{E_0}.$$

By continuing the process as above, we can construct a sequence $\{\alpha_n\}$ such that $\alpha_n(c) \in T\phi_{n-1}, \alpha_n \in \mathcal{R}_c$ and

$$\|\alpha_n(c) - \alpha_{n+1}(c)\|_E \quad < \quad \omega(\|\alpha_{n-1} - \alpha_n\|_{E_0})\|\alpha_{n-1} - \alpha_n\|_{E_0},$$

for each $n \in \mathbb{N}$. Since $\omega(t) < 1$ for all $t \in [0, \infty)$ and $\mathcal{R}_c$ is algebraically closed with respect to the difference, we have

$$\|\alpha_n - \alpha_{n+1}\|_{E_0} = \|\alpha_n(c) - \alpha_{n+1}(c)\|_E < \|\alpha_{n-1} - \alpha_n\|_{E_0}.$$

It follows that $\{\|\alpha_n - \alpha_{n+1}\|_{E_0}\}$ is a nonincreasing sequence in $[0, \infty)$. Therefore $\{\|\alpha_n - \alpha_{n+1}\|_{E_0}\}$ is convergent. Since $\limsup_{r\to t^+} \omega(r) < 1$, we obtain that

$$\limsup_{n\to\infty} \omega(\|\alpha_n - \alpha_{n+1}\|_{E_0}) = s \text{ for some } s \in [0, 1).$$

This implies that for each $k \in (s, 1)$, there is $N \in \mathbb{N}$ such that

$$\omega(\|\alpha_{n-1} - \alpha_n\|_{E_0}) < k, \text{ for all } n \geq N.$$

For each $n \geq N$, we obtain that

$$\begin{aligned} \|\alpha_n - \alpha_{n+1}\|_{E_0} &< \omega(\|\alpha_{n-1} - \alpha_n\|_{E_0})\|\alpha_{n-1} - \alpha_n\|_{E_0} \\ &< k\|\alpha_{n-1} - \alpha_n\|_{E_0}. \end{aligned}$$

Thus for each $m > n \geq N$, we have

$$\begin{aligned} \|\alpha_n - \alpha_m\|_{E_0} &\leq \|\alpha_n - \alpha_{n+1}\|_{E_0} + \cdots + \|\alpha_{m-1} - \alpha_m\|_{E_0} \\ &\leq [k^{n-N} + k^{n-N+1} + \cdots + k^{m-N-1}]\|\alpha_N - \alpha_{N+1}\|_{E_0} \\ &\leq \frac{k^{n-N}}{1-k}\|\alpha_N - \alpha_{N+1}\|_{E_0}. \end{aligned}$$

Therefore,

$$\lim_{n,m\to\infty} \|\alpha_n - \alpha_m\|_{E_0} = 0. \tag{3.3}$$

This implies that $\{\alpha_n\}$ is a Cauchy sequence. Therefore $\{f\phi_n\}$ is also a Cauchy sequence in $f(\mathcal{R}_c)$. By the completeness of $f(\mathcal{R}_c)$, we have $\{f\phi_n\}$ is a convergent sequence. Suppose that $\lim_{n\to\infty} f\phi_n = \phi^*$ for some $\phi^* \in f(\mathcal{R}_c)$. Therefore, there exists $\phi \in \mathcal{R}_c$ such that $\phi^* = f\phi$. That is

$$\lim_{n\to\infty} f\phi_n = f\phi. \tag{3.4}$$

Therefore, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(f\phi_{n+1}(c), T\phi) &\leq H_E(T\phi_n, T\phi) \\ &\leq \varphi(\|f\phi_n - f\phi\|_{E_0})\|f\phi_n - f\phi\|_{E_0} \\ &< \|f\phi_n - f\phi\|_{E_0}. \end{aligned}$$

By taking the limit as $n \to \infty$, we have $f\phi(c) \in T\phi$. Hence, T and f have PPF dependent coincidence point in $\mathcal{R}_c$. □

Corollary 3.5. *Suppose that $T : E_0 \to CB(E)$ is a multi-valued mapping and $f : \mathcal{R}_c \to \mathcal{R}_c$ is a single-valued mapping satisfying the following conditions:*

(i) $T(E_0) \subseteq f(\mathcal{R}_c)(c)$,
(ii) $f(\mathcal{R}_c)$ *is complete,*
(iii) $H_E(T\phi, T\alpha) \leq k\|f\phi - f\alpha\|$, *for all* $\phi, \alpha \in \mathcal{R}_c$ *where* $k \in [0, 1)$.

If $\mathcal{R}_c$ is algebraically closed with respect to the difference, then T and f have a PPF dependent coincidence point in $\mathcal{R}_c$.

Proof. Define $\varphi : [0, \infty) \to [0, 1)$ by $\varphi(t) = k$ for all $t \in [0, \infty)$. It follows that all assumptions in Theorem 3.4 are now satisfied. Hence, the proof is complete. □

The following theorem assures the existence of the PPF dependent fixed point in $\mathcal{R}_c$.

Theorem 3.6. *Suppose that $T : E_0 \to CB(E)$ is a multi-valued mapping satisfying the following conditions:*

(i) $T(E_0) \subseteq E_0(c)$,
(ii) *there exists a function* $\varphi : [0, \infty) \to [0, 1)$ *such that*

$$\limsup_{r \to t^+} \varphi(r) < 1 \text{ for all } t \in [0, \infty), \tag{3.5}$$

and for all $\phi, \alpha \in \mathcal{R}_c$,

$$H(T\phi, T\alpha) \leq \varphi(\|\phi - \alpha\|_{E_0})\|\phi - \alpha\|_{E_0}. \tag{3.6}$$

If $\mathcal{R}_c$ is algebraically closed with respect to the difference, then T has a PPF dependent fixed point in $\mathcal{R}_c$.

Proof. Define a function $\omega : [0, \infty) \to [0, 1)$ such that $\omega(t) = \frac{\varphi(t)+1}{2}$, for all $t \in [0, \infty)$. This implies that

$$\limsup_{r \to t^+} \omega(r) < 1, \ \varphi(t) < \omega(t) \text{ and } 0 < \omega(t) < 1, \text{ for all } t \in [0, \infty).$$

Let $\phi_0 \in \mathcal{R}_c$. Since $T\phi_0 \subseteq E$, there exists $x_1 \in E$ such that $x_1 \in T\phi_0$. Using the fact that $T(\phi_0) \subseteq E_0(c)$, we can choose $\phi_1 \in \mathcal{R}_c$ such that

$$\phi_1(c) = x_1 \in T\phi_0.$$

By applying (3.6), we obtain that

$$\begin{aligned} H(T\phi_0, T\phi_1) &\leq \varphi(\|\phi_0 - \phi_1\|_{E_0})\|\phi_0 - \phi_1\|_{E_0} \\ &< \omega(\|\phi_0, -\phi_1\|_{E_0})\|\phi_0 - \phi_1\|_{E_0}. \end{aligned}$$

If $\phi_0 = \phi_1$, then T has a PPF dependent fixed point. Assume that $\phi_0 \neq \phi_1$. By Lemma 3.3, there exists $x_2 \in T\phi_1$ such that

$$\|x_1 - x_2\|_E < \omega(\|\phi_0 - \phi_1\|_{E_0})\|\phi_0 - \phi_1\|_{E_0}.$$

Since $T(\phi_1) \subseteq E_0(c)$, we can choose $\phi_2 \in \mathcal{R}_c$ such that

$$\phi_2(c) = x_2 \in T\phi_1.$$

Therefore,

$$\|\phi_1(c) - \phi_2(c)\|_E \quad < \quad \omega(\|\phi_0 - \phi_1\|_{E_0})\|\phi_0 - \phi_1\|_{E_0}.$$

By continuing the process as above, we can construct a sequence $\{\phi_n\}$ in $\mathcal{R}_c$ such that $\phi_n(c) \in T\phi_{n-1}$ and

$$\|\phi_n(c) - \phi_{n+1}(c)\|_E \quad < \quad \omega(\|\phi_{n-1} - \phi_n\|_{E_0})\|\phi_{n-1} - \phi_n\|_{E_0},$$

for each $n \in \mathbb{N}$. Since $\omega(t) < 1$ for all $t \in [0, \infty)$ and $\mathcal{R}_c$ is algebraically closed with respect to the difference, we have

$$\|\phi_n - \phi_{n+1}\|_{E_0} = \|\phi_n(c) - \phi_{n+1}(c)\|_E < \|\phi_{n-1} - \phi_n\|_{E_0}.$$

It follows that $\{\|\phi_n - \phi_{n+1}\|_{E_0}\}$ is a nonincreasing sequence in $[0, \infty)$. Therefore, $\{\|\phi_n - \phi_{n+1}\|_{E_0}\}$ is convergent. As in the proof of Theorem 3.4, we obtain that $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c$. By the completeness of $\mathcal{R}_c$, we have $\{\phi_n\}$ is a convergent sequence. Suppose that $\lim_{n\to\infty} \phi_n = \phi$ for some $\phi \in \mathcal{R}_c$. Therefore, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(\phi_{n+1}(c), T\phi) &\leq H_E(T\phi_n, T\phi) \\ &\leq \varphi(\|\phi_n - \phi\|_{E_0})\|\phi_n - \phi\|_{E_0} \\ &< \|\phi_n - \phi\|_{E_0}. \end{aligned}$$

By taking the limit as $n \to \infty$, we have $\phi(c) \in T\phi$. Hence T has a PPF dependent fixed point in $\mathcal{R}_c$. □

Corollary 3.7. *Suppose that $T : E_0 \to CB(E)$ is a multi-valued mapping satisfying*

(i) $T(E_0) \subseteq E_0(c)$,
(ii) $H_E(T\phi, T\alpha) \leq k\|\phi - \alpha\|$, *for all $\phi, \alpha \in \mathcal{R}_c$ where $k \in [0, 1)$*

Then T has a PPF dependent fixed point in $\mathcal{R}_c$.

Definition 3.8. Let $T, S : E_0 \to CB(E)$ be two multi-valued mappings. A point $\phi \in E_0$ is said to be a PPF dependent common fixed point or a common fixed point with PPF dependence of T and S if $\phi(c) \in T\phi$ and $\phi(c) \in S\phi$ for some $c \in I$.

We next prove the existence of PPF dependent common fixed point theorems for multi-valued mappings satisfying some generalized contractive conditions in Banach spaces.

Theorem 3.9. *Suppose that $T, S : E_0 \to CB(E)$ are multi-valued mappings satisfying*

(3.7)

$$H_E(T\phi, S\alpha) \leq kM(\phi,\alpha) + L\min\{d(\phi(c), T\phi), d(\alpha(c), S\alpha), d(\phi(c), S\alpha), d(\alpha(c), T\phi)\},$$

for all $\phi, \alpha \in \mathcal{R}_c$ where $k \in [0,1)$, $L \geq 0$ and

$$M(\phi,\alpha) = \max\{\|\phi-\alpha\|_{E_0}, d(\phi(c), T\phi), d(\alpha(c), S\alpha), \frac{1}{2}[d(\phi(c), S\alpha) + d(\alpha(c), T\phi)]\}.$$

If $\mathcal{R}_c$ is algebraically closed with respect to the difference, then T and S have a PPF dependent common fixed point in $\mathcal{R}_c$. Moreover, if T or S is a single-valued mapping, then T and S have a unique PPF dependent common fixed point in $\mathcal{R}_c$.

Proof. Let $\varepsilon > 0$ be such that $k + \varepsilon < 1$. Let $\phi_0 \in \mathcal{R}_c$ and $x_1 \in S\phi_0$. We can choose $\phi_1 \in \mathcal{R}_c$ such that

$$\phi_1(c) = x_1 \in S\phi_0.$$

It is easily seen that, if $M(\phi_0,\phi_1) = 0$, then $\phi_0 = \phi_1$ and ϕ_0 is a common fixed point of T and S. Suppose that $M(\phi_0,\phi_1) > 0$. By Lemma 2.9, there exists $x_2 \in T\phi_1$ such that

$$\|x_2 - x_1\|_E \leq H_E(T\phi_1, S\phi_0) + \varepsilon M(\phi_0,\phi_1).$$

We can choose $\phi_2 \in \mathcal{R}_c$ such that

$$\phi_2(c) = x_2 \in T\phi_1.$$

Therefore

$$\|\phi_2(c) - \phi_1(c)\|_E \leq H_E(T\phi_1, S\phi_0) + \varepsilon M(\phi_0,\phi_1).$$

If $M(\phi_1,\phi_2) = 0$, then $\phi_1 = \phi_2$ and ϕ_1 is a common fixed point of T and S. We assume that $M(\phi_1,\phi_2) > 0$. By Lemma 2.9, there exists $x_3 \in S\phi_2$ such that

$$\|x_2 - x_3\|_E \leq H_E(T\phi_1, S\phi_2) + \varepsilon M(\phi_1,\phi_2).$$

Choose $\phi_3 \in \mathcal{R}_c$ such that

$$\phi_3(c) = x_3 \in S\phi_2.$$

Therefore

$$\|\phi_3(c) - \phi_2(c)\|_E \leq H_E(T\phi_1, S\phi_2) + \varepsilon M(\phi_1,\phi_2).$$

Continuing the process as above, we can construct a sequence $\{\phi_n\}$ in $\mathcal{R}_c$ such that $\phi_{2n+1}(c) \in S\phi_{2n}$ and $\phi_{2n+2}(c) \in T\phi_{2n+1}$ and $M(\phi_n,\phi_{n+1}) > 0$ with

$$\|\phi_{2n+1}(c) - \phi_{2n}(c)\|_E \leq H_E(T\phi_{2n-1}, S\phi_{2n}) + \varepsilon M(\phi_{2n-1},\phi_{2n})$$

and

$$\|\phi_{2n+2}(c) - \phi_{2n+1}(c)\|_E \leq H_E(T\phi_{2n+1}, S\phi_{2n}) + \varepsilon M(\phi_{2n},\phi_{2n+1}).$$

By applying (3.7), we obtain that

$$\begin{aligned}\|\phi_{2n+1}-\phi_{2n}\|_{E_0} &= \|\phi_{2n+1}(c)-\phi_{2n}(c)\|_E\\ &\le H_E(T\phi_{2n-1},S\phi_{2n})+\varepsilon M(\phi_{2n-1},\phi_{2n})\\ &\le kM(\phi_{2n-1},\phi_{2n})+\varepsilon M(\phi_{2n-1},\phi_{2n})+L\min\{d(\phi_{2n-1}(c),T\phi_{2n-1}),\\ &\quad d(\phi_{2n}(c),S\phi_{2n}),d(\phi_{2n-1}(c),S\phi_{2n}),d(\phi_{2n}(c),T\phi_{2n-1})\},\\ &\le (k+\varepsilon)M(\phi_{2n-1},\phi_{2n})+L\min\{\|\phi_{2n-1}(c)-\phi_{2n}(c)\|_E,\\ &\quad \|\phi_{2n}(c)-\phi_{2n+1}(c)\|_E,\|\phi_{2n-1}(c)-\phi_{2n+1}(c)\|_E,\|\phi_{2n}(c)-\phi_{2n}(c)\|_E\}.\\ &= (k+\varepsilon)M(\phi_{2n-1},\phi_{2n}),\end{aligned}$$

where

$$\begin{aligned}M(\phi_{2n-1},\phi_{2n}) &= \max\{\|\phi_{2n-1}-\phi_{2n}\|_{E_0},d(\phi_{2n-1}(c),T\phi_{2n-1}),d(\phi_{2n}(c),S\phi_{2n}),\\ &\quad \frac{1}{2}[d(\phi_{2n-1}(c),S\phi_{2n})+d(\phi_{2n}(c),T\phi_{2n-1})]\}\\ &\le \max\{\|\phi_{2n-1}-\phi_{2n}\|_{E_0},\|\phi_{2n-1}(c)-\phi_{2n}(c)\|_E,\|\phi_{2n}(c)-\phi_{2n+1}(c)\|_E,\\ &\quad \frac{1}{2}[\|\phi_{2n-1}(c)-\phi_{2n+1}(c)\|_E+\|\phi_{2n}(c)-\phi_{2n}(c)\|_E]\}\\ &= \max\{\|\phi_{2n-1}-\phi_{2n}\|_{E_0},\|\phi_{2n}-\phi_{2n+1}\|_{E_0},\frac{1}{2}\|\phi_{2n-1}-\phi_{2n+1}\|_{E_0}\}\\ &\le \max\{\|\phi_{2n-1}-\phi_{2n}\|_{E_0},\|\phi_{2n}-\phi_{2n+1}\|_{E_0},\\ &\quad \frac{1}{2}[\|\phi_{2n-1}-\phi_{2n}\|_{E_0}+\|\phi_{2n}-\phi_{2n+1}\|_{E_0}\}]\\ &\le \max\{\|\phi_{2n-1}-\phi_{2n}\|_{E_0},\|\phi_{2n}-\phi_{2n+1}\|_{E_0}\}.\end{aligned}$$

If there exists $n\in\mathbb{N}$ such that $M(\phi_{2n-1},\phi_{2n})=\|\phi_{2n}-\phi_{2n+1}\|_{E_0}$, then

$$\|\phi_{2n+1}-\phi_{2n}\|_{E_0}\le(k+\varepsilon)\|\phi_{2n}-\phi_{2n+1}\|_{E_0}<\|\phi_{2n}-\phi_{2n+1}\|_{E_0},$$

which leads to a contradiction. By setting $a=k+\varepsilon$, it follows that

$$\|\phi_{2n+1}-\phi_{2n}\|_{E_0}\le a\|\phi_{2n}-\phi_{2n-1}\|_{E_0}, \tag{3.8}$$

for each $n\in\mathbb{N}$. Similarly, we can prove that

$$\|\phi_{2n+2}-\phi_{2n+1}\|_{E_0}\le a\|\phi_{2n+1}-\phi_{2n}\|_{E_0}, \tag{3.9}$$

for each $n\in\mathbb{N}$. From (3.8) and (3.9), we can conclude that

$$\|\phi_{n+1}-\phi_n\|_{E_0}\le a\|\phi_n-\phi_{n-1}\|_{E_0} \text{ for all } n\in\mathbb{N}. \tag{3.10}$$

For each $n\in\mathbb{N}$, we obtain that

$$\|\phi_{n+1}-\phi_n\|_{E_0}\le a^n\|\phi_1-\phi_0\|_{E_0}. \tag{3.11}$$

Let $m,n\in\mathbb{N}$ with $m>n$. By applying (3.11), we have

$$\begin{aligned}\|\phi_n-\phi_m\|_{E_0} &\le \|\phi_n-\phi_{n+1}\|_{E_0}+\|\phi_{n+1}-\phi_{n+2}\|_{E_0}+\cdots+\|\phi_{m-1}-\phi_m\|_{E_0}\\ &\le [a^n+a^{n+1}+\cdots+a^{m-1}]\|\phi_1-\phi_0\|_{E_0}\\ &\le \frac{a^n}{1-a}\|\phi_1-\phi_0\|_{E_0}.\end{aligned}$$

It follows that

$$\lim_{n,m\to\infty} \|\phi_n - \phi_m\|_{E_0} = 0. \tag{3.12}$$

This implies that $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c$. Since $\mathcal{R}_c$ is algebraically closed with respect to the norm topology and so it is complete, we have

$$\lim_{n\to\infty} \phi_n = \phi \text{ for some } \phi \in \mathcal{R}_c. \tag{3.13}$$

We will prove that ϕ is a PPF dependent fixed point of S. By Using (3.7), we have

$$\begin{aligned}
d(\phi_{2n+2}(c),S\phi) &\le H(T\phi_{2n+1}, S\phi)\\
&\le kM(\phi_{2n+1},\phi) + L\min\{d(\phi_{2n+1}(c),T\phi_{2n+1}), d(\phi(c),S\phi),\\
&\quad d(\phi_{2n+1}(c),S\phi), d(\phi(c),T\phi_{2n+1})\}\\
&\le k\max\{\|\phi_{2n+1}-\phi\|_{E_0}, d(\phi_{2n+1}(c),T\phi_{2n+1}), d(\phi(c),S\phi),\\
&\quad \frac{1}{2}[d(\phi_{2n+1}(c),S\phi) + d(\phi(c),T\phi_{2n+1})]\}\\
&\quad +L\min\{d(\phi_{2n+1}(c),T\phi_{2n+1}), d(\phi(c),S\phi),\\
&\quad d(\phi_{2n+1}(c),S\phi), d(\phi(c),T\phi_{2n+1})\}\\
&\le k\max\{\|\phi_{2n+1}-\phi\|_{E_0}, \|\phi_{2n+1}(c)-\phi_{2n+2}(c)\|_E, d(\phi(c),S\phi),\\
&\quad \frac{1}{2}[d(\phi_{2n+1}(c),S\phi) + \|\phi(c)-\phi_{2n+2}(c)\|_E]\}\\
&\quad +L\min\{\|\phi_{2n+1}(c)-\phi_{2n+2}(c)\|_E, d(\phi(c),S\phi),\\
&\quad d(\phi_{2n+1}(c),S\phi), \|\phi(c)-\phi_{2n+2}(c)\|_E\}\\
&\le k\max\{\|\phi_{2n+1}-\phi\|_{E_0}, \|\phi_{2n+1}(c)-\phi(c)\|_E+\|\phi(c)-\phi_{2n+2}(c)\|_E,\\
&\quad d(\phi(c),S\phi), \frac{1}{2}[d(\phi_{2n+1}(c),S\phi) + \|\phi(c)-\phi_{2n+2}(c)\|_E]\}\\
&\quad +L\min \|\phi_{2n+1}(c)-\phi(c)\|_E + \|\phi(c)-\phi_{2n+2}(c)\|_E, d(\phi(c),S\phi),\\
&\quad d(\phi_{2n+1}(c),S\phi), \|\phi(c)-\phi_{2n+2}(c)\|_E\}\\
&\le k\max\{\|\phi_{2n+1}-\phi\|_{E_0}, \|\phi_{2n+1}-\phi\|_{E_0} + \|\phi-\phi_{2n+2}\|_{E_0},\\
&\quad d(\phi(c),S\phi), \frac{1}{2}[d(\phi_{2n+1}(c),S\phi) + \|\phi-\phi_{2n+2}\|_{E_0}]\}\\
&\quad +L\min \|\phi_{2n+1}-\phi\|_{E_0} + \|\phi-\phi_{2n+2}\|_{E_0}, d(\phi(c),S\phi),\\
&\quad d(\phi_{2n+1}(c),S\phi), \|\phi-\phi_{2n+2}\|_{E_0}\}.
\end{aligned}$$

Therefore, taking the limit as $n\to\infty$, this yields

$$d(\phi(c),S\phi) \le kd(\phi(c),S\phi).$$

Since $k\in[0,1)$, we obtain that $d(\phi(c),S\phi)=0$. Therefore $\phi(c)\in S\phi$. Similarly, we can prove that $\phi(c)\in T\phi$. Therefore T and S have a PPF common fixed point. We next prove that if T is a single-valued mapping, then the PPF common fixed point of T and S is unique. Assume that $\alpha \in \mathcal{R}_c$ is another PPF common fixed point of T and S. By using (3.7), we have

$$\begin{aligned}
\|\alpha - \phi\|_{E_0} &= \|\alpha(c) - \phi(c)\|_E \\
&\leq H(\{\alpha(c)\}, S\phi) \\
&= H(\{T\alpha\}, S\phi) \\
&\leq k \max\{\|\alpha - \phi\|_{E_0}, \|\alpha(c) - T\alpha\|_E, d(\phi(c), S\phi), \frac{1}{2}[d(\alpha(c), S\phi) + \|\phi(c) \\
&\quad -T\alpha\|_E]\} + L \min\{\|\alpha(c) - T\alpha\|_E, d(\phi(c), S\phi), d(\alpha(c), S\phi), \|\phi(c) - T\alpha\|_E\} \\
&\leq k \max\{\|\alpha - \phi\|_{E_0}, \|\alpha(c) - \alpha(c)\|_E, \|\phi(c) - \phi(c)\|_E, \\
&\quad \frac{1}{2}[\|\alpha(c) - \phi(c)\|_E + \|\phi(c) - \alpha(c)\|_E]\} + L \min\{\|\alpha(c) - \alpha(c)\|_E, \|\phi(c) \\
&\quad -\phi(c)\|_E, \|\alpha(c) - \phi(c)\|_E, \|\alpha(c) - \alpha(c)\|_E\} \\
&\leq k\|\alpha - \phi\|_{E_0}.
\end{aligned}$$

Since $k \in [0, 1)$, then we have $\|\alpha - \phi\|_{E_0} = 0$. Hence $\alpha = \phi$. This implies that T and S have a unique PPF common fixed point. □

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