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FLAG-TRANSITIVE POINT-PRIMITIVE $(v, k, 4)$ SYMMETRIC DESIGNS WITH EXCEPTIONAL SOCLE OF LIE TYPE

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Abstract. Let G be an automorphism group of a 2-$(v, k, 4)$ symmetric design $\mathcal{D}$. In this paper, we prove that if G is flag-transitive point-primitive, then the socle of G cannot be an exceptional group of Lie type.
Keywords: Symmetric design, flag-transitive, point-primitive, exceptional simple group.
MSC(2010): Primary: 05B05; Secondary: 20B15, 20B25.

1. Introduction

A 2-(v, k, λ) design $\mathcal{D}$ is a pair $(P, \mathcal{B})$, where P is a v-set and $\mathcal{B}$ is a family of b k-subsets (blocks) of P such that each element of P is contained in exactly r blocks, and any 2-subset of P is contained in exactly λ blocks. The numbers v, b, r, k and λ are parameters of $\mathcal{D}$. A 2-(v, k, λ) design with $v = b$ (or equivalently, $r = k$) is a symmetric (v, k, λ) design, and is nontrivial if $\lambda < k < v - 1$. An automorphism of a design $\mathcal{D}$ is a permutation of the point set that preserves the block set. The group of all automorphisms of $\mathcal{D}$ under composition of automorphisms is the full automorphism group of $\mathcal{D}$, denoted by $Aut(\mathcal{D})$. Let $G \leq Aut(\mathcal{D})$. Then $\mathcal{D}$ is called point-primitive if G is a primitive permutation group on the point set P. A flag in a symmetric design is an incident point-block pair, $\mathcal{D}$ is called flag-transitive if G is transitive on the set of flags.

Flag-transitive symmetric designs with a small λ have been studied by many researchers. For the flag-transitive projective planes (i.e. $\lambda = 1$), Kantor [9] proved that either $\mathcal{D}$ is a Desarguesian projective plane and $PSL_3(n) \trianglelefteq G$, or G is a sharply flag-transitive Frobenius group of odd order $(n^2+n+1)(n+1)$, where n is even and $n^2 + n + 1$ is prime. In [26–29], Regueiro reduced the classification of flag-transitive biplanes (i.e. $\lambda = 2$) to the situation where the automorphism group is a one-dimensional affine group.

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In 2009, Law, Praeger and Reichard [13] suggested the following problem:

Problem 1.1. Reduce the classification of flag-transitive symmetric (v, k, λ) designs with $\lambda = 3$ or 4 to the case of one-dimensional affine automorphism groups.

For the case $\lambda = 3$, Problem 1.1 has been solved in [33–36]. For the case $\lambda = 4$, Fang and et al. in [7] and Regueiro in [25] obtained independently the following reduction theorem: If $\mathcal{D}$ is a $(v, k, 4)$ symmetric design admitting a flag-transitive primitive automorphism group G, then G must be an affine or almost simple type. Furthermore, it was proved in [7] that if G is almost simple then the socle of G cannot be a sporadic simple group. More recently in [5,37] the flag-transitive primitive $(v, k, 4)$ symmetric designs when $Soc(G) = A_n$, $PSL_2(q)$ were classified. Here we use the following group theoretic notations. The socle of a finite group is the product of its all minimal normal subgroups; it is denoted by $Soc(G)$. A finite group is almost simple if its socle is a non-abelian simple group, and is affine if its socle is elementary abelian.

Here we continue to study Problem 1.1, and consider the case that $Soc(G)$ is an exceptional group of Lie type. Note that the order of the simple exceptional group is given in [12, Table 5.1.B]. Our main result is the following theorem.

Theorem 1.2. *There is no $(v, k, 4)$ symmetric design admitting a flag-transitive, point-primitive almost simple automorphism group with exceptional socle of Lie type.*

2. Preliminary results

In this section, we start with a few preliminary results which will be used in this paper.

Corollary 2.1. *Let $\mathcal{D}$ be a $(v, k, 4)$ symmetric design. Then*

(1) $k(k-1) = 4(v-1)$.
(2) $k \equiv 0$ *or* $1 \pmod 4$.

Proof. Part (1) is obvious. Part (2) is follows from [26, Lemma 3]. □

Corollary 2.2. *If $\mathcal{D}$ is a flag-transitive $(v, k, 4)$ symmetric design, then $4v < k^2$, and hence $4|G| < |G_x|^3$, where x is a point in P.*

Proof. The equality $k(k-1) = 4(v-1)$ implies $k^2 = 4v - 4 + k$, so clearly $4v < k^2$. Since $v = |G : G_x|$ and $k \le |G_x|$, the result follows. □

Remark 2.3. From this corollary we have $|G_x| > \sqrt[3]{|G|}$, which is called the cube root bound.

Corollary 2.4 ([26, Corollary 2])**.** *If G is a flag-transitive automorphism group of a $(v, k, 4)$ symmetric design $\mathcal{D}$, then $k \mid 4(v-1, |G_x|)$.*

Corollary 2.5 ([6], [7, Lemma 1.4])**.** *If $\mathcal{D}$ is a flag-transitive $(v,k,4)$ symmetric design, then $k \mid 4d$, where d is any subdegree of G.*

Corollary 2.6 (Tits Lemma, [31, 1.6])**.** *If X is a simple group of Lie type in characteristic p, then any proper subgroup of index prime to p is contained in a parabolic subgroup of X.*

Corollary 2.7. *Suppose $\mathcal{D}$ is a $(v,k,4)$ symmetric design with a point-primitive, flag-transitive automorphism group G with simple socle X of Lie type in characteristic p, and the stabilizer G_x is not a parabolic subgroup of G. If p is odd, then $(p,k)=1$, and if $p=2$, then $(k,2)=1$ or $4\|k$. Hence $|G|<4|G_x||G_x|_{p'}^2$.*

Proof. By Corollary 2.2 we have $|G|<|G_x|^3$. Now, by Lemma 2.6, $p \mid v=[G:G_x]$, and so $(p,v-1)=1$. Since $k \mid 4(v-1)$, if p is odd then $(k,p)=1$, and if $p=2$ then $(k,2)=1$ or $4\|k$. Hence $k \mid 4|G_x|_{p'}$, and since $4v<k^2$, we have $|G|<4|G_x||G_x|_{p'}^2$. □

Corollary 2.8. *Let $\mathcal{D}$ be a $(v,k,4)$-symmetric design with a flag-transitive, point-primitive group G. Suppose p divides v, and G_x contains a normal subgroup of characteristic p which is quasisimple and $p \nmid |Z(H)|$. Then k is divisible by $[H:P]$, for some parabolic subgroup P of H.*

Proof. Since $\lambda=4$, this can be proved as Lemma 6 in [29]. □

Corollary 2.9. ([15]) *If X is a simple group of Lie type in odd characteristic, and X is neither $PSL_d(q)$ nor $E_6(q)$, then the index of any parabolic subgroup is even.*

Corollary 2.10. ([18]) *If X is a group of Lie type in characteristic p, acting on the set of cosets of a maximal parabolic subgroup, and X is not $PSL_d(q)$, $P\Omega_{2m}^{+}(q)$ (with m odd), nor $E_6(q)$, then there is a unique subdegree which is a power of p.*

We need the following results concerning the maximal subgroups of exceptional groups of Lie type.

Theorem 2.11 ([20, Theorem 2, Table III])**.** *If X is a finite simple exceptional group of Lie type such that $X \le G \le Aut(X)$, and G_x is a maximal subgroup of G such that $X_0=Soc(G_x)$ is not simple, then one of the following holds:*

(1) *G_x is a parabolic subgroup.*
(2) *G_x is a subgroup of maximal rank, given by* [17].
(3) *$G_x=N_G(E)$, where E is an elementary abelian group given in* [3, Theorem 1(II)].
(4) *G_x is the centralizer of a graph, field, or graph-field automorphism of X of prime order.*
(5) *$X=E_8(q)(p>5)$, and X_0 is either $A_5\times A_6$ or $A_5\times L_2(q)$.*
(6) *X_0 is one of the cases listed in Table 1.*

TABLE 1. The list of X_0

X	X_0
$F_4(q)$	$L_2(q) \times G_2(q)(p > 2, q > 3)$
$E_6^\epsilon(q)$	$L_3(q) \times G_2(q), U_3(q) \times G_2(q)(q > 2)$
$E_7(q)$	$L_2(q) \times L_2(q)(p > 3), L_2(q) \times G_2(q)(p > 2, q > 3),$ $L_2(q) \times F_4(q)(q > 3), G_2(q) \times PSp_6(q)$
$E_8(q)$	$L_2(q) \times L_3^\epsilon(q)(p > 3), L_2(q) \times G_2(q) \times G_2(q)(p > 2, q > 3),$ $G_2(q) \times F_4(q), L_2(q) \times G_2(q^2)(p > 2, q > 3)$

The notation $E_6^\epsilon(q)(\epsilon = \pm)$ denotes $E_6(q)$ if $\epsilon = +$, ${}^2E_6(q)$ if $\epsilon = -$; similarly $L_3^\epsilon(q)$ is $L_3(q)$ or $U_3(q)$ respectively if $\epsilon = +$ or $\epsilon = -$.

Theorem 2.12. ([19]) *Let X be a finite simple exceptional group of Lie type, with $X \leq G \leq Aut(X)$, and G_x is a maximal subgroup of G, and $X_0 = Soc(G_x)$ is a simple group of Lie type over F_q ($q = p^e > 2$) such that $\frac{1}{2}\mathrm{rank}(X) < \mathrm{rank}(X_0)$; assume also that (X, X_0) is not $(E_8, {}^2A_5(5))$ or $(E_8, {}^2D_5(3))$. Then one of the following holds:*

1. *X_0 is a subgroup of maximal rank.*
2. *X_0 is a subfield or twisted subgroups.*
3. *$X = E_6^\epsilon(q)$ and $X_0 = C_4(q)$ (q odd) or $F_4(q)$.*

Theorem 2.13 ([22, Theorem 1.2])**.** *Let X be a finite exceptional group of Lie type such that $X \leq G \leq Aut(X)$, and G_x is a maximal subgroup of G such that $X_0 = Soc(G_x)$ is a simple group of Lie type over F_q with $q = p^e$ such that $\mathrm{rank}(X_0) \leq \frac{1}{2}\mathrm{rank}(X)$. Then $|G_x| < 4eq^{20}, 4eq^{28}, 4eq^{30}$ or $12eq^{56}$, according as $X = F_4(q), E_6^\epsilon(q), E_7(q)$ or $E_8(q)$, respectively. In all cases, $|G_x| < 5e|G|^{\frac{5}{13}}$.*

3. **Proof of Theorem 1.2**

In this section, we shall prove Theorem 1.2 by a series of lemmas. Throughout this paper we assume that the following hypothesis holds:

Hypothesis: Let $\mathcal{D}$ be a (v, k, λ) symmetric design, G be a flag-transitive, point-primitive automorphism group of $\mathcal{D}$ with $X = Soc(G)$ be an exceptional simple group of Lie type.

Corollary 3.1. *The group X is not a Suzuki group ${}^2B_2(q)$, with $q = 2^{2c+1} > 2$.*

Proof. Suppose that $X = {}^2B_2(q)$ with $q = 2^{2c+1}$. Then $|G| = f|X| = f(q^2 + 1)q^2(q-1)$, where $f \mid 2c+1$, and so the order of any point stabilizer G_x is one of the following ([32]):

(1) $fq^2(q-1)$
(2) $4f(q+\sqrt{2q}\epsilon+1)$ with $\epsilon=\pm$
(3) $f(q_0^2+1)q_0^2(q_0-1)$, where $8\leq q_0^m=q$ with $m\geq 3$.

Case (1). By putting $v=q^2+1$, from the basic equation $k(k-1)=4(v-1)$, we have $k(k-1)=4q^2=2^{4c+4}$. This is impossible.

Case (2). By Corollary 2.2, $4|G|<|G_x|^3$, we have $4f(q^2+1)q^2(q-1)<4^3f^3(q+\sqrt{2q}\epsilon+1)^3$ and it follows that

$$4f\Big(\frac{7}{8}q^5\Big)<4^3f^3(2q+1)^3\leq 4^3f^3\Big(\frac{17}{8}q\Big)^3.$$

So

$$q^2<\frac{17^3f^2}{28}<176f^2\leq 176e^2,$$

which implies $q\leq 2^5$.

First assume $q=32$. Then $v=198400$ and 325376 when $\epsilon=+$ and $-$ respectively. When $\epsilon=+$, $(|G_x|,v-1)=41$, and $\epsilon=-$, $(|G_x|,v-1)=25$ or 125, depending on whether $f=1$ or 5. In all cases we see $k^2<v$, a contradiction.

Next assume that $q=8$. Then $v=560$ or 1456, and $(|G_x|,v-1)=13$ or 5 when $\epsilon=+$ or $-$ respectively, therefore k is again too small.

Case (3). Here $|G_x|=f(q_0^2+1)q_0^2(q_0-1)$, so $(q_0,v-1)=1$. By Corollary 2.7, $|G|<4|G_x||G_x|_{p'}^2$ and we obtain

$$(q_0^{2m}+1)q_0^{2m}(q_0^m-1)<4f^2(q_0^2+1)^3q_0^2(q_0-1)^3.$$

Now from $q_0^{5m-1}<(q_0^{2m}+1)q_0^{2m}(q_0^m-1)$ and $4f^2(q_0^2+1)^3q_0^2(q_0-1)^3=4f^2q_0^2(q_0^3-q_0^2+q_0-1)^3<f^2q_0^{13}$, we have that

$$q_0^{5m-1}<f^2q_0^{13}<q_0^{13+m},$$

which forces $m=3$. Then

$$v=(q_0^4-q_0^2+1)q_0^4(q_0^2+q_0+1),$$

and by Lemma 2.4 we obtain $k\leq 4|G_x|_{p'}\leq 4fq_0^3<4q_0^{9/2}$. The inequality $v<k^2$ forces $q_0=2,4,8$, so $q=2^3,2^6,2^9$ respectively.

If $q_0=2$, then $v=1456$, and $|G_x|=20f$ with $f=1$ or 3. Hence $(v-1,|G_x|)=5f$, and therefore $k^2<v$, which is a contradiction.

If $q_0=4$, then $v=1295616$, and $|G_x|=816f$ with $f\mid 6$. Hence $(v-1,|G_x|)=f$ and then $k^2<v$.

If $q_0=8$, then $v=1205899264$, and $|G_x|=29120f$ with $f\mid 9$. Hence $(v-1,|G_x|)=f$ and then $k^2<v$. □

Corollary 3.2. *The point stabilizer G_x is not a parabolic subgroup of G.*

Proof. Firstly, by Lemma 3.1, $X\neq {}^2B_2(q)$. Secondly, we assume that $X={}^2G_2(q)$ with $q=3^{2e+1}$. The parabolic subgroup of ${}^2G_2(q)$ is isomorphic to $[q^3]:Z_{q-1}$. Then $v=q^3+1$. Since $k(k-1)=4(v-1)=4q^3$, so $q^3\mid k(k-1)$

with $q = 3^{2e+1}$. It follows that $q^3 \mid k$ or $q^3 \mid k-1$, and then $q^3 = v-1 \leq k$ or $k-1$, which contradicts the fact that $\mathcal{D}$ is non-trivial.

Thirdly, if $X \neq E_6(q)$, then by Lemma 2.10 there is a unique subdegree which is a power of p. Therefore, by Lemma 2.5, k divides 4 times a power of p, but it also divides $4(v-1)$, so it is too small to satisfy $k^2 < v$. For example, assume that $X = {}^3D_4(q)$ with $q = p^e$. If $X \cap G_x \cong P_a$, then $v = (q^8+q^4+1)(q+1)$ and $v-1 = q(q^8+q^7+q^4+q+1)$. By Lemma 2.5, k divides 4 times a power of p, also k divides $4(v-1)$, therefore k divides $4q$, k is too small. If $X \cap G_x \cong P_b$, then $v = (q^8+q^4+1)(q^3+1)$ and $v-1 = q^3(q^8+q^5+q^4+q+1)$. By Lemma 2.10, k divides 4 times a power of p, also k divides $4(v-1)$, therefor k divides $4q^3$, k is too small.

Finally, we assume that $X = E_6(q)$. If G contains a graph automorphism or $X \cap G_x = P_i$ with $i = 2$ or 4. Then there is a unique subdegree which is a power of p (cf. [30, p.345]) and again k is too small. If $X \cap G_x = P_3$, the A_1A_4 type parabolic, then

$$P_3 = \frac{1}{d}q^{36}(q-1)^6(q+1)^3(q^2+1)(q^2+q+1)(q^4+q^3+q^2+q+1),$$

and

$$v = (q^3+1)(q^4+1)(q^6+1)(q^4+q^2+1)(q^8+q^7+\cdots+q+1).$$

Clearly, $q \mid v-1$,

$$v-1 \equiv -1 \pmod{q+1}, \qquad v-1 \equiv -1 \pmod{q^2+1},$$
$$v-1 \equiv -1 \pmod{q^2+q+1}, \qquad v-1 \equiv 0 \pmod{q^4+q^3+q^2+q+1}.$$

Since k divides $4(|G_x|, v-1)$, k divides $4q(q^4+q^3+q^2+q+1)(q-1)^6 \cdot 2de$, where $d = (3, q-1)$. Hence $k^2 < v$, which is a contradiction. If $X \cap G_x = P_1$, then $v = (q^4-q^2+1)(q^4+q^2+1)(q^6+q^3+1)(q^2+q+1)$ and the nontrivial subdegrees are (cf. [16]):

$$q(q^3+1)(q^7+q^6+\cdots+q+1) \quad \text{and} \quad q^8(q^4+q^3+q^2+q+1)(q^4+1).$$

It follows from Lemma 2.5 that k divides

$$4\big(q(q^3+1)(q^7+q^6+\cdots+q+1), q^8(q^4+q^3+q^2+q+1)(q^4+1)\big) = 4q(q^4+1),$$

we see that $k^2 < v$. □

Corollary 3.3. *The group X is not a Ree group ${}^2G_2(q)$, where $q = 3^{2c+1} > 3$.*

Proof. Suppose for the contrary that $X = {}^2G_2(q)$ with $q = 3^{2c+1} > 3$. A complete list of maximal subgroups of G can be found in [11].

First by Lemma 3.2, $X \cap G_x$ is not the maximal parabolic subgroup $[q^3] : Z_{q-1}$.

Now suppose $G_x \cap X = 2 \times L_2(q)$. Then $v = q^2(q^2-q+1)$, and by Lemma 2.4 we have k divides $4(|G_x|, v-1)$. But $\big(q(q^2-1), q^4-q^3+q^2-2\big) = q-1$, which is too small.

The groups $X \cap G_x = N_X(S_2)$ of order $2^3 \cdot 3 \cdot 7$ where $S_2 \in Syl_2(X)$, and $L_2(8)$ are ruled out, since the cube root bound forces $q = 3$.

If $X \cap G_x = {}^2G_2(q_0)$, with $q_0^m = q$ and m prime, then

$$v = q_0^{3(m-1)}(q_0^{3(m-1)} - q_0^{3(m-2)} + \cdots + (-1)^m q_0^3 + (-1)^{m-1}) \times (q_0^{m-1} + q_0^{m-2} + \cdots + q_0 + 1).$$

Now k divides $4|G_x| = 4fq_0^3(q_0^3+1)(q_0-1)$, but since $(q_0, v-1) = 1$, $q_0 \nmid k$, so in fact $k \leq 4f(q_0^3+1)(q_0-1)$, and the inequality $v < k^2$ forces $m = 2$, which is impossible.

If $X \cap G_x = Z_{q \pm \sqrt{3q}+1} : Z_6$, then the cube root bound is not satisfied, since $q \geq 27$.

Finally, if $X \cap G_x = (2^2 \times D_{(q+1)/2}) : 3$, then the cube root bound is also not satisfied. □

Corollary 3.4. *The group X is not a Ree group ${}^2F_4(q)$.*

Proof. Suppose $X = {}^2F_4(q)$ with $q = 2^{2c+1}$. Then $|G| = f|X| = fq^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$, where $f \mid 2c+1$. The complete list of maximal subgroups of ${}^2F_4(q)$, $q = 2^{2c+1} \geq 8$, is given by Malle [24]. Then from [24] we see that if $q \neq 2$, there are no maximal subgroups satisfying the inequality $4|G_x||G_x|_{p'}^2 > |G|$ in Corollary 2.7, except the parabolic subgroups. But the parabolic subgroups are ruled out by Lemma 3.2. If $q = 2$, only $L_3(3).2$, $5^2.4A_4$ or $L_2(25)$ can be satisfied $4|G_x||G_x|_{p'}^2 > |G|$. For the cases $5^2.4A_4$ and $L_2(25)$, by Lemma 2.4, k must divide $4(v-1, |G_x|)$, it is too small. If $X \cap G_x = L_3(3).2$, then $|G_x| = 2^5 \cdot 13 \cdot 3^3 f$, and $v = 1600$. However, there is no integer k satisfying the basic equation $k(k-1) = 4(v-1)$. □

Corollary 3.5. *The group X is not ${}^3D_4(q)$.*

Proof. Suppose by contradiction that $X = {}^3D_4(q)$ with $q = p^e$. The maximal subgroups of G can be found in [10]. By Corollary 2.7, we see that $4|G_x||G_x|_{p'}^2 > |G|$, and so $X \cap G_x$ must be one of the groups $G_2(q)$, $(SL_2(q^3) \circ SL_2(q)).(2, q-1)$ or ${}^3D_4(q^{1/2})$, and its order is $q^6(q^2-1)(q^6-1)$, $q^4(q^2-1)(q^6-1)$ or $q^6(q^3-1)^2(q^2-q+1)$, respectively.

Case (1). If $X \cap G_x = G_2(q)$, then $v = q^6(q^8+q^4+1) = q^6(q^4+q^2+1)(q^4-q^2+1)$. Since $(v-1, q) = 1$ and $(v-1, q^4+q^2+1) = 1$, by Lemma 2.4, k divides

$$4(v-1, |G_x|) = 4\big(v-1, fq^6(q^2-1)(q^6-1)\big) = 4\big(v-1, f(q^2-1)^2\big),$$

where $f \mid 3e$. It is obvious that k is too small to satisfy $4v < k^2$.

Case (2). Here $v = q^8(q^8+q^4+1) = q^8(q^4+q^2+1)(q^4-q^2+1)$, $(v-1, q) = 1$ and $(v-1, q^4+q^2+1) = 1$. So

$$\begin{aligned} 4(v-1, |G_x|) &= 4\big((q^4+1)(q^{12}+q^4-1), fq^4(q^2-1)^2(q^4+q^2+1)\big) \\ &= 4\big((q^4+1)(q^{12}+q^4-1), f(q^2-1)^2\big) \\ &\le 4f(q^2-1)^2. \end{aligned}$$

Hence k is too small to satisfy $4v < k^2$.

Case (3). If $X \cap G_x = {}^3D_4(q^{1/2})$, then $v = q^6(q+1)^2(q^2-q+1)(q^4-q^2+1) > \frac{1}{2}q^{14}$. Since $(v-1, q) = 1$, $(v-1, q^2-q+1) = 1$, and $k \mid 4(v-1, |G_x|)$, we get $k \le 4f(q^3-1)^2 \le 12eq^6$, which is too small to satisfy $k^2 > 4v$. □

Corollary 3.6. *The group X is not a Chevalley group $G_2(q)$ with $q > 2$.*

Proof. Suppose that $X = G_2(q)$, where $q = p^e$. The maxiaml subgroups of X can be found in [11] for q odd and in [4] for q even.

First consider the case where $X \cap G_x = SL_3^\epsilon(q).2$. Then $v = \frac{q^3(q^3+\epsilon)}{2}$. We rule out this case using the method in [26]. From the factorization $\Omega_7(q) = G_2(q)N_1^\epsilon$ (cf. [14]), it follows that the suborbits of $\Omega_7(q)$ are unions of G_2-suborbits, and so by Lemma 2.5, k divides 4 times each of the $\Omega_7(q)$-subdegrees.

If q is odd, as in [28], we assume that $G_\alpha = N_i^\epsilon \in \mathcal{C}_1$, the stabilizer of a nonsingular i-dimensional subspace W of V of sign ϵ. Let $i = 1$. Then the $\Omega_7(q)$-subdegrees are $(q^3-\epsilon)(q^3+\epsilon)$, $\frac{q^2(q^3-\epsilon)}{2}$ and $\frac{q^2(q^3-\epsilon)(q-3)}{2}$ (cf. [23]). By Lemma 2.5 and $k \mid 4(v-1)$, we have $k \mid 2(q^3-\epsilon)$. Let $k = \frac{2(q^3-\epsilon)}{u}$ for some integer u. Then from the basic equation $k(k-1) = 4(v-1)$ we get

$$\frac{1}{u}\Big(\frac{2(q^3-\epsilon)}{u} - 1\Big) = q^3 + 2\epsilon.$$

So

$$q^3 = \frac{u+6\epsilon}{2-u^2} - 2\epsilon,$$

which forces $u = 1$ when $\epsilon = +$, or $u = 2$ when $\epsilon = -$, and then $q^3 = 4$ or 5 respectively, which is a contradiction.

If q is even, the subdegrees for $Sp_6(q)$ are $(q^3-\epsilon)(q^4+\epsilon)$ and $\frac{q^2(q-1)(q^3-\epsilon)}{2}$ (see [23] or [2]). So by Lemma 2.5 we have $k \mid 4(q^3-\epsilon)(q-1, q^4+\epsilon)$, and since $k \mid 4(v-1)$, it follows that $k \mid 4((q^3-\epsilon)(q-1, q^4+\epsilon), v-1) = 4(q^3-\epsilon)$. Let $k = \frac{4(q^3-\epsilon)}{u}$ for some integer u. Then from the basic equation $k(k-1) = 4(v-1)$ we get

$$\frac{2}{u}\Big(\frac{4(q^3-\epsilon)}{u} - 1\Big) = q^3 + 2\epsilon.$$

So

$$q^3 = \frac{u+24\epsilon}{8-u^2} - 2\epsilon,$$

which forces $u = 2$ and $q^3 = 5$ when $\epsilon = +$, or $u = 3, 4$ and $q^3 = 20$, 4 when $\epsilon = -$, which is a contradiction.

If $X \cap G_x = G_2(q_0) < G_2(q)$ or ${}^2G_2(q) < G_2(q)$, then $(p, k) = 1$, so by Lemma 2.8, k is divisible by the index of a parabolic subgroup of G_x, which is equal to $\frac{q_0^6-1}{q_0-1}$ for the former case, or $q^3 + 1$ for the latter case. But this is impossible, since k also divides $4(v - 1, |G_x|)$.

If $X \cap G_x = (SL_2(q) \circ SL_2(q)) \cdot 2$, then $|X \cap G_x| = q^2(q^2 - 1)^2$ and $v = q^4(q^4 + q^2 + 1)$. Since $(q^2, v - 1) = 1$, $v - 1 = (q^2 - 1)^2(q^4 + 3q^2 + 6) + (9q^2 - 7)$ and $81(q^2 - 1)^2 = (9q^2 - 7)(9q^2 - 11) + 4$, we have

$$4(v - 1, |X \cap G_x|) = 4\big((q^2 - 1)^2, 9q^2 - 7\big) = 4(9q^2 - 7, 4).$$

It follows from Lemma 2.4 that k divides $2^4 f$, and is too small.

If $X \cap G_x = J_2$ with $q = 4$, then $v = 2^5 \cdot 13$. So that $k \mid 4(v - 1, |G_x|) = 20$, and is too small.

If $X \cap G_x = G_2(2)$ with $q = p \geq 5$, then $|X \cap G_x| = 2^6 \cdot 3^3 \cdot 7$. The cube root bound implies $q^{13} < q^6(q^2 - 1)(q^6 - 1) < 2^{18} \cdot 3^9 \cdot 7^3 f^2$, and it follows that $q = 5$ or 7. In both cases $4(v - 1, |G_x|)$ is too small.

If $X \cap G_x = PGL_2(q)$ or $L_2(8)$, then the cube root bound is not satisfied.

If $X \cap G_x = L_2(13)$ with $p \neq 13$, then $|X \cap G_x| = 2^2 \cdot 3 \cdot 7 \cdot 13$. The cube root bound implies $q^6(q^2 - 1)(q^6 - 1) < 2^6 \cdot 3^3 \cdot 7^3 \cdot 13^3 f^2$, and so $q = 3$, 5. If $q = 3$, then $(v - 1, |G_x|) = 13$, hence k is too small. If $q = 5$, then v is not an integer.

If $X \cap G_x = J_1$ with $q = 11$, then $|X \cap G_x| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ and $v = 2^3 \cdot 3^2 \cdot 5 \cdot 11^5 \cdot 37$, $(v - 1, |X \cap G_x|) = 1$, hence the inequality $v < k^2$ cannot be satisfied.

If $X \cap G_x = 2^3.L_3(2)$, then the cube root bound implies $q^6(q^6 - 1)(q^2 - 1) < 1344^3 f^2$, and hence $q = 3$, 5. In both cases, k is too small.

There is no other maximal subgroup G_x satisfying the cube root bound. □

Now we use Theorems 2.11, 2.12, 2.13 to rule out the remaining cases:

$$X \in \{F_4(q), E_6^\epsilon(q), E_7(q), E_8(q)\},$$

where $q = p^e$ and p is a prime (and $q > 2$ if $X = F_4(q)$ or $E_6^\epsilon(q)$). We first give the following lemma.

Corollary 3.7. *Let X, G, G_x and X_0 as in Theorem 2.11. Let $\mathcal{D}$ be a symmetric (v, k, λ) design and $G \leq Aut(\mathcal{D})$ be flag-transitive point-primitive. Then the point stabilizer G_x is not the centralizer of a graph, field, or graph-field automorphism of X of prime order.*

Proof. Suppose that G_x is the centralizer of a graph, field, or graph-field automorphism of X of prime order. Then the conjugacy classes of such automorphisms are known (see [1, §19], [3, Prop. 2.7] or [8, 9-1]). By checking the orders of G_x, it implies that they do not satisfy the cube root bound. □

Corollary 3.8. *The group X is not $F_4(q)$.*

Proof. Suppose by contradiction that $X = F_4(q)$. First assume that $X_0 = Soc(G_x)$ is not simple. Then by Theorem 2.11, Lemmas 3.2 and 3.7, one of the following holds:

(1) G_x is a subgroup of maximal rank;

(2) $G_x = 3^3.SL_3(3)$;

(3) $X_0 = L_2(q) \times G_2(q)$.

Case (1). The possibilities for the first case are given in [17, Table 5.1] (the groups in [17, Table 5.2] are too small). In every case there exists a large power of q dividing v, so $(v-1, q) = 1$. By Lemma 2.4, $k \mid 4(|G_x|, v-1)$, and in each case $(|G_x|_{p'}, v-1)$ is too small for k to satisfy $k^2 > 4v$.

Case (2). It can be ruled out by the cube root bound.

Case (3). Assume that $X_0 = L_2(q) \times G_2(q)$. Clearly, G_x is not simple. By [3, Theorem 1] we know that G_x is not local. Then G_x must be a maximal rank subgroup (also see [30, p.346]) which has been ruled out in Case (1).

Hence X_0 is simple. First assume $X_0 \notin Lie(p)$. Then by [21, Table 1] the possibilities of X_0 are the following:

A_{7-10}, $L_2(17)$, $L_2(25)$, $L_2(27)$, $L_3(3)$, $U_4(2)$, $Sp_6(2)$, $\Omega_8^+(2)$, ${}^3D_4(2)$, J_2, $A_{11}(p = 11)$, $L_3(4)(p = 3)$, $L_4(3)(p = 2)$, ${}^2B_2(8)(p = 5)$, $M_{11}(p = 11)$.

The only possibilities for X_0 that could satisfy the cube root bound are A_9, $A_{10}(q = 2)$, $Sp_6(2)(p = 2)$, $\Omega_8^+(2)(p = 2, 3)$, ${}^3D_4(2)(p = 2, 3)$, $J_2(q = 2)$, $L_4(3)(q = 2)$. However, by Lemma 2.4, $k \mid 4(|G_x|, v-1)$, in all these cases $k^2 < v$.

Now assume $X_0 = X_0(r) \in Lie(p)$. If $rank(X_0) > \frac{1}{2} rank(G)$, then if $r > 2$ by Theorem 2.12, $X \cap G_x$ is a subfield subgroup. The only possibilities that satisfy the cube root bound are $F_4(q^{\frac{1}{2}})$ and $F_4(q^{\frac{1}{3}})$. For the former case, $v = q^{12}(q^6+1)(q^4+1)(q^3+1)(q+1) > q^{26}$. Now $k \mid |F_4(q^{\frac{1}{2}})|$, and $(k, v) \mid 4$. Since $(q, k) \le 4$, then k divides

$$4(f(q^6-1)(q^4-1)(q^3-1)(q-1), v-1) < q^{13},$$

and so $k^2 < v$, a contradiction.

For the latter case, $v = \frac{q^{16}(q^{12}-1)(q^4+1)(q^6-1)}{(q^{\frac{8}{3}}-1)(q^{\frac{2}{3}}-1)}$. But $k < 4q^7 q^{\frac{10}{3}}$, and then $k^2 < v$.

If $r = 2$, then the subgroups $X_0(2)$ with $\mathrm{rank}(X_0) > \frac{1}{2}\mathrm{rank}(G)$ that satisfy the cube root bound are $B_3(2)$, $B_4(2)$, $C_3(2)$, $C_4(2)$, $D_4^{\epsilon}(2)$. However, in all cases $k \mid 4(|G_x|, v-1)$ forces $k^2 < v$.

If $\mathrm{rank}(X_0) \le \frac{1}{2}\mathrm{rank}(G)$, then Theorem 2.13 implies $|G_x| < 4eq^{20}$. By checking the order of groups of Lie type, we see that if $|G_x| < 4eq^{20}$, then $|G_x|_{p'} < 4e$, and so $4|G_x||G_x|_{p'}^2 < |G|$, contradicting Corollary 2.7. □

Corollary 3.9. *The group X is not $E_6^{\epsilon}(q)$.*

Proof. Suppose by contradiction that $X = E_6^\epsilon(q)$. First assume X_0 is not simple. Then by Theorem 2.11, Lemmas 3.2 and 3.7, one of the following holds:

(1) G_x is a subgroup of maximal rank;

(2) $G_x = 3^{3+3}.SL_3(3)$;

(3) $X_0 = L_3(q) \times G_2(q)$, $U_3(q) \times G_2(q)(q > 2)$.

Case (1). The possibilities for the case are given in [17, Table 5.1]. Some cases can be ruled out by the cube root bound, and in each the remaining cases, calculating $4(|G_x|, v-1)$ we get $k^2 < v$.

Case (2). It can be ruled out by the cube root bound.

Case (3). Assume that Case (3) holds. We have known that G_x is not local, and it is also not simple. Then G_x must be a maximal rank subgroup (also see [30, p.346]), a case already considered.

Hence X_0 is simple. First consider the case $X_0 \notin \mathrm{Lie}(p)$. Then we find the possibilities of X_0 in [21, Table 1]. The cases which satisfy Corollary 2.2 are A_{11}, $U_4(3)$, ${}^2F_4(2)'$, A_{12}, $\Omega_7(3)$, J_3, $F_{i_{22}}$, $\Omega_8^+(2)$, ${}^3D_4(2)$, $L_4(3)(p = 2)$. In the cases of A_{11}, A_{12}, $\Omega_7(3)$, J_3, $F_{i_{22}}$ have orders that does not divide $|E_6(2)|$. In other cases which k is too small to satisfy $v < k^2$.

Now assume $X_0 = X_0(r) \in \mathrm{Lie}(p)$. If $\mathrm{rank}(X_0) > \frac{1}{2}\mathrm{rank}(G)$, then if $r > 2$ by Theorem 2.12 the only possibilities are $E_6^\epsilon(q^{\frac{1}{s}})$ with $s = 2$ or 3, $C_4(q)$ and $F_4(q)$. In all cases k is too small. If $r = 2$, then the possibilities satisfying the cube root bound with order dividing $|E_6^\epsilon(2)|$ are $A_5^\epsilon(2)$, $B_4(2)$, $C_4(2)$, $D_4^\epsilon(2)$ and $D_5(2)$. However, in all cases $k \mid 4(|G_x|, v-1)$ forces $k^2 < v$, which is a contradiction.

If $\mathrm{rank}(X_0) \leq \frac{1}{2}\mathrm{rank}(G)$, then Theorem 2.13 implies $|G_x| < 4eq^{28}$. By checking the p-part and p'-part of the order of the possible subgroups, we see that the p'-part is always less than $4e$ and so $|G_x|_{p'} < 4e$, so $4|G_x||G_x|_{p'}^2 < |G|$, contradicting Corollary 2.7. □

Corollary 3.10. *The group X is not $E_7(q)$.*

Proof. Suppose by contradiction that $X = E_7(q)$ with $q = p^e$. First assume X_0 is not simple. Then by Theorem 2.11, Lemmas 3.2 and 3.7, one of the following holds:

(1) G_x is a subgroup of maximal rank;

(2) $X_0 = L_2(q) \times L_2(q)(p > 3)$, $L_2(q) \times G_2(q)(p > 2, q > 3)$, $L_2(q) \times F_4(q)(q > 3)$, $G_2(q) \times PSp_6(q)$.

Case (1). From [17, Table 5.1] the only subgroups of maximal rank satisfy the cube root bound are $d.(L_2(q) \times P\Omega_{12}^+(q)).d$, $h.(L_8^\epsilon(q).g.(2 \times (2/h))$ and $c.(E_6^\epsilon(q) \times (q - \epsilon/c)).c.2$, where $d = (2, q-1)$, $\epsilon = \pm$, $h = (4, q-\epsilon)/d$, $c = (3, q-\epsilon)$ and $g = (8, q - \epsilon)/d$.

If $G_x = d.(L_2(q) \times P\Omega_{12}^+(q)).d$, then

$$\begin{aligned} |G_x| &= \frac{1}{d} q^{31}(q^2-1)^2(q^4-1)(q^6-1)^2(q^8-1)(q^{10}-1), \\ v &= q^{32}(q^4-q^2+1)(q^{12}+q^{10}+q^8+q^6+q^4+q^2+1) \\ &\quad \times(q^{16}+q^{14}+q^{12}+q^{10}+q^8+q^6+q^4+q^2+1). \end{aligned}$$

Clearly, from $v_p = q^{32}$ and $q^4-q^2+1 \mid v$ we have $(k, q^4-q^2+1) = 1$. Since $(k,v) \mid 4$ and

$$v-1 \equiv 2\times 31 \pmod{q\pm 1}, \qquad v-1 \equiv 2 \pmod{q^2+1},$$
$$v-1 \equiv 0 \pmod{q^4+1}, \qquad v-1 \equiv -1 \pmod{q^4+q^2+1},$$

by combining these with the fact k divides $4(v-1, |G_x|)$ (Lemma 2.4), we obtain

$$k \le 2^{16}\cdot 31^{14} e(q^4+1)(q^8+q^6+q^4+q^2+1),$$

where $f \mid de$. From $4v < k^2$ we get the pairs of (p,e) are $(2,1)$, $(2,2)$, $(2,3)$, $(2,4)$, $(3,1)$, $(3,2)$, $(5,1)$, $(7,1)$, $(11,1)$, $(13,1)$, $(17,1)$ with $q = p^e$. By computing the value of k when $q = p^e$ we known that k is too small to satisfy $k^2 > 4v$, a contradiction. Similarly, $G_x \ncong h.(L_8^\epsilon(q).g.(2\times(2/h))$, $c.(E_6^\epsilon(q)\times(q-\epsilon/c)).c.2$.

Case (2). Assume that X_0 is one of the groups listed in (2). Then it follows from [3, Theorem 1] that G_x is not local, and it is also not simple, so G_x must be a maximal subgroup, which has been ruled out in Case (1). Hence X_0 is simple.

Assume that $X_0 \notin \mathrm{Lie}(p)$. Then by [21, Table 1], the only group satisfying $|G_x|^3 > 4|G|$ is $Fi_{22}(p=2)$, but simple calculation implies k is too small. Now assume $X_0 = X_0(r) \in \mathrm{Lie}(p)$. If $\mathrm{rank}(X_0) \le \frac{1}{2}\mathrm{rank}(G)$, then by Theorem 2.13 we have $|G_x|^3 \le |G|$, which contradicts the cube root bound. So $\mathrm{rank}(X_0) > \frac{1}{2}\mathrm{rank}(G)$. If $r > 2$, then by Theorem 2.12, $G_x \cap X = E_7(q^{\frac{1}{s}})$ with $s = 2$ or 3. However in both cases $(v,k) \mid 4$ forces $k^2 < v$. If $r = 2$, then $rank(X_0) \ge 5$. The groups satisfying the cube root bound and having order dividing $|E_7(2)|$ are $A_6^\epsilon(2)$, $A_7^\epsilon(2)$, $B_5(2)$, $C_5(2)$, $D_5^\epsilon(2)$, and $D_6(2)$. However, in all cases $k^2 < v$. □

Corollary 3.11. *The group X is not $E_8(q)$.*

Proof. Suppose by contradiction that $X = E_8(q)$. First assume X_0 is not simple. Then by Theorem 2.11, Lemmas 3.2 and 3.7, one of the following holds:

(1) G_x is a subgroup of maximal rank;
(2) $(X, X\cap G_x) = (E_8(p), 2^{5+10}.SL_5(2))$ or $(E_8(p^a), 5^3.SL_3(5))$, where $p \ne 2,5$, $a = 1$ if $5 \mid q^2-1$ and $a = 2$ if $5 \mid q^2+1$;
(3) $X \cap G_x = (A_5\times A_6).2^2$;
(4) $X_0 = L_2(q)\times L_3^\epsilon(q)(p>3)$, $G_2(q)\times F_4(q)$, $L_2(q)\times G_2(q)\times G_2(q)$ $(p>2, q>3)$, or $L_2(q)\times G_2(q^2)(p>2, q>3)$.

Case (1). By [17, Table 5.1] and Corollary 2.7, the only subgroups of maximal rank such that $4|G_x||G_x|_{p'}^2 > |G|$ are $d.P\Omega_{16}^{+}(q).d$, $d.(L_2(q) \times E_7(q)).d$, where $d = (2, q-1)$, $\epsilon = \pm 1$. If $G_x = d.P\Omega_{16}^{+}(q).d$, then

$$\begin{aligned} |G_x| =& q^{56}(q^2-1)^6(q^2+1)^4(q^4+1)^2(q^4+q^2+1) \\ & \times (q^8+q^6+q^4+q^2+1)(q^8+q^4+1)(q^{14}-1), \\ v =& q^{64}(q^8+q^6+q^4+q^2+1)(q^{20}+q^{10}+1)(q^8-q^4+1) \\ & \times (q^8+q^4+1)(q^8-q^6+q^4-q^2+1)(q^{12}+q^6+1). \end{aligned}$$

On the one hand, since q^8+q^4+1, $q^8+q^6+q^4+q^2+1$ and q can divide $(v, |X \cap G_x|)$, by Lemma 2.4, $k \mid 4(v-1, |G_x|)$ and we get $k \leq 4f(q^2-1)^6(q^2+1)^4(q^4+1)^2(q^{14}-1) < 4fq^{42}$. On the other hand, $v > q^{128}$, and so $k^2 < v$, a contradiction. Similarly, $G_x \ncong d.(L_2(q) \times E_7(q)).d$.

Case (2)-(3). These cases can be ruled out by the cube root bound.

Case (4). This case can be ruled out as Case (2) in Lemma 3.10.

Hence X_0 is simple. First suppose that $X_0 \notin \mathrm{Lie}(p)$. Then by [21, Table 1] the possibilities X_0 in every case the cube root bound is not satisfied.

Now suppose that $X_0 \in \mathrm{Lie}(p)$. If $\mathrm{rank}(X_0) \leq \frac{1}{2}\mathrm{rank}(G)$, then by Theorem 2.13 we have $|G_x|^3 < |G|$, which contradicts the cube root bound. So $\mathrm{rank}(X_0) > \frac{1}{2}\mathrm{rank}(G)$. If $r > 2$, then by Theorem 2.12, $G_x \cap X$ is a subfield subgroup. The only cases in which the cube root bound can be satisfied are when $q = q_0^2$ or $q = q_0^3$, but in all cases we have $k^2 < 4v$. If $r = 2$, then $\mathrm{rank}(X_0) \geq 5$. The groups satisfy the cube root bound are $A_8^{\epsilon}(2)$, $B_7(2)$, $C_7(2)$, $D_8(2)$, and $D_7^{\epsilon}(2)$. However, in all cases, it is easy to know that k is too small. □

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References

[1] M. Aschbacher and G.M. Seitz, Involutions in Chevalley groups over fields of even order, *Nagoya Math. J.* **63** (1976) 1–91.

[2] E. Bannai, S. Hao and S.Y. Song, Character tables of the association schemes of finite orthogonal groups on the non-isotropic points, *J. Combin. Theory Ser. A* **54** (1990) 164–200.

[3] A.M. Cohen, M.W. Liebeck, J. Saxl, and G.M. Seitz, The local maximal subgroups of exceptional groups of Lie type, finite and algebraic, *Proc. Lond. Math. Soc.* **64** (1992) 21–48.

[4] B.N. Cooperstein, Maximal subgroups of $G_2(2^n)$, *J. Algebra* **70** (1981) 23–36.

[5] H. Dong and S. Zhou, Alternating groups and flag-transitive 2-$(v, k, 4)$ symmetric designs, *J. Combin. Des.* **19** (2011) 475–483.
[6] H. Davies, Flag-transitive and primitivity, *Discrete Math.* **63** (1987), no. 1, 91–93.
[7] W. Fang, H. Dong and S. Zhou, Flag-transitive 2-$(v, k, 4)$ symmetric designs, *Ars Combin.* **95** (2010) 333–342.
[8] D. Gorenstein and R. Lyons, The local structure of finite groups of characteristic 2 type, *Mem. Amer. Math. Soc.* **42** (1983), no. 276, 731 pages.
[9] W.M. Kantor, Primitive permutation groups of odd degree, and an application to finite projective planes, *J. Algebra* **106** (1987), no. 1, 15–45.
[10] P.B. Kleidman, The maximal subgroups of the Steinberg groups ${}^3D_4(q)$ and of their automorphism groups, *J. Algebra* **115** (1988), no. 1, 182–199.
[11] P.B. Kleidman, The maximal subgroups of the Chevalley groups $G_2(q)$ with q odd, the Ree groups ${}^2G_2(q)$, and their automorphism groups, *J. Algebra* **117** (1998), no. 1, 30–71.
[12] P.B. Kleidman and M.W. Liebeck, The Subgroup Structure of the Finite Classical Groups, London Math. Soc. Lecture Note Ser. 129, Cambridge Univ. Press, Cambridge, 1990.
[13] M. Law, C.E. Praeger and S. Reichard, Flag-transitive symmetric 2-$(96, 20, 4)$-designs, *J. Combin. Theory Ser. A* **116** (2009), no. 5, 1009–1022.
[14] M.W. Liebeck, C.E. Praeger and J. Saxl, The maximal factorizations of the finite simple groups and their automorphism groups, *Mem. Amer. Math. Soc.* **86** (1990), no. 432, 151 pages.
[15] M.W. Liebeck and J. Saxl, The primitive permutation groups of odd degree, *J. Lond. Math. Soc.* **31** (1985), no. 2, 250–264.
[16] M.W. Liebeck and J. Saxl, The finite primitive permutation groups of rank three, *Bull. Lond. Math. Soc. (2)* **18** (1986), no. 2, 165–172.
[17] M.W. Liebeck, J. Saxl and G.M. Seitz, Subgroups of maximal rank in finite exceptional groups of Lie type, *Proc. Lond. Math. Soc. (3)* **65** (1992), no. 2, 297–325.
[18] M.W. Liebeck, J. Saxl and G.M. Seitz, On the overgroups of irreducible subgroups of the finite classical groups, *Proc. Lond. Math. Soc. (3)* **55** (1987), no. 3, 507–537.
[19] M.W. Liebeck, J. Saxl and D.M. Testerman, Simple subgroups of large rank in groups of Lie type, *Proc. Lond. Math. Soc. (3)* **72** (1996), no. 2, 425–457.
[20] M.W. Liebeck and G.M. Seitz, Maximal subgroups of exceptional groups of Lie type, finite and algebraic, *Geom. Dedicata* **35** (1990), no. 1-3, 353–387.
[21] M.W. Liebeck and G.M. Seitz, On finite subgroups of exceptional algebraic groups, *J. Reine Angew. Math.* **515** (1999) 25–72.
[22] M.W. Liebeck and A. Shalev, The probability of generating a finite simple group, *Geom. Dedicata* **56** (1995), no. 1, 103–113.
[23] M.W. Liebeck, C.E. Praeger and J. Saxl, On the 2-closures of finite permutation groups, *J. Lond. Math. Soc. (2)* **37** (1988), no. 2, 241–252.
[24] G. Malle, The maximal subgroups of ${}^2F_4(q^2)$, *J. Algebra* **139** (1991), no. 1, 52–69.
[25] E. O'Reilly-Regueiro, Reduction for primitive flag-transitive $(v, k, 4)$-symmetric designs, *Des. Codes Cryptogr.* **56** (2010), no. 1, 61–63.
[26] E. O'Reilly-Regueiro, On primitivity and reduction for flag-transitive symmetric designs, *J. Combin. Theory Ser. A* **109** (2005), no. 1, 135–148.
[27] E. O'Reilly-Regueiro, Biplanes with flag-transitive automorphism groups of almost simple type, with alternating or sporadic socle, *European J. Combin.* **26** (2005), no. 5, 577–584.
[28] E. O'Reilly-Regueiro, Biplanes with flag-transitive automorphism groups of almost simple type, with classical socle, *J. Algebraic Combin.* **26** (2007), no. 4, 529–552.

[29] E. O'Reilly-Regueiro, Biplanes with flag-transitive automorphism groups of almost simple type, with exceptional socle of Lie type, *J. Algebraic Combin.* **27** (2008), no. 4, 479–491.
[30] J. Saxl, On finite linear spaces with almost simple flag-transitive automorphism groups, *J. Combin. Theory Ser. A* **100** (2002), no. 2, 322–348.
[31] G.M. Seitz, Flag-transitive subgroups of Chevalley groups, *Ann. of Math. (2)* **97** (1973), no. 1, 27–56.
[32] M. Suzuki, On a class of doubly transitive groups, *Ann. of Math. (2)* **75** (1962) 105–145.
[33] S. Zhou and H. Dong, Sporadic groups and flag-transitive triplanes, *Sci. China Ser. A* **52** (2009), no. 2, 394–400.
[34] S. Zhou and H. Dong, Exceptional groups and flag-transitive triplanes, *Sci. China Math.* **53** (2010), no. 2, 447–456.
[35] S. Zhou and H. Dong, Finite classical groups and flag-transitive triplanes, *Discrete Math.* **309** (2009), no. 16, 5183–5195.
[36] S. Zhou and H. Dong, Alternating groups and flag-transitive triplanes, *Des. Codes Cryptogr.* **57** (2010), no. 2, 117–126.
[37] S. Zhou and D. Tian, Flag-transitive point-primitive 2-$(v,k,4)$ symmetric designs and the two dimensional classical groups, *Appl. Math. J. Chinese Univ. Ser. B* **26** (2011), no. 3, 334–341.

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