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CONNECTIONS BETWEEN LABELLINGS OF TREES

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Abstract. There are many long-standing conjectures related with some labellings of trees. It is important to connect labellings that are related with conjectures. We find some connections between known labellings of simple graphs.
Keywords: Trees, (odd-)graceful labellings, felicitous lalbellings, (k,d)-graceful labellings.
MSC(2010): Primary: 05C78.

1. **Introduction and concepts**

As known, the cycle C_5 of length 5 has no graceful, odd-graceful and (k,d)-graceful labellings, but C_5 admits edge-magic total labellings; the complete graph K_4 does not admit odd-graceful and edge-magic total labellings, however it has a graceful labelling. On the other hands, there are many long-standing conjectures related with some labellings of trees. In this paper, we will show a couple of connections among known labellings of trees. Standard terminology and notation of graph theory are used here. Graphs mentioned have no multiple edges, and are loopless, undirected and finite. A (p,q)-graph has p vertices and q edges. The cardinality of elements of a set S is denoted as $|S|$. The shorthand symbol $[m,n]$ stands for an integer set $\{m, m+1, \dots, n\}$, where m and n are integers with $0 \leq m < n$. In Definition 1.1 we restate several known labellings, that can be found in [2] and [4].

Definition 1.1. Suppose that a connected (p,q)-graph G admits a mapping $\theta : V(G) \to \{0,1,2,\dots\}$. For edges $xy \in E(G)$ the induced edge labels are defined by $\theta(xy) = |\theta(x) - \theta(y)|$. Write $\theta(V(G)) = \{\theta(u) : u \in V(G)\}$, $\theta(E(G)) = \{\theta(xy) : xy \in E(G)\}$. There are the following constraints:
(a) $|\theta(V(G))| = p$.
(b) $|\theta(E(G))| = q$.

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(c) $\theta(V(G)) \subseteq [0, q]$, $\min \theta(V(G)) = 0$.
(d) $\theta(V(G)) \subset [0, 2q-1]$, $\min \theta(V(G)) = 0$.
(e) $\theta(E(G)) = \{\theta(xy) : xy \in E(G)\} = [1, q]$.
(f) $\theta(E(G)) = \{\theta(xy) : xy \in E(G)\} = \{1, 3, 5, \dots, 2q-1\}$.
(g) G is a bipartite graph with the bipartition (X, Y) such that $\max\{\theta(x) : x \in X\} < \min\{\theta(y) : y \in Y\}$ ($\theta(X) < \theta(Y)$ for short).
(h) G is a tree containing a perfect matching M such that $\theta(x) + \theta(y) = q$ for each edge $xy \in M$.
(i) G is a tree having a perfect matching M such that $\theta(x) + \theta(y) = 2q - 1$ for each edge $xy \in M$.

Then θ is a *graceful labelling* if it holds (a), (c) and (e); θ is a *set-ordered graceful labelling* if it holds (a), (c), (e) and (g); θ is a *strongly graceful labelling* if it holds (a), (c), (e) and (h); θ is a *strongly set-ordered graceful labelling* if it holds (a), (c), (e), (g) and (h). Also θ is an *odd-graceful labelling* if it holds (a), (d) and (f); θ *set-ordered odd-graceful labelling* if it holds (a), (d), (f) and (g); θ is a *strongly odd-graceful labelling* if it holds (a), (d), (f) and (i); θ is a *strongly set-ordered odd-graceful labelling* if it holds (a), (d), (f), (g) and (i). □

In [6], the authors showed that a connected bipartite graph H admits a (strongly) set-ordered graceful labelling if and only if H admits a (strongly) set-ordered odd-graceful labelling. Definition 1.2 presents the graph labellings that will be used in this article.

Definition 1.2. Let G be a (p, q)-graph having p vertices and q edges, and let $S_{k,d} = \{k, k+d, \dots, k+(q-1)d\}$ for integers $k \geq 1$, $d \geq 1$.

(1) [2] A *felicitous labelling* f of G hold $f(V(G)) \subseteq [0, q]$, $f(x) \neq f(y)$ for distinct $x, y \in V(G)$ and $f(E(G)) = \{f(uv) = f(u) + f(v) \pmod{q} : uv \in E(G)\} = [0, q-1]$; and furthermore, f is *super* if $f(V(G)) = [0, p-1]$.

(2) [3] A *(k, d)-graceful labelling* f of G hold $f(V(G)) \subseteq [0, k+(q-1)d]$, $f(x) \neq f(y)$ for distinct $x, y \in V(G)$ and $\pi(E(G)) = \{|\pi(u) - \pi(v)|;\ uv \in E(G)\} = S_{k,d}$. Especially, a $(k, 1)$-graceful labelling is also a k-graceful labelling.

(3) [2] An *edge-magic total labelling* f of G hold $f(V(G) \cup E(G)) = [1, p+q]$ such that for any edge $uv \in E(G)$, $f(u) + f(v) + f(uv) = c$, where the magic constant c is a fixed integer; and furthermore f is *super* if $f(V(G)) = [1, p]$.

(4) [2] A *(k, d)-edge antimagic total labelling* f of G hold $f(V(G) \cup E(G)) = [1, p+q]$ and $\{f(u) + f(v) + f(uv) : uv \in E(G)\} = S_{k,d}$, and furthermore f is *super* if $f(V(G)) = [1, p]$.

(5) [5] An *odd-elegant labelling* f of G hold $f(V(G)) \subset [0, 2q-1]$, $f(u) \neq f(v)$ for distinct $u, v \in V(G)$, and $f(E(G)) = \{f(uv) = f(u) + f(v) \pmod{2q} : uv \in E(G)\} = \{1, 3, 5, \dots, 2q-1\}$.

(6) [1] A labeling f of G is said to be *(k, d)-arithmetic* if $f(V(G)) \subseteq [0, k+(q-1)d]$, $f(x) \neq f(y)$ for distinct $x, y \in V(G)$ and $\{f(u) + f(v) : uv \in E(G)\} = S_{k,d}$.

(7) [2] A *harmonious labelling* f of G hold $f(V(G)) \subseteq [0,q-1]$, $\min f(V(G)) = 0$ and $f(E(G)) = \{f(uv) = f(u)+f(v) \pmod{q} : uv \in E(G)\} = [0,q-1]$ such that (i) if G is not a tree, $f(x) \neq f(y)$ for distinct $x, y \in V(G)$; (ii) if G is a tree, $f(x) \neq f(y)$ for distinct $x, y \in V(G) \setminus \{w\}$, and $f(w) = f(x_0)$ for some $x_0 \in V(G) \setminus \{w\}$.

2. Main results

Theorem 2.1. *Let T be a tree on p vertices, and let (X, Y) be its bipartition. For all values of integers $k \geq 1$ and $d \geq 1$, the following assertions are mutually equivalent:*

(1) *T admits a set-ordered graceful labelling f with $f(X) < f(Y)$.*

(2) *T admits a super felicitous labelling α with $\alpha(X) < \alpha(Y)$.*

(3) *T admits a (k,d)-graceful labelling β with $\beta(x) < \beta(y) - k + d$ for all $x \in X$ and $y \in Y$.*

(4) *T admits a super edge-magic total labelling γ with $\gamma(X) < \gamma(Y)$ and a magic constant $|X| + 2p + 1$.*

(5) *T admits a super $(|X| + p + 3, 2)$-edge antimagic total labelling θ with $\theta(X) < \theta(Y)$.*

(6) *T has an odd-elegant labelling η with $\eta(x) + \eta(y) \leq 2p - 3$ for every edge $xy \in E(T)$.*

(7) *T has a (k,d)-arithmetic labelling ψ with $\psi(x) < \psi(y) - k + d \cdot |X|$ for all $x \in X$ and $y \in Y$.*

(8) *T has a harmonious labelling φ with $\varphi(X) < \varphi(Y \setminus \{y_0\})$ and $\varphi(y_0) = 0$.*

Proof. Let T be a tree having p vertices and the bipartition (X, Y), where $X = \{u_i : i \in [1,s]\}$ and $Y = \{v_j : j \in [1,t]\}$ with $s + t = p$. Suppose that T has a set-ordered graceful labelling f with $f(u_i) = i - 1$ for $i \in [1,s]$ and $f(v_j) = s - 1 + j$ for $j \in [1,t]$, and $f(u_i v_j) = f(v_j) - f(u_i) = s + j - i - 2$ for each edge $u_i v_j \in E(T)$. Clearly, $f(v_j) + f(v_{t-j+1}) = 2s + t - 1 = s + p - 1$ for $j \in [1,t]$. Notice that $|E(T)| = p - 1$.

(1) $\Rightarrow$ (2) T has a labelling g_1 defined as: $g_1(u_i) = f(u_i)$ for $i \in [1,s]$, $g_1(v_j) = f(v_{t-j+1})$ for $j \in [1,t]$. For each edge $u_i v_j \in E(T)$,

$$\begin{aligned} g_1(u_i) + g_1(v_j) &= f(u_i) + f(v_{t-j+1}) \\ &= f(u_i) + s + p - 1 - f(v_j) \\ &= s + p - 1 - \big[f(v_j) - f(u_i)\big] \\ &= s + p - 1 - f(u_i v_j), \end{aligned} \tag{2.1}$$

we obtain $U_1 = \{s+p-1-1, s+p-1-2, \ldots, s+p-1-(s-1), s+p-1-s\}$ and $U_2 = \{p-2, p-3, \ldots, s\}$. Under modulo $(p-1)$, U_1 distributes a set $U'_1 = [0, s-1]$ from (2.1). Therefore, $f(E(T)) = \{f(u_i v_j) = f(u_i) + f(v_j) \pmod{p-1} : u_i v_j \in E(T)\} = U'_1 \cup U_2 = [0, p-2]$. Clearly, f is a super felicitous labelling, as desired.

(2) $\Rightarrow$ (1) Let T have a super felicitous labelling α with $\alpha(X) < \alpha(Y)$, which induces that $\alpha(u_i) = i-1$ for $i \in [1,s]$ and $\alpha(v_j) = s-1+j$ for $j \in [1,t]$. It is easy to deduce $\alpha(v_j)+\alpha(v_{t-j+1}) = (s-1+j)+(s-1+t-j+1) = s+p-1$, for $j \in [1,t]$. We define a labelling h_1 of T as: $h_1(u_i) = \alpha(u_i) = i-1$, for $i \in [1,s]$, $h_1(v_j) = \alpha(v_{t-j+1}) = p-j$ for $j \in [1,t]$. Clearly, $h_1(X) < h_1(Y)$. We have $h_1(u_iv_j) = |h_1(u_i)-h_1(v_j)| = \alpha(v_{t-j+1})-\alpha(u_i) = s+p-1-\big[\alpha(v_j)+\alpha(u_i)\big] = s+p-1-\big[s-1+j+i-1\big] = p+1-(i+j)$ for each edge $u_iv_j \in E(T)$, which produces $h_1(V(T)) = [1,p-1]$. So, h_1 is a set-ordered graceful labelling.

(1) $\Rightarrow$ (3) Necessity. We extend the set-ordered graceful labelling f to another labelling g_2 of T as follows. Define $g_2(u_i) = d\cdot f(u_i)$ for $i \in [1,s]$, $g_2(v_j) = k+d\cdot\big[f(v_j)-1\big]$ for $j \in [1,t]$. Therefore, $g_2(u_iv_j) = |g_2(u_i)-g_2(v_j)| = k+d\cdot\big[f(v_j)-f(u_i)-1\big]$ for each edge $u_iv_j \in E(T)$, which yields the set $g_2(E(T)) = S_{k,d}$ defined in Definition 1.2. So g_2 is a (k,d)-graceful labelling g_2 with $g_2(x) < g_2(y)-k+d$ for all $x \in X$ and $y \in Y$.

(3) $\Rightarrow$ (1) Suppose that T has a (k,d)-graceful labelling β with $\beta(X) < \beta(Y)$ for all values of integers $k,d \geq 1$. In a path $u_{i_1}v_{j_2}u_{i_3}$ of T, if $\beta(u_{i_1}) = d\cdot a$, then we have $\beta(v_{j_2}) = k+d\cdot c$ and $\beta(u_{i_3}) = d\cdot b$, since $\beta(u_{i_1}v_{j_2}), \beta(v_{j_2}u_{i_3}) \in S_{k,d}$; if $\beta(u_{i_1}) = k+d\cdot a$, it must be that $\beta(v_{j_2}) = d\cdot c$ and $\beta(u_{i_3}) = k+d\cdot b$. Notice that $\beta(X) = \{d\cdot a_i : i \in [1,s]\}$, $\beta(Y) = \{k+d\cdot b_j : j \in [1,t]\}$, and $\{a_i, b_j : i \in [1,s], j \in [1,t]\} = [0,s+t-1] = [0,p-1]$. Therefore, $\beta(u_i) = d\cdot(i-1)$ for $i \in [1,s]$, $\beta(v_j) = k+d\cdot(s-1+j)$ for $i \in [1,t]$, since $\beta(u_i) < \beta(v_j)-k+d$ for all $u_i \in X$ and $v_j \in Y$. We extend the labelling β to a labelling h_2 of T by setting $h_2(u_i) = \frac{1}{d}\beta(u_i)$ for $i \in [1,s]$ and $h_2(v_j) = \frac{1}{d}\big[\beta(v_j)-k\big]+1$ for $i \in [1,t]$. Notice that $h_2(X) < h_2(Y)$, and for each edge $u_iv_j \in E(T)$

(2.2)

$$h_2(u_iv_j) = |h_2(u_i)-h_2(v_j)| = \frac{1}{d}\big[\beta(v_j)-\beta(u_i)-k\big]+1 = \frac{1}{d}\big[\beta(u_iv_j)-k\big]+1.$$

Since every $\beta(u_iv_j) \in S_{k,d}$, the form (2.2) distributes $h_2(E(T)) = [1,p-1]$. Therefore, h_2 is a set-ordered graceful labelling.

(1) $\Rightarrow$ (4) To show that T has a super edge-magic labelling g_3, we define $g_3(u_i) = f(u_i)+1$ for $i \in [1,s]$, $g_3(v_j) = f(v_{t-j+1})+1$ for $j \in [1,t]$, and $g_3(u_iv_j) = p+f(u_iv_j)$ for each edge $u_iv_j \in E(T)$. We compute

$$\begin{aligned} g_3(u_i)+g_3(u_iv_j)+g_3(v_j) &= f(u_i)+p+f(u_iv_j)+f(v_{t-j+1})+2 \\ &= f(u_i)+p+f(u_iv_j)+s+p-f(v_j)+1 \\ &= s+2p+1, \end{aligned}$$

which implies that g_3 is a super edge-magic total labelling having $g_3(X) < g_3(Y)$ and a magic constant $s+2p+1$.

(4) $\Rightarrow$ (1) Suppose that T admits a super edge-magic total labelling γ with $\gamma(X) < \gamma(Y)$ and a magic constant $|X|+2p+1$. Notice that $\gamma(V(T)) = [1,p]$ and $\gamma(E(T)) = [p+1,p+p-1]$. So, $\gamma(u_i) = i$ for $i \in [1,s]$, $\gamma(v_j) = s+j$

for $j \in [1,t]$, and $\gamma(u_i)+\gamma(u_iv_j)+\gamma(v_j) = |X|+2p+1 = s+2p+1$ for each edge $u_iv_j \in E(T)$. We can define a labelling h_3 of T in the way that $h_3(u_i) = \gamma(u_i)-1$ for $i \in [1,s]$, and $h_3(v_j) = \gamma(v_{t-j+1})-1$ for $j \in [1,t]$. For each edge $u_iv_j \in E(T)$ we have

$$\begin{aligned} h_3(u_iv_j) &= h_3(v_j)-h_3(u_i)\\ &= \gamma(v_{t-j+1})-\gamma(u_i)\\ &= s+p+1-\gamma(v_j)-\gamma(u_i)\\ &= s+p+1-[s+2p+1-\gamma(u_iv_j)]\\ &= \gamma(u_iv_j)-p, \end{aligned}$$

which distributes $h_3(E(T)) = [1,p-1]$, since $\gamma(u_iv_j) \in [p+1,p+p-1]$. Hence, h_3 is a set-ordered graceful labelling.

(1) $\Rightarrow$ (5) We define a labelling g_4 in the way that $g_4(u_i) = f(u_i)+1$ for $i \in [1,s]$, $g_4(v_j) = f(v_{t-j+1})+1 = s+p-f(v_j)$ for $j \in [1,t]$, and $g_4(u_iv_j) = 2p-f(u_iv_j)$ for each edge $u_iv_j \in E(T)$. Notice that $g_4(V(T)) = [1,p]$. We have $g_4(u_i)+g_4(u_iv_j)+g_4(v_j) = s+3p+1-2f(u_iv_j)$, which induces a set $\{p+s+3, p+s+3+2, p+s+3+4, \dots, p+s+3+2(p-2)\}$. Therefore, g_4 is a super $(s+p+3,2)$-edge antimagic total labelling.

(5) $\Rightarrow$ (1) Suppose that T admits a super $(|X|+p+3,2)$-edge antimagic total labelling θ with $\theta(X) < \theta(Y)$. Notice that $\theta(u_i) = i$ for $i \in [1,s]$, $\theta(v_j) = s+j$ for $j \in [1,t]$. Since $\theta(u_iv_j) \in [p+1,p+p-1]$ for each edge $u_iv_j \in E(T)$, we can write $\theta(u_iv_j) = p+\lambda_{i,j}$ for $\lambda_{i,j} \in [1,p-1]$. For each edge $u_iv_j \in E(T)$, the form $\theta(u_i)+\theta(u_iv_j)+\theta(v_j) = s+p+i+j+\lambda_{i,j} \in W$, where $W = \{p+s+3, p+s+3+2, p+s+3+4, \dots, p+s+3+2(p-2)\}$, induces $i+j+\lambda_{i,j} \in \{3,5,7,\dots,3+2(p-2)\}$. Hence $i+j \in [2,p]$. Next, we define a labelling h_4 of T as: $h_4(u_i) = \theta(u_i)-1 = i-1$ for $i \in [1,s]$, $h_4(v_j) = \theta(v_{t-j+1})-1 = s+p-\theta(v_j) = p-j$ for $j \in [1,t]$. Furthermore, all edges $u_iv_j \in E(T)$ hold $h_4(u_iv_j) = h_4(v_j)-h_4(u_i) = p+1-(i+j)$, which yields $h_4(E(T)) = [1,p-1]$. So, h_4 is a set-ordered graceful labelling, as desired.

(1) $\Rightarrow$ (6) We define a labelling g_5 of T by setting $g_5(u_i) = 2f(u_i)$ for $i \in [1,s]$, $g_5(v_j) = 2p-1-2f(v_j)$ for $j \in [1,t]$. Hence, $g_5(u_i)+g_5(v_j) = 2p-1-2[f(v_j)-f(u_i)] = 2p-1-2f(u_iv_j)$ for each edge $u_iv_j \in E(T)$, which implies $g_5(E(T)) = \{1,3,5,\dots,2p-3\}$. So, g_5 is an odd-elegant labelling with $g_5(u_i)+g_5(v_j) \leq 2p-3$ for each edge $u_iv_j \in E(T)$.

(6) $\Rightarrow$ (1) Suppose that T admits an odd-elegant labelling η with $\eta(u_i)+\eta(v_j) \leq 2p-3$ for every edge $u_iv_j \in E(T)$. Since $\eta(E(T)) = \{\eta(u_iv_j) = \eta(u_i)+\eta(v_j) \pmod{2p-2} : u_iv_j \in E(T)\} = \{1,3,5,\dots,2p-3\}$, so the vertices of X have the same parity, so do the vertices of Y. Without loss of generality, we may assume that each vertex $u_i \in X$ is even, and each vertex $v_j \in Y$ is odd.

It is straightforward to define a labelling h_5 of T as: $h_5(u_i) = \frac{1}{2}\eta(u_i)$ for $i \in [1,s]$, and $h_5(v_j) = \frac{1}{2}\big[2p-1-\eta(v_j)\big]$ for $j \in [1,t]$. Notice that $2p-1-\eta(v_j) >$

$\eta(u_i)$ by the assumption of $\eta(u_i)+\eta(v_j)\leq 2p-3$ for each edge $u_iv_j\in E(T)$. Since $h_5(u_iv_j)=|h_5(u_i)-h_5(v_j)|=\frac{1}{2}\big[2p-1-\eta(v_j)-\eta(u_i)\big]$, we can confirm $h_5(E(T))=[1,p-1]$. Hence, h_5 is a graceful labelling with $h_5(X)<h_5(Y)$.

(1) $\Rightarrow$ (7) We extend the set-ordered graceful labelling f to another labelling g_6 of T by setting $g_6(u_i)=d\cdot f(u_i)$ for $i\in[1,s]$, and $g_6(v_j)=k+d\cdot\big[f(v_{t-j+1})-s\big]=k+d\cdot\big[p-1-f(v_j)\big]$ for $j\in[1,t]$. Hence, $g_6(u_iv_j)=g_6(u_i)+g_6(v_j)=k+d\cdot(p-1)-d\cdot\big[f(v_j)-f(u_i)\big]$ for each edge $u_iv_j\in E(T)$, which yields the set $g_6(E(T))=S_{k,d}$. It follows that g_6 is a (k,d)-arithmetic labelling g_6 with $g_6(x)<g_6(y)-k+d\cdot s$ for all $x\in X$ and $y\in Y$.

(7) $\Rightarrow$ (1) Suppose that T has a (k,d)-arithmetic labelling ψ with $\psi(x)<\psi(y)-k+d\cdot s$ for all $x\in X$ and $y\in Y$, and all values of integers $k,d\geq 1$. For every path xyz of T, if $\psi(x)=d\cdot a_x$, we have $\psi(y)=k+d\cdot c_y$ and $\psi(z)=d\cdot b_z$, since $\psi(xy),\psi(yz)\in S_{k,d}$ (resp. on the other hands, if $\psi(x)=k+d\cdot a_x$, it must be that $\psi(y)=d\cdot c_y$ and $\psi(z)=k+d\cdot b_z$). Therefore, $\psi(X)=\{d\cdot a_i: i\in[1,s]\}$, $\psi(Y)=\{k+d\cdot b_j: j\in[1,t]\}$, and $\{a_i,b_j:\ i\in[1,s],j\in[1,t]\}=[0,s+t-1]=[0,p-1]$. So, we have $\psi(u_i)=d\cdot(i-1)$ for $i\in[1,s]$, $\psi(v_j)=k+d\cdot(t-j)$ for $j\in[1,t]$ since $\psi(u_i)<\psi(v_j)-k+d\cdot s$ for all $u_i\in X$ and $v_j\in Y$. Notice that $\psi(v_j)+\psi(v_{t-j+1})=2k+d\cdot(t-1)$ for $j\in[1,t]$. We extend the labelling ψ to a labelling h_6 of T by setting $h_6(u_i)=\frac{1}{d}\psi(u_i)$ for $i\in[1,s]$ and $h_6(v_j)=\frac{1}{d}\big[\psi(v_{t-j+1})-k\big]+s$ for $i\in[1,t]$. Clearly, $h_6(X)<h_6(Y)$. For each edge $u_iv_j\in E(T)$, we have $h_6(u_i)<h_6(v_j)$ and

$$\begin{aligned}h_6(u_iv_j)&=|h_6(u_i)-h_6(v_j)|\\&=\frac{1}{d}\big[\psi(v_j)-\psi(u_i)-k\big]+s\\&=\frac{1}{d}\big[2k+d\cdot(t-1)-\psi(v_j)-\psi(u_i)-k\big]+s\\&=\frac{1}{d}\big[k+d\cdot(p-1)-\psi(u_iv_j)\big].\end{aligned}\tag{2.3}$$

Since $\psi(E(T))=S_{k,d}$, the form (2.3) distributes $h_6(E(T))=[1,p-1]$. We conclude that h_6 is a set-ordered graceful labelling.

(1) $\Leftrightarrow$ (8) To show the proof of "if", we define a labelling g_7 of T in the way that $g_7(u_i)=f(u_i)$ for $i\in[1,s]$, $g_7(v_j)=f(v_{t-j+1})$ for $j\in[1,t-1]$, and $g_7(v_t)=0$. For each edge $u_iv_j\in E(T)$, We have (2.1) if $j\neq t$. For each edge $u_kv_t\in E(T)$, we have

$$\begin{aligned}g_7(u_k)+g_7(v_t)&=g_7(u_k)+0=g_7(u_k)+(p-1)\ (\mathrm{mod}\ \ p-1)\\&=f(u_k)+f(v_t)\ (\mathrm{mod}\ \ p-1)\\&=f(u_k)+s+p-1-f(v_1)\ (\mathrm{mod}\ \ p-1)\\&=s+p-1-\big[f(v_1)-f(u_k)\big]\ (\mathrm{mod}\ \ p-1)\\&=s-f(u_kv_1).\end{aligned}\tag{2.4}$$

Two forms (2.1) and (2.4) give us two sets $\{s+p-1-1, s+p-1-2, \dots, s+p-1-(s-1), s+p-1-s\}$ and $\{p-2, p-3, \dots, s\}$. Under modulo $(p-1)$, $g_7(E(T)) = \{g_7(u_iv_j) = g_7(u_i)+g_7(v_j) \pmod{p-1} : u_iv_j \in E(T)\} = [0, p-2]$. Therefore, g_7 is a harmonious labelling.

To show the proof of "only if", we take a harmonious labelling φ of T with $\varphi(X) < \varphi(Y\{v_t\})$ and $\varphi(v_t) = 0$, which induces that $\varphi(u_i) = i-1$ for $i \in [1, s]$ and $\varphi(v_j) = s-1+j$ for $j \in [1, t-1]$. We define a new labelling φ' by setting $\varphi'(x) = \varphi(x)$ for $x \in V(T) \setminus \{v_t\}$ and $\varphi'(v_t) = p-1$. Clearly, $\varphi'(v_j)+\varphi'(v_{t-j+1}) = (s-1+j)+(s-1+t-j+1) = s+p-1$ for $j \in [1,t]$ for $j \neq t$. We define a labelling h_7 of T as: $h_7(u_i) = \varphi'(u_i) = i-1$ for $i \in [1, s]$, $h_7(v_j) = \varphi'(v_{t-j+1}) = s+p-1-\varphi'(v_j) = p-j$ for $j \in [1, t]$. So, $h_7(X) < h_7(Y)$. Furthermore, we have $h_7(u_iv_j) = |h_7(u_i) - h_7(v_j)| = \varphi'(v_{t-j+1}) - \varphi'(u_i) = s+p-1-\left[\varphi'(v_j)+\varphi'(u_i)\right] = s+p-1-\left[s-1+j+i-1\right] = p+1-(i+j)$ for each edge $u_iv_j \in E(T)$, which induces $h_7(E(T)) = [1, p-1]$. As a result, h_7 is a set-ordered graceful labelling. Now the proof of Theorem 2.1 is completed. □

Based on the proof of Theorem 2.1 we can prove the following results.

Corollary 2.2. *Let T be a tree having p vertices and a perfect matching M, and let (X, Y) be its bipartition. For all values of integers $k \geq 1$ and $d \geq 1$, the following assertions are mutually equivalent:*

(1) *T admits a strongly set-ordered graceful labelling f, with $f(X) < f(Y)$.*

(2) *T admits a super felicitous labelling α such that $\alpha(X) < \alpha(Y)$ and $\alpha(y) - \alpha(x) = |X|$ for $xy \in M$, with $x \in X$ and $y \in Y$.*

(3) *T admits a (k, d)-graceful labelling β such that $\beta(x) < \beta(y) - k + d$ for all $x \in X$ and $y \in Y$ and $\beta(u) + \beta(v) = k + (p-2)d$ for $uv \in M$.*

(4) *T admits a super edge-magic total labelling γ such that $\gamma(X) < \gamma(Y)$ and a magic constant $|X| + 2p + 1$ as well as $\gamma(v) - \gamma(u) = |X|$ for $uv \in M$, with $u \in X$ and $v \in Y$.*

(5) *T admits a super $(|X| + p + 3, 2)$-edge antimagic total labelling θ such that $\theta(X) < \theta(Y)$ and, $\theta(v) - \theta(u) = |X|$ for $uv \in M$, with $u \in X$ and $v \in Y$.*

(6) *T has an odd-elegant labelling η such that $\eta(x) + \eta(y) \leq 2p - 3$ for every edge $xy \in E(T)$ and, $2\eta(v) - \eta(u) = 2p$ for $uv \in M$, with $u \in X$ and $v \in Y$.*

(7) *T has a (k, d)-arithmetic labelling ψ such that $\psi(x) < \psi(y) - k + d \cdot |X|$ for all $x \in X$ and $y \in Y$ as well as $\psi(v) - \psi(u) = |X|$ for $uv \in M$, with $u \in X$ and $v \in Y$.*

(8) *T has a harmonious labelling φ such that $\varphi(X) < \varphi(Y \setminus \{y_0\})$ and $\varphi(y_0) = 0$, and $\varphi(v) - \varphi(u) = |X|$ for $uv \in M$, with $u \in X$ and $v \in Y \setminus \{y_0\}$.*

Corollary 2.3. *Let G be a bipartite graph with the bipartition (X, Y), and let f be a mapping $V(G) \to \{0, 1, 2, \dots\}$ such that $f(u) \neq f(v)$ for all distinct $u, v \in V(G)$, and $f(xy) = f(y) - f(x) \geq 1$ for each edge $xy \in E(G)$, with $x \in X$ and $y \in Y$. Write $f(V(G)) = \{f(w) : w \in V(G)\}$, and $M = \max f(V(G)) + \min f(V(G))$. Then we have* *(i) G has a labelling g_1 induced by f such that $g_1(u) \neq g_1(v)$ for all distinct $u, v \in V(G)$, and $g_1(x) + g_1(y) = M - f(xy)$ for each edge $xy \in E(G)$, with $x \in X$ and $y \in Y$.*

(ii) for all values of positive integers d and k, G has a labelling g_2 such that $g_2(u) \neq g_2(v)$ for all distinct $u, v \in V(G)$, and $g_2(y) - g_2(y) = k + d \cdot f(xy)$ for each edge $xy \in E(G)$, with $x \in X$ and $y \in Y$.

(iii) there are a labelling g_3 and a constant $\lambda > 0$ such that $g_3(u) \neq g_3(v)$ for all distinct $u, v \in V(G)$, and $g_3(x) + g_3(xy) + g_3(y) = \lambda$ for each edge $xy \in E(G)$.

(iv) G has a labelling g_4 such that $g_4(u) \neq g_4(v)$ for all distinct $u, v \in V(G)$, and $g_4(x) + g_4(y) = k + d \cdot \big[M - f(xy)\big]$ for each edge $xy \in E(G)$, with $x \in X$ and $y \in Y$.

3. **Further works**

We do not know the nature of trees having no set-ordered graceful labellings, and have not methods to construct such trees. We can provide a kind of trees that have no set-ordered graceful labellings. A 2-star $S(m)$ has its own vertex set $V(S(m)) = \{u_0, x_i, y_i : i \in [1, m]\}$ and edge set $E(S(m)) = \{u_0y_i, x_iy_i : i \in [1, m]\}$. Clearly, a 2-star $S(m)$ is a tree.

Theorem 3.1. *Every 2-star $S(m)$ has no set-ordered graceful labellings.*

Proof. Let (X, Y) be the bipartition of a 2-star $S(m)$, where $X = \{u_0, x_i : i \in [1, m]\}$ and $Y = \{y_i : i \in [1, m]\}$. Suppose that $S(m)$ admits a set-ordered graceful labelling f with $f(X) < f(Y)$. Notice that $f(V(S(m))) = [0, 2m]$ and $f(E(S(m))) = [1, 2m]$. Furthermore, $f(X) = [0, m]$ and $f(Y) = [m + 1, 2m]$, by the definition of a set-ordered graceful labelliing. Let $f(u_0) = a \in f(X)$.

If $a = 0$, we have an edge label set $U_1 = \{f(u_0y_i) : i \in [1, m]\} = [m+1, 2m]$ such that $f(E(S(m))) \setminus U_1 = \{f(y_ix_i) : i \in [1, m]\} = [1, m]$. For the purpose of convenience, we assume that $f(x_{i_{j+1}}) = 1 + f(x_{i_j})$ for $j \in [1, m - 1]$ with $f(x_{i_1}) = 1$. Hence, it must be that $f(x_{i_k}y_k) = k$ for $k \in [1, m]$; it is impossible.

If $a \geq 1$, we may assume another edge label set $U_2 = \{f(u_0y_i) : i \in [1, m]\} = [m+1-a, 2m-a]$ such that $f(E(S(m))) \setminus U_2 = \{f(y_ix_i) : i \in [1, m]\} = [1, m-a] \cup [2m-a+1, 2m]$. Without loss of generality, we have that $f(x_{i_j}) = j-1$ for $j \in [1, a]$, and $f(x_{i_j}) = j$ for $j \in [a + 1, m]$. But, $f(x_{i_1}y_m) = f(y_m) - f(x_{i_1}) = 2m - 0 = 2m$, and $f(x_{i_2}y_{m-1}) = f(y_{m-1}) - f(x_{i_2}) = 2m - 1 - 1 = 2m - 2$, we have no edge $x_{i_j}y_k$ such that $f(x_{i_j}y_k) = 2m - 1$. □

Our researching works will focus on the following problem: Let G be a tree having vertices $u_1, u_2, \dots, u_p$. Suppose G admits a labelling θ, determine all possible groups of trees $T_1, T_2, \dots, T_p$ having the labellings that are as the same as θ such that for each $i \in [1, p]$ identifying arbitrarily a vertex of T_i with the vertex u_i of G produces a tree having the labelling θ.

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