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ARENS REGULARITY OF BILINEAR MAPS AND BANACH MODULE ACTIONS

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Abstract. Let X, Y and Z be Banach spaces and let $f : X \times Y \longrightarrow Z$ be a bounded bilinear map. In this paper we study the relation between Arens regularity of f and the reflexivity of Y. We also give some conditions under which the Arens regularity of a Banach algebra A implies the Arens regularity of certain Banach right module action of A .
Keywords: Banach algebra, bilinear map, Arens product, second dual, Banach module action.
MSC(2010): Primary 46H25; Secondary 47A07.

1. Introduction and preliminaries

For a normed space X, we denote by X' and X'' the first and second duals of X, respectively. We usually identify an element of X with its canonical image in X''.

Let X, Y and Z be normed spaces and let $f : X \times Y \longrightarrow Z$ be a bounded bilinear map. In [1], R. Arens showed that f has two natural, but different, extensions f''' and $f^{r'''r}$ from $X'' \times Y''$ to Z''. The adjoint $f' : Z' \times X \longrightarrow Y'$ of f is defined by $\langle f'(z', x), y\rangle = \langle z', f(x, y)\rangle$ $(x \in X, y \in Y, z' \in Z')$, which is also a bounded bilinear map. Similarly by setting $f'' = (f')'$ and continuing in this way, the mapping $f'' : Y'' \times Z' \longrightarrow X'$, $f''' : X'' \times Y'' \longrightarrow Z''$ are also bounded bilinear mappings.
We also denote by f^r the reverse map of f, that is, the bounded bilinear map $f^r : Y \times X \longrightarrow Z$ defined by $f^r(y, x) = f(x, y)$, $(x \in X, y \in Y)$, and it may be extended as above to $f^{r'''r} : X'' \times Y'' \longrightarrow Z''$.

The map f is called Arens regular when the equality $f''' = f^{r'''r}$ holds. Two natural extensions π''' and $\pi^{r'''r}$, of the multiplication map $\pi : A \times A \longrightarrow A$ of a Banach algebra (A, π), are the so-called first and second Arens products,

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which will be denoted by $\Box$ and $\Diamond$, respectively. The Banach algebra (A, π) is said to be Arens regular if the multiplication map π is Arens regular. For example, let G be a locally compact topological group, then $L^1(G)$ is Arens regular if and only if G is finite [7].

Let (A, π) be a Banach algebra, and X be a Banach space. Suppose that $\pi_r : X \times A \longrightarrow X$ is a bounded bilinear map. Then the pair (X, π_r) is said to be a right Banach A-module if π_r is associative, i.e. $\pi_r(x, \pi(a, b)) = \pi_r(\pi_r(x, a), b)$, for every $a, b \in A$, $x \in X$. A left A-module (π_l, X) can be defined similarly.

For a bounded linear map $T : X \longrightarrow Y$ we define the adjoint $T^* : Y' \longrightarrow X'$ by $T^*(y') = y'oT$. Then T^* is also a bounded linear map.

2. **Arens regularity of bilinear maps and reflexivity**

Let X, Y and Z be normed spaces. In this section we study the relation between the Arens regularity of a bilinear map $f : X \times Y \longrightarrow Z$ and the reflexivity of Y. If Y is reflexive, then obviously f is Arens regular, however, the Arens regularity of f does not imply the reflexivity of Y; for example, it is known that the multiplication map of every non-reflexive C^*-algebra is Arens regular. We quote the following result from [5] characterizing the Arens regularity of a bounded bilinear map.

Proposition 2.1 ([5, Theorem 2.1])**.** *For a bounded bilinear map $f : X \times Y \longrightarrow Z$ the following statements are equivalent:*

(i) *f is Arens regular;*

(ii) *$f'''' = f^{r''''r}$;*

(iii) *$f''''(Z', X'') \subseteq Y'$;*

(iv) *the linear map $x \longmapsto f'(z', x) : X \longrightarrow Y'$ is weakly compact for every $z' \in Z'$.*

In the following result we show that under certain conditions the Arens regularity of f implies the reflexivity of Y.

Theorem 2.2. *Let X be a Banach space and let $f : X \times Y \longrightarrow Z$ be an Arens regular bounded bilinear map. If $f'(z', X) = Y'$ for some $z' \in Z'$, then Y is reflexive.*

Proof. We define the map $f_{z'}$ from X to Y' by $f_{z'}(x) = f'(z', x)$. Since $f'(z', X) = Y'$, $f_{z'}$ is onto and this implies that $f_{z'}^{**} : X'' \longrightarrow Y'''$ is onto. Since for every $y'' \in Y'', x \in X$,

$$\begin{aligned} \langle f_{z'}^*(y''), x\rangle = \langle y'', f_{z'}(x)\rangle &= \langle y'', f'(z', x)\rangle \\ &= \langle f''(y'', z'), x\rangle, \end{aligned}$$

we have

$$\begin{aligned}\langle f_{z'}^{**}(x''), y''\rangle = \langle x'', f_{z'}^{*}(y'')\rangle &= \langle x'', f''(y'', z')\rangle \\ &= \langle f'''(x'', y''), z'\rangle \\ &= \langle z', f'''(x'', y'')\rangle \\ &= \langle f''''(z', x''), y''\rangle.\end{aligned}$$

Thus, for every $x'' \in X''$, $f_{z'}^{**}(x'') = f''''(z', x'')$. Now by Proposition 2.1, the Arens regularity of f implies that $f''''(Z', X'') \subseteq Y'$. Let $y''' \in Y'''$, so there exists a $x'' \in X''$ such that $y''' = f_{z'}^{**}(x'') = f''''(z', x'') \in Y'$. Hence, Y is reflexive, as claimed. □

Example 2.3. Let A and B be two Banach algebras and $\mathcal{B}(A, B)$ the Banach space of all bounded linear operators from A to B. Then, the mapping $f : \mathcal{B}(A, B) \times A \longrightarrow B$ defined by $f(T, a) = T(a)$ is a bounded bilinear map. If B unital, then by the Hahn-Banach theorem there exists a $b' \in B'$ such that $b'(1_B) = 1$. We show that $f'(b', B(A, B)) = A'$. Let $a' \in A'$. Then for the bounded linear map $T_{a'} : A \longrightarrow B$ defined by $T_{a'}(a) = a'(a)1_B$, we have

$$\langle f'(b', T_{a'}), a\rangle = \langle b', T_{a'}(a)\rangle = \langle b', a'(a)1_B\rangle = \langle a', a\rangle.$$

Thus by Theorem 2.2, f is Arens regular if and only if A is reflexive.

We use Theorem 2.2 to prove the following result that was proved in [6, Corollary 3.2] by a different method.

Corollary 2.4. *For a Banach space X, the bilinear map $f : X' \times X \longrightarrow \mathbb{C}$ defined by $f(x', x) = \langle x', x\rangle$ is Arens regular if and only if X is reflexive.*

Proof. Note that for $f' : \mathbb{C} \times X' \longrightarrow X'$ we have $f'(1, X') = X'$. So, by Theorem 2.2 f is Arens regular if and only if X is reflexive. □

As another consequence of Theorem 2.2 we present the following result .

Corollary 2.5. *Let A be a Banach algebra with a bounded approximate identity and let π denote the multiplication of A. Then π' is Arens regular if and only if A is reflexive.*

Proof. Since A has a bounded approximate identity, there exists an $e'' \in A''$ such that $\pi''(e'', A') = A'$. By Theorem 2.2, π' is Arens regular if and only if A is reflexive. □

Applying Corollary 2.5 for the group convolution algebra $L^1(G)$ and also for a C^*- algebra, we arrive at the following corollary which has already proved in [3].

Corollary 2.6. (1) *Let π denote the multiplication of the group algebra $L^1(G)$ on a locally compact group G. Then the bilinear map π' is Arens regular if and only if G is finite.*

(2) *Let π denote the multiplication of a C^*-algebra A. Then the bilinear map π' is Arens regular if and only if A is finite dimensional.*

Let X and A be normed spaces. Following [4], a bounded bilinear map $g : X \times A \longrightarrow X$ is said to be approximately unital if there exists a bounded net (e_α) in A such that $\lim_\alpha g(x, e_\alpha) = x$, for all $x \in X$. We present the next result as a consequence of Theorem 2.2.

Corollary 2.7 ([4, Theorem 4.1])**.** *Let X and A be normed spaces. Then the adjoint g' of an approximately unital bounded bilinear map $g : X \times A \longrightarrow X$ is Arens regular if and only if X is reflexive.*

Proof. Let e'' be a w^*-cluster point of a bounded net (e_α) in A satisfying $\lim_\alpha g(x, e_\alpha) = x$, for all $x \in X$. It follows that $g''(e'', x') = x'$ for each $x' \in X'$. Applying Theorem 2.2 for $g' : X' \times X \longrightarrow A'$, we deduce that, g' is Arens regular if and only if X is reflexive. □

3. Arens regularity of Banach algebras and module actions

Suppose that A is a Banach algebra. It is worth to mention that, in general, there is no relation between the Arens regularity of A and the Arens regularity of the right Banach $A-$modules. For example, let A be the C^*-algebra of compact operators on a separable, infinite-dimensional Hilbert space H and let X be the trace-class operators on H. Then, a direct verification reveals that the usual $A-$module action on X is not Arens regular [2].

On the other hand, an arbitrary Banach algebra A can be viewed as a right Banach $A-$module under the module action $\pi_r(a, b) = \varphi(a)b$ (for a fixed $\varphi \in A'$ with $\|\varphi\| = 1$), which is trivially Arens regular.

The following results provide an interrelation between the Arens regularity of A and that of certain $A-$module actions.

Theorem 3.1. *Let A be a Banach algebra and let (X, π_r) be a right Banach A-module. If π_r' is onto and π_r is Arens regular, then π is Arens regular.*

Proof. By Proposition 2.1, the Arens regularity of π_r implies that $\pi_r''''(X', X'') \subseteq A'$. Let π denote the multiplication of A, $a' \in A', a'' \in A''$ and $b'' \in A''$. Since π_r' is onto, there exist $x' \in X'$, $x \in X$ such that $\pi_r'(x', x) = a'$. Further,

$$\begin{aligned}
\langle \pi''''(a', a''), b'' \rangle &= \langle \pi''''(\pi_r'(x', x), a''), b'' \rangle \\
&= \langle \pi''''(\pi_r''''(x', x), a''), b'' \rangle \\
&= \langle \pi_r''''(x', x), \pi'''(a'', b'') \rangle \\
&= \langle x', \pi_r'''(x, \pi'''(a'', b'')) \rangle \\
&= \langle x', \pi_r'''(\pi_r'''(x, a''), b'') \rangle \\
&= \langle \pi_r''''(x', (\pi_r'''(x, a'')), b'' \rangle.
\end{aligned}$$

Thus $\pi''''(a', a'') = \pi_r''''(x', (\pi_r'''(x, a'')) \in A'$. This implies that π is Arens regular. □

Theorem 3.2. *Let A be a Banach algebra and let (X, π_r) be a right Banach A-module. If A is Arens regular and $\pi_r(x, A) = X$ for some $x \in X$, then π_r is Arens regular.*

Proof. For the Arens regularity of π_r, it is enough to show that $\pi_r^{\prime\prime\prime\prime}(X', X'') \subseteq A'$. The map $\pi_x : A \longrightarrow X$ defined by $\pi_x(a) = \pi_r(x, a)$ is onto. Therefore, $\pi_x^{**} : A'' \longrightarrow X''$ is onto. Let $x' \in X', x'' \in X''$ and $b'' \in A''$. Since π_x^{**} is onto, there exists an element $a'' \in A''$ such that $x'' = \pi_x^{**}(a'') = \pi_r^{\prime\prime\prime}(x, a'')$. Let π denote the multiplication of A. Then

$$\begin{aligned}
\langle \pi_r^{\prime\prime\prime\prime}(x', x''), b'' \rangle &= \langle \pi_r^{\prime\prime\prime\prime}(x', (\pi_r^{\prime\prime\prime}(x, a'')), b'' \rangle \\
&= \langle x', \pi_r^{\prime\prime\prime}(\pi_r^{\prime\prime\prime}(x, a''), b'') \rangle \\
&= \langle x', \pi_r^{\prime\prime\prime}(x, \pi^{\prime\prime\prime}(a'', b'')) \rangle \\
&= \langle \pi_r^{\prime\prime\prime\prime}(x', x), \pi^{\prime\prime\prime}(a'', b'') \rangle \\
&= \langle \pi_r'(x', x), \pi^{\prime\prime\prime}(a'', b'') \rangle \\
&= \langle \pi^{\prime\prime\prime\prime}(\pi_r'(x', x), a''), b'' \rangle.
\end{aligned}$$

Thus $\pi_r^{\prime\prime\prime\prime}(x', x'') = \pi^{\prime\prime\prime\prime}(\pi_r'(x', x), a'') \in A'$; that is, π_r is Arens regular. □

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