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MULTIPLIERS OF CONTINUOUS G-FRAMES IN HILBERT SPACES

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ABSTRACT. In this paper we introduce continuous g-Bessel multipliers in Hilbert spaces and investigate some of their properties. We provide some conditions under which a continuous g-Bessel multiplier is a compact operator. Also, we show the continuous dependency of continuous g-Bessel multipliers on their parameters.
Keywords: g-frames, continuous frames, continuous g-frames, multiplier of frames, multiplier of continuous g-frames.
MSC(2010): Primary: 41A58; Secondary: 47A58, 42C45.

1. Introduction and preliminaries

In 1952, Duffin and Schaeffer [13] introduced the concept of *discrete frames* in Hilbert spaces. G-frame as a natural generalization of frames in Hilbert space, introduced by Sun in [18]. The concept of g-frame includes several generalizations of frame.

Definition 1.1. Let $\mathcal{H}$ be a Hilbert space and $\{\mathcal{K}_i\}_{i\in I}$ be a sequence of Hilbert spaces. We call $\{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i) : i \in I\}$ a *g-frame* for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$, or simply, a g-frame for $\mathcal{H}$, if there exist two positive constants A and B such that

$$A\|f\|^2 \le \sum_{i\in I} \|\Lambda_i f\|^2 \le B\|f\|^2, \quad f \in \mathcal{H}.$$

We call A and B the lower and upper g-frame bounds, respectively.

We can refer to [1, 3, 4, 15] for some properties of g- frames. In 2007, P. Balazs [7] introduced Bessel and frame multipliers for Hilbert spaces.

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Definition 1.2. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Let $\{\phi_i\}_{i\in I} \subset \mathcal{H}_1$ and $\{\psi_i\}_{i\in I} \subset \mathcal{H}_2$ be Bessel sequences. Fix $m = \{m_i\}_{i\in I} \in \ell^\infty$. The operator $M_{m,\{\phi_i\},\{\psi_i\}} : \mathcal{H}_1 \to \mathcal{H}_2$ defined by

$$M_{m,\{\phi_i\},\{\psi_i\}}(f) = \sum_{i\in I} m_i \langle f, \phi_i\rangle \psi_i \,, \tag{1.1}$$

is called the *Bessel multiplier* for $\{\phi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$.

Multipliers are not only interesting from a theoretical point of view, but also used in applications. Multipliers have applications in computational auditory scene analysis [19], virtual acoustics [14], sound morphing [12] and psychoacoustics [9]. In 2009, Rahimi [16] introduced the multipliers of g-Bessel sequences and investigated some of their properties (see also [17]).

Definition 1.3. Suppose $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i), i \in I\}$ and $\Phi = \{\Phi_i \in B(\mathcal{H}, \mathcal{K}_i), i \in I\}$ are g-Bessel sequences. If $m = \{m_i\}_{i\in I} \subseteq \ell^\infty$, then the operator

$$M_{m,\Lambda,\Phi} : \mathcal{H} \to \mathcal{H}, \qquad M_{m,\Lambda,\Phi}(f) = \sum_{i\in I} m_i \Lambda_i^* \Phi_i(f)$$

is called the g-Bessel multiplier of Λ, Φ with respect to m.

Ali, Antoine and Gazeau [5], generalized the concept of frame to a family of vectors indexed by a measurable space and introduced the *continuous frames*.

Definition 1.4. Let $\mathcal{H}$ be a complex Hilbert space and (Ω, μ) be a measure space. The mapping $F : \Omega \to \mathcal{H}$ is called a continuous frame with respect to (Ω, μ), if
(i) F is weakly-measurable, i.e., for all $f \in \mathcal{H}$, $\omega \to \langle f, F(\omega)\rangle$ is a measurable function on Ω,
(ii) there exist constants $A, B > 0$ such that

$$A\|f\|^2 \le \int_\Omega |\langle f, F(\omega)\rangle|^2 \, d\mu_\omega \le B\|f\|^2, \quad f \in \mathcal{H}. \tag{1.2}$$

If in (1.2), the right hand inequality holds for all $f \in \mathcal{H}$, then we call the mapping $F : \Omega \to \mathcal{H}$ a Bessel mapping with respect to (Ω, μ). If F is a Bessel mapping from Ω to $\mathcal{H}$, then

$$T_F : L^2(\Omega, \mu) \to \mathcal{H}, \quad \langle T_F \varphi, h\rangle = \int_\Omega \varphi(\omega) \langle F(\omega), h\rangle d\mu_\omega, \quad h \in \mathcal{H}$$

is well-defined linear and bounded operator and its adjoint is given by

$$T_F^* : \mathcal{H} \to L^2(\Omega, \mu), \quad (T_F^* h)(\omega) = \langle h, F(\omega)\rangle, \quad \omega \in \Omega.$$

The operators T_F and T_F^* are called the synthesis and analysis operator of F.

In [8], Multipliers of continuous frame defined by Balazes, Bayer and Rahimi. They studied some properties of multiplier of continuous frame and proved some statements on the compactness of these kinds of multipliers.

Definition 1.5 ([8]). Let F and G be Bessel mappings for $\mathcal{H}$ with respect to (Ω, μ) and $m : \Omega \to \mathbb{C}$ be a measurable function. The operator $M_{m,F,G} : \mathcal{H} \to \mathcal{H}$ defined by

$$\langle M_{m,F,G} f, g\rangle = \int_\Omega m(\omega)\langle f, F(\omega)\rangle\langle G(\omega), g\rangle d\mu_\omega, \quad f, g \in \mathcal{H},$$

is called the continuous Bessel multiplier of F and G with respect to m.

Continuous g-frame in Hilbert spaces as a common generalization of g-frame and continuous frame defined by Abdollahpour and Faroughi [2].
In the following, suppose (Ω, μ) is a measure space with positive measure μ and $\mathcal{H}$ is a Hilbert space and $\{K_\omega\}_{\omega\in\Omega}$ is a family of Hilbert spaces. We say that $F \in \Pi_{\omega\in\Omega}\mathcal{K}_\omega$ is strongly measurable if F as a mapping of Ω into $\bigoplus_{\omega\in\Omega}\mathcal{K}_\omega$ is measurable.

Definition 1.6. A family of operators $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega), \omega \in \Omega\}$ is a *continuous generalized frame*, or simply, *continuous g-frame* with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ if
(i) for each $f \in \mathcal{H}$, $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable,
(ii) there are two constants $0 < A_\Lambda \le B_\Lambda < \infty$ such that

$$A_\Lambda||f||^2 \le \int_\Omega ||\Lambda_\omega f||^2 d\mu_\omega \le B_\Lambda||f||^2, f \in \mathcal{H}. \tag{1.3}$$

The family $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega), \omega \in \Omega\}$ is called a continuous g-Bessel family with bound B_Λ if the right hand inequality in (1.3) holds for all $f \in \mathcal{H}$.

It is proved in [2], if $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega), \omega \in \Omega\}$ is a continuous g-frame, then there is a unique positive invertible operator $S_\Lambda : \mathcal{H} \to \mathcal{H}$ such that for each $f, g \in \mathcal{H}$,

$$\langle S_\Lambda f, g\rangle = \int_\Omega \langle f, \Lambda_\omega^*\Lambda_\omega g\rangle d\mu_\omega,$$

and $A_\Lambda I \le S_\Lambda \le B_\Lambda I$. The operator S_Λ is called the *continuous g-frame operator* of $\{\Lambda_\omega\}_{\omega\in\Omega}$ and we write $S_\Lambda f = \int_\Omega \Lambda_\omega^*\Lambda_\omega f d\mu_\omega$.

Definition 1.7. We consider the space

$$\widehat{\mathcal{K}} = \left\{F \in \prod_{\omega\in\Omega}\mathcal{K}_\omega : \text{F is strongly measurable, } \int_\Omega \|F(\omega)\|^2 d\mu_\omega < \infty\right\}.$$

It is clear that $\widehat{\mathcal{K}}$ is a Hilbert space with point-wise operations and with inner product given by

$$\langle F, G\rangle = \int_\Omega \langle F(\omega), G(\omega)\rangle d\mu_\omega.$$

We state the following proposition, which is used in the rest of this paper and it is proved in [2].

Proposition 1.8. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous g-Bessel family with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ with bound B. Then the mapping $T_\Lambda : \widehat{\mathcal{K}} \to \mathcal{H}$ defined by*

$$\langle T_\Lambda F, g\rangle = \int_\Omega \langle \Lambda_\omega^* F(\omega), g\rangle d\mu_\omega,\ F \in \widehat{\mathcal{K}}, g \in \mathcal{H}$$

is linear and bounded with $\|T_\Lambda\| \le \sqrt{B}$. Furthermore for each $g \in \mathcal{H}$ and $\omega \in \Omega$

$$T_\Lambda^*(g)(\omega) = \Lambda_\omega g.$$

Now, we recall the construction of interpolation spaces, usually called the complex interpolation method. A compatible couple of Banach spaces is a pair $\overline{X} = (X_0, X_1)$ of Banach spaces X_0 and X_1 both continuously embedded in a Hausdorff topological vector space Y. In this case the intersection $X_0 \cap X_1$ is a subspace of Y, and it is a Banach space with the norm

$$\|x\|_{X_0\cap X_1} = \max\{\|x\|_{X_0}, \|x\|_{X_1}\}.$$

Also the subspace

$$X_0 + X_1 = \{x \in X : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}$$

is a Banach space with the norm

$$\|x\|_{X_0+X_1} = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}.$$

A Banach space X is said to be an intermediate space with respect to $\overline{X} = (X_0, X_1)$ if

$$X_0 \cap X_1 \subset X \subset X_0 + X_1$$

and both inclusions are continuous.
An interpolation space between X_0 and X_1 is any intermediate space X such that for every $T \in B(X_0 + X_1)$ whose restriction to X_0 belongs to $B(X_0)$ and whose restriction to X_1 belongs to $B(X_1)$, the restriction of T to X belongs to $B(X)$.

The complex interpolation method requires the space $\mathcal{F}(\overline{X})$ of all functions f from the closed strip $S = \{z \in \mathbb{C} : 0 \le Rez \le 1\}$ into $X_0 + X_1$ such that
i) $f(z)$ is bounded and continuous on S.
ii) $f(z)$ is analytic relative to the norm of $X_0 + X_1$ on the interior of S.
ii) $t \to f(j + it)$ is continuous and bounded from the real line into X_j for $j = 0, 1$.
The vector space $\mathcal{F}(\overline{X})$ is a Banach space with the following norm

$$\|f\|_{\mathcal{F}(\overline{X})} = max\{sup_{t\in\mathbb{R}}\|f(it)\|_{X_0}, sup_{t\in\mathbb{R}}\|f(1 + it)\|_{X_1}\}.$$

For $0 \le \theta \le 1$, the complex interpolation space $[X_0, X_1]_\theta$ consists of all $x \in X_0 + X_1$ such that $x = f(\theta)$ for some $f \in \mathcal{F}(\overline{X})$, equips with the complex interpolation norm

$$\|x\|_{[\theta]} = inf\{\|f\|_{\mathcal{F}(\overline{X})} : f(\theta) = x, f \in \mathcal{F}(\overline{X})\}.$$

It was proved in Theorem 2.2.4 of [20], $[X_0, X_1]_\theta$ is a Banach space and $[X_0, X_1]_\theta$ is an interpolation space between X_0 and X_1, for $0 < \theta < 1$. Also, if $\overline{X} = (X_0, X_1)$ and $\overline{Y} = (Y_0, Y_1)$ are compatible couples of Banach spaces and

$$T : X_0 + X_1 \to Y_0 + Y_1$$

is a linear bounded operator such $T \in B(X_j, Y_j)$ for $j = 0, 1$, then for all $0 < \theta < 1$, $T : [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta$ is bounded and

$$\|T\|_{[X_0,X_1]_\theta,[Y_0,Y_1]_\theta} \leq \|T\|^{1-\theta}_{X_0,Y_0} \|T\|^{\theta}_{X_1,Y_1}.$$

2. Multiplier of continuous frames

The authors of [6] introduced *continuous Riesz basis* and gave some equivalent conditions for a continuous frame to be a continuous Riesz basis. Here, we review some definitions and basic properties of continuous Riesz basis.

Suppose (Ω, μ) is a measure space and $\mathcal{H}$ is a Hilbert space. We denote by $L^2(\Omega, \mu, \mathcal{H})$ the set of all mappings $F : \Omega \to \mathcal{H}$ such that for all $f \in \mathcal{H}$, the function $\omega \to \langle f, F(\omega)\rangle$ is defined almost everywhere on Ω, and belongs to $L^2(\Omega, \mu)$. A Bessel mapping $F : \Omega \to \mathcal{H}$ is called *μ-complete* if $f \in \mathcal{H}$ so that $\langle f, F(\omega)\rangle = 0$ for almost all $\omega \in \Omega$, then $f = 0$.

Definition 2.1. Let (Ω, μ) be a measure space. A mapping $F \in L^2(\Omega, \mu, \mathcal{H})$ is called continuous Riesz basis for $\mathcal{H}$ with respect to (Ω, μ), if $\{F(\omega)\}_{\omega\in\Omega}$ is μ-complete and there are two positive numbers A and B such that

$$A \left(\int_{\Omega_1} |m(\omega)|^2 d\mu_\omega \right)^{\frac{1}{2}} \leq \left\| \int_{\Omega_1} m(\omega)F(\omega)d\mu_\omega \right\| \leq B \left(\int_{\Omega_1} |m(\omega)|^2 d\mu_\omega \right)^{\frac{1}{2}}$$

for every $m \in L^2(\Omega, \mu)$ and for every measurable subset Ω_1 of Ω with $\mu(\Omega_1) < \infty$. The constant A and B are called Riesz basis bounds.

Definition 2.2. A Bessel mapping $F \in L^2(\Omega, \mu, \mathcal{H})$ is said to be L^2-independent if $\int_\Omega m(\omega)F(\omega)d\mu_\omega = 0$ for $m \in L^2(\Omega, \mu)$, implies that $m = 0$ almost everywhere.

Theorem 2.3 ([6])**.** *Let $\mathcal{H}$ be a Hilbert space and (Ω, μ) be a measure space. A continuous frame $F \in L^2(\Omega, \mu, \mathcal{H})$ is a continuous Riesz basis for $\mathcal{H}$ if and only if F is μ-complete and L^2-independent.*

If (Ω, μ) is a σ-finite measure space and $F \in L^2(\Omega, \mu, \mathcal{H})$ is a continuous Riesz basis for $\mathcal{H}$ with respect to (Ω, μ), then

$$A \left(\int_{\Omega} |m(\omega)|^2 d\mu_\omega \right)^{\frac{1}{2}} \leq \left\| \int_{\Omega} m(\omega)F(\omega)d\mu_\omega \right\| \leq B \left(\int_{\Omega} |m(\omega)|^2 d\mu_\omega \right)^{\frac{1}{2}}.$$

Since, in this case there is $\{\Omega_n\}_{n=1}^{\infty}$ a family of disjoint measurable subsets of Ω such that $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$, $\mu(\Omega_n) < \infty$ for all $n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$ we have

$$A\left(\sum_{k=1}^{n}\int_{\Omega_k}|m(\omega)|^2d\mu_\omega\right)^{\frac{1}{2}} \leq \left\|\sum_{k=1}^{n}\int_{\Omega_k}m(\omega)F(\omega)d\mu_\omega\right\| \leq B\left(\sum_{k=1}^{n}\int_{\Omega_k}|m(\omega)|^2d\mu_\omega\right)^{\frac{1}{2}}.$$

Therefore,

$$A\left(\sum_{k=1}^{\infty}\int_{\Omega_k}|m(\omega)|^2d\mu_\omega\right)^{\frac{1}{2}} \leq \left\|\sum_{k=1}^{\infty}\int_{\Omega_k}m(\omega)F(\omega)d\mu_\omega\right\| \leq B\left(\sum_{k=1}^{\infty}\int_{\Omega_k}|m(\omega)|^2d\mu_\omega\right)^{\frac{1}{2}},$$

and so

$$A\left(\int_{\Omega}|m(\omega)|^2d\mu_\omega\right)^{\frac{1}{2}} \leq \left\|\int_{\Omega}m(\omega)F(\omega)d\mu_\omega\right\| \leq B\left(\int_{\Omega}|m(\omega)|^2d\mu_\omega\right)^{\frac{1}{2}}.$$

Now, we give the following propositions as new results on multipliers of continuous frames.

Proposition 2.4. *Let (Ω,μ) be a σ-finite measure space and $m \in L^\infty(\Omega,\mu)$. Let G be a continuous Riesz basis and F be a Bessel mapping with non zero elements. The mapping $m \to M_{m,F,G}$ is injective.*

Proof. Let $M_{m,F,G} = 0$. Then for any $f \in \mathcal{H}$, $M_{m,F,G}f = 0$. So

$$A\left(\int_{\Omega}|m(\omega)\langle f,F(\omega)\rangle|^2d\mu_\omega\right)^{\frac{1}{2}} \leq \left\|\int_{\Omega}m(\omega)\langle f,F(\omega)\rangle G(\omega)d\mu_\omega\right\| = 0.$$

Therefore $m(\omega)\langle f,F(\omega)\rangle = 0$ a.e. $\omega \in \Omega$. Then $m(\omega) = 0$ almost everywhere, and so $m \to M_{m,F,G}$ is injective. □

Proposition 2.5. *Let $m : \Omega \to \mathbb{C}$ be a measurable function such that $0 < inf_{\omega\in\Omega}|m(\omega)| \leq sup_{\omega\in\Omega}|m(\omega)| < \infty$. Let G and F be continuous Riesz bases. Then the multiplier $M_{m,F,G}$ is invertible.*

Proof. We have $M_{m,F,G} = T_G D_m T_F^*$, where $D_m : L^2(\Omega,\mu) \to L^2(\Omega,\mu)$ is defined by $(D_m\varphi)(\omega) = m(\omega)\varphi(\omega)$. Since T_G, D_m and T_F^* are invertible, $M_{m,F,G}$ is also invertible. □

3. Multiplier of continuous G-frames

In this section, we intend to define continuous g-Bessel multipliers and investigate some of their properties. We start with the following elementary lemma.

Lemma 3.1. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ and $\{\Phi_\omega\}_{\omega\in\Omega}$ be continuous g-Bessel families with bounds B_Λ and B_Φ, respectively, with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ and $m \in L^\infty(\Omega,\mu)$. Then the operator $M_{m,\Lambda,\Phi} : \mathcal{H} \to \mathcal{H}$ defined by*

$$\langle M_{m,\Lambda,\Phi} f, g\rangle = \int_\Omega m(\omega)\langle \Lambda_\omega^* \Phi_\omega f, g\rangle d\mu_\omega, \quad (f, g \in \mathcal{H})$$

is a bounded operator.

Proof. For any $f, g \in \mathcal{H}$, we have

$$\begin{aligned}
|\langle M_{m,\Lambda,\Phi} f, g\rangle| &= \Big|\int_\Omega m(\omega)\langle \Lambda_\omega^* \Phi_\omega f, g\rangle d\mu_\omega\Big| \\
&\le \int_\Omega |m(\omega)||\langle \Lambda_\omega^* \Phi_\omega f, g\rangle| d\mu_\omega \\
&\le \|m\|_\infty \int_\Omega |\langle \Phi_\omega f, \Lambda_\omega g\rangle| d\mu_\omega \\
&\le \|m\|_\infty \int_\Omega \|\Phi_\omega f\|\|\Lambda_\omega g\| d\mu_\omega \\
&\le \|m\|_\infty \Big(\int_\Omega \|\Phi_\omega f\|^2 d\mu_\omega\Big)^{\frac{1}{2}} \Big(\int_\Omega \|\Lambda_\omega g\|^2 d\mu_\omega\Big)^{\frac{1}{2}} \\
&\le \sqrt{B_\Lambda B_\Phi}\|f\|\|g\|\|m\|_\infty.
\end{aligned}$$

This shows that $\|M_{m,\Lambda,\Phi}\| \le \|m\|_\infty \sqrt{B_\Lambda B_\Phi}$ and so $M_{m,\Lambda,\Phi}$ is a bounded operator. □

Now we are ready to introduce the concept of continuous g-Bessel multipliers.

Definition 3.2. Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ and $\{\Phi_\omega\}_{\omega\in\Omega}$ be continuous g-Bessel families with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ and $m \in L^\infty(\Omega,\mu)$. The operator $M_{m,\Lambda,\Phi}$: $\mathcal{H} \to \mathcal{H}$ defined by

$$\langle M_{m,\Lambda,\Phi} f, g\rangle = \int_\Omega m(\omega)\langle \Lambda_\omega^* \Phi_\omega f, g\rangle d\mu_\omega, \quad f, g \in \mathcal{H}$$

is called the *continuous g-Bessel multiplier* of Λ, Φ with respect to m. For simply, we write $M_{m,\Lambda,\Phi} f = \int_\Omega m(\omega)\Lambda_\omega^* \Phi_\omega f d\mu_\omega$.

We mention that every continuous Bessel multiplier is a continuous g-Bessel multiplier. In fact if F and G are Bessel mapping for $\mathcal{H}$ with respect to (Ω,μ) and $m \in L^\infty(\Omega,\mu)$ and we consider $\Lambda_\omega = \langle \cdot, F(\omega)\rangle$ and $\Phi_\omega = \langle \cdot, G(\omega)\rangle$ for any

$\omega \in \Omega$, then $\{\Lambda_\omega\}_{\omega\in\Omega}$ and $\{\Phi_\omega\}_{\omega\in\Omega}$ are g-Bessel families for $\mathcal{H}$ respect to $\mathbb{C}$ and

$$\begin{aligned}\langle M_{m,\Lambda,\Phi} f, g\rangle &= \int_\Omega m(\omega)\langle \Lambda_\omega^* \Phi_\omega f, g\rangle d\mu_\omega \\ &= \int_\Omega m(\omega)\langle \Phi_\omega f, \Lambda_\omega g\rangle d\mu_\omega \\ &= \int_\Omega m(\omega)\Big\langle \langle f, G(\omega)\rangle, \langle g, F(\omega)\rangle \Big\rangle d\mu_\omega \\ &= \int_\Omega m(\omega)\langle f, G(\omega)\rangle \cdot \overline{\langle g, F(\omega)\rangle} d\mu_\omega \\ &= \int_\Omega m(\omega)\langle f, G(\omega)\rangle \cdot \langle F(\omega), g\rangle d\mu_\omega \\ &= \langle M_{m,G,F} f, g\rangle,\end{aligned}$$

for all $f, g \in \mathcal{H}$.

In the next proposition we show that under some conditions a continuous g-Bessel multiplier could be positive (invertible) operator.

Proposition 3.3. *Let $m \in L^\infty(\Omega,\mu)$ and $m(\omega) > 0$ a.e.. Then $M_{m,\Lambda,\Lambda}$ is a positive operator, for any continuous g-Bessel family $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$. If $m(\omega) \geq \delta > 0$ a.e., then for any continuous g-frame $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for complex Hilbert space $\mathcal{H}$, the multiplier $M_{m,\Lambda,\Lambda}$ is a positive invertible operator.*

Proof. For any $f \in \mathcal{H}$ and any continuous g-Bessel family $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$, we have

$$\langle M_{m,\Lambda,\Lambda} f, f\rangle = \int_\Omega m(\omega)\langle \Lambda_\omega^* \Lambda_\omega f, f\rangle d\mu_\omega = \int_\Omega m(\omega)||\Lambda_\omega f||^2 d\mu_\omega \geq 0.$$

If $m(\omega) \geq \delta$ for some positive constant δ and $||m||_\infty < \infty$, then we have

$$\begin{aligned}\delta A \parallel f \parallel^2 \leq \delta \int_\Omega \parallel \Lambda_\omega f \parallel^2 d\mu_\omega &\leq \int_\Omega m(\omega) \parallel \Lambda_\omega f \parallel^2 d\mu_\omega \\ &\leq \parallel m \parallel_\infty \int_\Omega \parallel \Lambda_\omega f \parallel^2 d\mu_\omega \\ &\leq B_\Lambda \parallel m \parallel_\infty \parallel f \parallel^2 .\end{aligned}$$

So $\Gamma = \{\sqrt{m(\omega)}\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous g-frame and $S_\Gamma = M_{m,\Lambda,\Lambda}$. Therefore the multiplier $M_{m,\Lambda,\Lambda}$ is a positive invertible operator. □

Proposition 3.4. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ and $\{\Phi_\omega\}_{\omega\in\Omega}$ be continuous g-Bessel families with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ and $m \in L^\infty(\Omega,\mu)$. Then*

$$M^*_{m,\Lambda,\Phi} = M_{\overline{m},\Phi,\Lambda}.$$

Proof. For any $f, g \in \mathcal{H}$, we have

$$\begin{aligned}\langle g, M^*_{m,\Lambda,\Phi} f\rangle &= \langle M_{m,\Lambda,\Phi} g, f\rangle \\ &= \int_\Omega m(\omega)\langle \Lambda^*_\omega \Phi_\omega g, f\rangle d\mu_\omega = \int_\Omega m(\omega)\langle g, \Phi^*_\omega \Lambda_\omega f\rangle d\mu_\omega \\ &= \int_\Omega \langle g, \overline{m(\omega)}\Phi^*_\omega \Lambda_\omega f\rangle d\mu_\omega = \langle g, M_{\overline{m},\Phi,\Lambda} f\rangle.\end{aligned}$$

□

Proposition 3.5. *If $m \in L^\infty(\Omega,\mu)$, then*

$$D_m : \widehat{\mathcal{K}} \longrightarrow \widehat{\mathcal{K}}, \quad D_m(\{f_\omega\}_{\omega\in\Omega}) = \{m(\omega)f_\omega\}_{\omega\in\Omega},$$

is a bounded linear operator and $\|D_m\| \le \|m\|_\infty$.

Proof. Let $m \in L^\infty(\Omega,\mu)$. Then $|m(\omega)| \le \|m\|_\infty$ a.e. $\omega \in \Omega$. If $\{f_\omega\}_{\omega\in\Omega} \in \widehat{\mathcal{K}}$, then

$$\int_\Omega \|m(\omega)f_\omega\|^2 d\mu_\omega \le \|m\|_\infty \int_\Omega \|f_\omega\|^2 d\mu_\omega < \infty.$$

This implies that D_m is bounded and $\|D_m\| \le \|m\|_\infty$. □

D_m is called the multiplication operator with the symbol m. By using synthesis and analysis operators, it can be implied

$$M_{m,\Lambda,\Phi} = T_\Lambda D_m T^*_\Phi.$$

Suppose $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous g-Bessel family with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$. We say Λ is norm bounded if there is a constant $M > 0$ such that $\|\Lambda_\omega\| \le M$ for every $\omega \in \Omega$.

Theorem 3.6. *Let $dim\mathcal{K}_\omega < \infty$ for all $\omega \in \Omega$. Let $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ and $\Phi = \{\Phi_\omega\}_{\omega\in\Omega}$ be continuous g-Bessel families with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ and let Λ or Φ be norm bounded with bound M. Let $m : \Omega \to \mathbb{C}$ be a bounded measurable function with support of a finite measure, i.e., there exists a subset $K \subseteq \Omega$ with $\mu(K) < \infty$ such that $m(\omega) = 0$ for almost every $\omega \in \Omega\backslash K$. Then $M_{m,\Lambda,\Phi}$ is a compact operator.*

Proof. At first, suppose that Φ is norm bounded and $\|\Phi_\omega\| \le M$ for all $\omega \in \Omega$. We prove $D_m T^*_\Phi : \mathcal{H} \to \widehat{\mathcal{K}}$ is compact. Let $f_n \to 0$ weakly. Then $\Phi_\omega(f_n) \to 0$ for every fixed $\omega \in \Omega$. On the other hand, there is a positive constant C such that for any $n \in \mathbb{N}$, $\|f_n\| \le C$. Therefore

$$|m(\omega)|^2\|\Phi_\omega(f_n)\|^2 \le \|m\|^2_\infty\|f_n\|^2\|\Phi_\omega\|^2 \le \|m\|^2_\infty C^2M^2, \;\; n \in \mathbb{N}, \omega \in \Omega.$$

So by Lebesgue's Dominated Convergence Theorem

$$\|D_m T^*_\Phi f_n\|^2 = \int_\Omega |m(\omega)|^2\|\Phi_\omega f_n\|^2 d\mu_\omega = \int_K |m(\omega)|^2\|\Phi_\omega f_n\|^2 d\mu_\omega \to 0.$$

Now, By Proposition VI.3.3 of [11], the operator $D_m T_\Phi^*$ is compact. Thus $M_{m,\Lambda,\Phi} = T_\Lambda D_m T_\Phi^*$ is compact. If Λ is norm bounded, then by $M_{m,\Lambda,\Phi}^* = M_{\overline{m},\Phi,\Lambda} = T_\Phi D_{\overline{m}} T_\Lambda^*$, we conclude that $M_{m,\Lambda,\Phi}^*$ is compact and so $M_{m,\Lambda,\Phi}$ is compact. □

Let $m : \Omega \to \mathbb{C}$ be a bounded measurable function. We recall that m vanishes at infinity if for every $\varepsilon > 0$ there exists a measurable subset $K \subseteq \Omega$ such that $\mu(K) < \infty$ and $m(\omega) \leq \varepsilon$, *a.e.* $\omega \in \Omega \setminus K$.

Corollary 3.7. *Let $dim\mathcal{K}_\omega < \infty$ for all $\omega \in \Omega$ and $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ and $\Phi = \{\Phi_\omega\}_{\omega\in\Omega}$ be continuous g-Bessel families with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ and let Λ or Φ be norm bounded. Let $m : \Omega \to \mathbb{C}$ be a bounded measurable function that vanishes at infinity. Then $M_{m,\Lambda,\Phi}$ is compact.*

Proof. For every $n \in \mathbb{N}$, choose a set $K_n \subset \Omega$ such that $\mu(K_n) < \infty$ and $|m(\omega)| \leq \frac{1}{n}$ for almost every $\omega \in \Omega \setminus K_n$. Let us consider

$$m_n(\omega) = m(\omega)\chi_{K_n}(\omega)$$

where χ_{K_n} denotes the characteristic function of the set K_n. Then $\|m_n - m\|_\infty \leq \frac{1}{n} \to 0$ and by Lemma 3.1, we have

$$\|M_{m_n,\Lambda,\Phi} - M_{m,\Lambda,\Phi}\| = \|M_{m_n-m,\Lambda,\Phi}\| \leq \|m_n - m\|_\infty \sqrt{B_\Lambda B_\Phi} \longrightarrow 0.$$

The functions m_n are bounded and of finite support, so $M_{m_n,\Lambda,\Phi}$ is compact for every $n \in \mathbb{N}$ and by Theorem 3.6, $M_{m,\Lambda,\Phi}$ is compact. □

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $T \in B(\mathcal{H},\mathcal{K})$ is a compact operator, then there exists a unique compact and non-negative operator S such that $S^2 = T^*T$. The eigenvalues of S are called singular values of T and they form a non-increasing sequence of non-negative numbers that either consists of only finitely many nonzero terms or converges to zero. The space of all compact operators T on $\mathcal{H}$ with its singular value sequence $\{\lambda_n\}$ belonging to ℓ_p is called Schatten p-class and denoted by $S_p(\mathcal{H})$ for $1 \leq p < \infty$. $S_1(\mathcal{H})$ is also called the trace-class of $\mathcal{H}$, and $S_2(\mathcal{H})$ is usually called the Hilbert-Schmidt class. The space $S_p(\mathcal{H})$ is a Banach space with the norm

$$\|T\|_{\mathcal{S}_p} = \Big(\sum_n |\lambda_n|^p\Big)^{1/p}.$$

We use the following lemma in the proof of Theorem 3.9 and its proof can be found in the literature.

Lemma 3.8. *Let $\mathcal{H}$ be a Hilbert space. A bounded operator $T : \mathcal{H} \to \mathcal{H}$ is trace class if and only if $\sum_n |\langle Te_n, e_n\rangle| < \infty$ for every orthonormal basis (e_n) of $\mathcal{H}$. Moreover,*

$$\|T\|_{\mathcal{S}_1} = \sup\left\{\sum_n |\langle Te_n e_n\rangle|, \quad \{e_n\} \text{ orthonormal basis}\right\}.$$

Theorem 3.9. *Let $M > 0$ be such that $dim(\mathcal{K}_\omega) \leq M$, for all $\omega \in \Omega$ and $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ and $\Phi = \{\Phi_\omega\}_{\omega\in\Omega}$ be norm bounded continuous g-Bessel families with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ with bounds L_Λ and L_Φ, respectively. Let $m \in L^1(\Omega, \mu)$. Then $M_{m,\Lambda,\Phi}$ is a bounded trace-class operator with*

$$\|M_{m.\Lambda,\Phi}\|_{\mathcal{S}_1} \leq \|m\|_1 L_\Lambda L_\Phi M.$$

Proof. We have

$$\begin{aligned}\left|\int_\Omega m(\omega)\langle\Lambda_\omega^*\Phi_\omega f, g\rangle d\mu_\omega\right| &\leq \int_\Omega |m(\omega)|\|\Lambda_\omega g\|\|\Phi_\omega f\| d\mu_\omega \\ &\leq \int_\Omega |m(\omega)|\|\Lambda_\omega\|\|g\|\|\Phi_\omega\|\|f\| d\mu_\omega \\ &\leq \|f\|\|g\| L_\Lambda L_\Phi \int_\Omega |m(\omega)| d\mu_\omega \\ &= \|f\|\|g\| L_\Lambda L_\Phi \|m\|_1,\end{aligned}$$

for all $f, g \in \mathcal{H}$. Thus $M_{m.\Lambda,\Phi}$ is a well-defined bounded linear operator. Let $\{e_n\}$ be an orthonormal basis of $\mathcal{H}$. Then by using Fubini's theorem and Cauchy Schwarz's inequality we have

$$\begin{aligned}\sum_n |\langle M_{m,\Lambda,\Phi} e_n, e_n\rangle| &= \sum_n \left|\int_\Omega m(\omega)\langle\Lambda_\omega^*\Phi_\omega e_n, e_n\rangle d\mu_\omega\right| \\ &\leq \sum_n \int_\Omega |m(\omega)||\langle\Phi_\omega e_n, \Lambda_\omega e_n\rangle| d\mu_\omega \\ &\leq \int_\Omega |m(\omega)| \sum_n |\langle\Phi_\omega e_n, \Lambda_\omega e_n\rangle| d\mu_\omega \\ &\leq \int_\Omega |m(\omega)| \Big(\sum_n \|\Lambda_\omega e_n\|^2\Big)^{\frac{1}{2}} \Big(\sum_n \|\Phi_\omega e_n\|^2\Big)^{\frac{1}{2}} d\mu_\omega.\end{aligned}$$

Now, let $\{h_{\omega i}\}_{i\in I_\omega}$ be an orthonormal basis for $\mathcal{K}_\omega$, for all $\omega \in \Omega$. Then

$$\begin{aligned}\sum_n \|\Lambda_\omega e_n\|^2 &= \sum_n \sum_{i\in I_\omega} |\langle\Lambda_\omega e_n, h_{\omega i}\rangle|^2 = \sum_{i\in I_\omega} \sum_n |\langle e_n, \Lambda_\omega^* h_{\omega i}\rangle|^2 \\ &= \sum_{i\in I_\omega} \|\Lambda_\omega^* h_{\omega i}\|^2 \leq M\|\Lambda_\omega^*\|^2 = M\|\Lambda_\omega\|^2,\end{aligned}$$

and similarly, $\sum_n \|\Phi_\omega e_n\|^2 \leq M\|\Phi_\omega\|^2$. So

$$\begin{aligned}\sum_n |\langle M_{m,\Lambda,\Phi} e_n, e_n\rangle| &\leq \int_\Omega |m(\omega)| \Big(\sum_n \|\Lambda_\omega e_n\|^2\Big)^{\frac{1}{2}} \Big(\sum_n \|\Phi_\omega e_n\|^2\Big)^{\frac{1}{2}} d\mu_\omega \\ &\leq M \int_\Omega |m(\omega)|\|\Lambda_\omega\|\|\Phi_\omega\| d\mu_\omega \\ &\leq \|m\|_1 L_\Lambda L_\Phi M.\end{aligned}$$

Hence $M_{m,\Lambda,\Phi}$ is a trace class operator and by Lemma 3.8,

$$\|M_{m,\Lambda,\Phi}\|_{\mathcal{S}_1} \leq \|m\|_1 L_\Lambda L_\Phi M.$$

□

Theorem 3.10. *Let $M > 0$ be such that $dim(\mathcal{K}_\omega) \leq M$, for all $\omega \in \Omega$. Let $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ and $\Phi = \{\Phi_\omega\}_{\omega\in\Omega}$ be norm bounded continuous g-Bessel families with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ with bounds L_Λ and L_Φ, respectively. Let $m \in L^p(\Omega,\mu), 1 < p < \infty$. Then $M_{m,\Lambda,\Phi}$ is a well-defined bounded operator that belongs to the Schatten p-class $S_p(\mathcal{H})$, with*

$$\|M_{m.\Lambda,\Phi}\|_{S_p} \leq \|m\|_p (L_\Lambda L_\Phi M)^{\frac{1}{p}} (B_\Lambda B_\Phi)^{\frac{1}{2q}}.$$

Proof. The function $\omega \to \langle \Lambda_\omega^* \Phi_\omega f, g\rangle$ is bounded for all $f, g \in \mathcal{H}$, since

$$|\langle \Lambda_\omega^* \Phi_\omega f, g\rangle| = |\langle \Phi_\omega f, \Lambda_\omega g\rangle| \leq \|\Phi_\omega f\|\|\Lambda_\omega g\| \leq L_\Phi L_\Lambda \|f\|\|g\|.$$

Furthermore,

$$\begin{aligned}\int_\Omega |\langle \Phi_\omega f, \Lambda_\omega g\rangle|^q d\mu_\omega &\leq L_\Phi^{q-1} L_\Lambda^{q-1} \|f\|^{q-1}\|g\|^{q-1} \int_\Omega |\langle \Phi_\omega f, \Lambda_\omega g\rangle| d\mu_\omega \\ &\leq L_\Phi^{q-1} L_\Lambda^{q-1} \|f\|^{q-1}\|g\|^{q-1} \Big(\int_\Omega \|\Phi_\omega f\|^2 d\mu_\omega\Big)^{\frac{1}{2}} \Big(\int_\Omega \|\Lambda_\omega g\|^2 d\mu_\omega\Big)^{\frac{1}{2}} \\ &\leq L_\Phi^{q-1} L_\Lambda^{q-1} \|f\|^q \|g\|^q \sqrt{B_\Phi B_\Lambda},\end{aligned}$$

for all $f, g \in \mathcal{H}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$\begin{aligned}|\langle M_{m,\Lambda,\Phi} f, g\rangle| &\leq \int_\Omega |m(\omega)||\langle \Phi_\omega f, \Lambda_\omega g\rangle| d\mu_\omega \\ &\leq \Big(\int_\Omega |m(\omega)|^p d\mu_\omega\Big)^{\frac{1}{p}} \Big(\int_\Omega |\langle \Phi_\omega f, \Lambda_\omega g\rangle|^q d\mu_\omega\Big)^{\frac{1}{q}} \\ &\leq \|m\|_p (L_\Phi L_\Lambda)^{\frac{q-1}{q}} \|f\|\|g\| (B_\Phi B_\Lambda)^{\frac{1}{2q}},\end{aligned}$$

for all $f, g \in \mathcal{H}$. This show that $M_{m,\Lambda,\Phi}$ is a well-defined bounded operator. By Lemma 3.1 the mapping

$$L^\infty(\Omega,\mu) \to \mathcal{B}(\mathcal{H}), \quad m \to M_{m,\Lambda,\Phi},$$

is a bounded linear operator and by Theorem 3.9,

$$L^1(\Omega,\mu) \to \mathcal{S}_1(\mathcal{H}), \quad m \to M_{m,\Lambda,\Phi},$$

is a bounded linear operator. Assume that $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{\infty}$. Then Theorem 5.1.1 of [10] implies that

$$[L_1(\Omega,\mu), L_\infty(\Omega,\mu)]_\theta = L_p(\Omega,\mu).$$

Also, by [20, Theorem 2.2.7], we have

$$[S_1(\mathcal{H}), B(\mathcal{H})]_\theta = S_p(\mathcal{H}).$$

Now, by [20, Theorem 2.2.4], we conclude that the mapping

$$L^p(\Omega,\mu) \to S_p(\mathcal{H}), \quad m \to M_{m,\Lambda,\Phi},$$

is a bounded linear operator and

$$\begin{aligned}\|M_{m,\Lambda,\Phi}\|_{S_p} &\leq \|m\|_p (L_\Phi L_\Lambda M)^{1-\theta}(\sqrt{B_\Phi B_\Lambda})^{\theta} \\ &= \|m\|_p (L_\Phi L_\Lambda M)^{\frac{1}{p}} (B_\Phi B_\Lambda)^{\frac{1}{2q}},\end{aligned}$$

where $\theta = 1 - \frac{1}{p} = \frac{1}{q}$. □

In the following results we show the continuous dependency of continuous g-Bessel multipliers on their parameters.

Corollary 3.11. *Let $M > 0$ be such that $dim(\mathcal{K}_\omega) \leq M$, for all $\omega \in \Omega$. Let $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ and $\Phi = \{\Phi_\omega\}_{\omega\in\Omega}$ be norm bounded continuous g-Bessel families with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ and $m : \Omega \to \mathbb{C}$ be a measurable function. Let $m^{(n)}$ be functions indexed by $n \in \mathbb{N}$ with $m^{(n)} \to m$ in $L^p(\Omega,\mu)$. Then*

$$\|M_{m^{(n)},\Lambda,\Phi} - M_{m,\Lambda,\Phi}\|_{S_p} \to 0, \quad n \to \infty.$$

Theorem 3.12. *Let $m \in L^2(\Omega,\mu)$ and $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ and $\Phi = \{\Phi_\omega\}_{\omega\in\Omega}$ be continuous g-Bessel families with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$. Let $\Lambda^{(n)} = \{\Lambda_\omega^{(n)} \in B(\mathcal{H},\mathcal{K}_\omega); \omega \in \Omega\}$ be a sequence of continuous g-Bessel families with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ and for given $\varepsilon > 0$, there exists N such that for all $\omega \in \Omega$ and $n \geq N$, $\|\Lambda_\omega^{(n)} - \Lambda_\omega\| < \varepsilon$. Then $M_{m,\Lambda^{(n)},\Phi}$ converges to $M_{m,\Lambda,\Phi}$ in the operator norm.*

Proof. We have

$$\begin{aligned}|\langle (M_{m,\Lambda^{(n)},\Phi} - M_{m,\Lambda,\Phi})f, g\rangle| &\leq \int_\Omega |m(\omega)| \|\Phi_\omega f\| \|(\Lambda_\omega^{(n)} - \Lambda_\omega)g\| d\mu_\omega \\ &\leq \Big(\int_\Omega |m(\omega)|^2 \|(\Lambda_\omega^{(n)} - \Lambda_\omega)g\|^2 d\mu_\omega\Big)^{\frac{1}{2}} \Big(\int_\Omega \|\Phi_\omega f\|^2 d\mu_\omega\Big)^{\frac{1}{2}} \\ &\leq \varepsilon \|g\| \|m\|_2 \sqrt{B_\Phi} \|f\|,\end{aligned}$$

for all $f, g \in \mathcal{H}$ and for all $n \geq N$. Thus,

$$\|M_{m,\Lambda^{(n)},\Phi} - M_{m,\Lambda,\Phi}\| \leq \varepsilon \|m\|_2 \sqrt{B_\Phi}, \quad n \geq N.$$

So $M_{m,\Lambda^{(n)},\Phi}$ converges to $M_{m,\Lambda,\Phi}$ in the operator norm. □

Theorem 3.13. *Let $m \in L^1(\Omega,\mu)$ and $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ and $\Phi = \{\Phi_\omega\}_{\omega\in\Omega}$ be continuous g-Bessel families with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$, and Φ be norm bounded with bound L_Φ. Let $\Lambda^{(n)} = \{\Lambda_\omega^{(n)} \in B(\mathcal{H},\mathcal{K}_\omega); \omega \in \Omega\}$ be a sequence of continuous g-Bessel families with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ and for given $\varepsilon > 0$, there exists N such that for all $\omega \in \Omega$ and $n \geq N$, $\|\Lambda_\omega^{(n)} - \Lambda_\omega\| < \varepsilon$. Then, $M_{m,\Lambda^{(n)},\Phi}$ converges to $M_{m,\Lambda,\Phi}$ in the operator norm.*

Proof. We have

$$\begin{aligned}\left|\langle (M_{m,\Lambda^{(n)},\Phi} - M_{m,\Lambda,\Phi})f, g\rangle\right| &\leq \int_\Omega |m(\omega)||\langle \Phi_\omega f, (\Lambda_\omega^{(n)} - \Lambda_\omega)g\rangle| d\mu_\omega \\ &\leq \int_\Omega |m(\omega)|\|\Phi_\omega\|\|f\|\|\Lambda_\omega^{(n)} - \Lambda_\omega\|\|g\| d\mu_\omega \\ &\leq \varepsilon L_\Phi \|g\|\|f\|\|m\|_1,\end{aligned}$$

for all $n \geq N$ and for all $f, g \in \mathcal{H}$. Hence,

$$\|M_{m,\Lambda^{(n)},\Phi} - M_{m,\Lambda,\Phi}\| < \varepsilon L_\Phi \|m\|_1, \quad n \geq N.$$

Thus, $M_{m,\Lambda^{(n)},\Phi}$ converges to $M_{m,\Lambda,\Phi}$ in the operator norm. □

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