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# A NEW RESULT ON CHROMATICITY OF $K_4$-HOMEOMORPHIC GRAPHS WITH GIRTH 9

N.S.A. KARIM, R. HASNI* AND G.C. LAU

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Abstract. For a graph $G$, let $P(G, \lambda)$ denote the chromatic polynomial of $G$. Two graphs $G$ and $H$ are chromatically equivalent if they share the same chromatic polynomial. A graph $G$ is chromatically unique if any graph chromatically equivalent to $G$ is isomorphic to $G$. A $K_4$-homeomorph is a subdivision of the complete graph $K_4$. In this paper, we determine a family of chromatically unique $K_4$-homeomorphs which have girth 9 and have exactly one path of length 1, and give sufficient and necessary condition for the graphs in this family to be chromatically unique.

**Keywords:** Chromatic polynomial, chromatically unique, $K_4$-homeomorphs.
**MSC(2010):** 05C15.

## 1. Introduction

All graphs considered here are simple graphs. For such a graph $G$, let $P(G, \lambda)$ denote the chromatic polynomial of $G$. Two graphs $G$ and $H$ are chromatically equivalent (or simply $\chi-$equivalent), denoted by $G \sim H$, if $P(G, l) = P(H, l)$. A graph $G$ is chromatically unique (or simply $\chi-$unique) if for any graph $H$ such as $H \sim G$, we have $H \cong G$, i.e, $H$ is isomorphic to $G$. The search for $\chi-$unique graphs has been the subject of much interest in chromatic graph theory (see [5, 10, 11]).

A $K_4$-homeomorph is a subdivision of the complete graph $K_4$. Such a homeomorph is denoted by $K_4(a, b, c, d, e, f)$ where the six edges of $K_4$ are replaced by the six paths of length $a, b, c, d, e$ and $f$, respectively, as shown in Figure 1. So far, the chromaticity of $K_4$-homeomorphs with girth $g$, where $3 \le g \le 7$ has been studied by many authors (see [4, 12–15]). Also the study of the chromaticity of $K_4$-homeomorphs with at least 2 paths of length 1 has been completed

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(see [6,12,16,20]). Recently, Shi et al. [19] studied the chromaticity of one family of $K_4$-homeomorphs with girth 8, that is, $K_4(2,3,3,d,e,f)$. In [18], Shi solved completely the chromaticity of $K_4$-homeomorphs with girth 8. By Ren [17], the chromaticity of $K_4$-homeomorphs with exactly 3 paths of same length has been obtained. Recently, Catada-Ghimire and Hasni [1] investigated the chromaticity of $K_4$-homeomorphs with exactly 2 paths of length 2. Hence, to completely determine the chromaticity of $K_4$-homeomorph with girth 9, there are only 6 more types to be solved, that is, $K_4(1,2,6,d,e,f)$, $K_4(1,3,5,d,e,f)$, $K_4(1,4,4,d,e,f)$, $K_4(2,3,4,d,e,f)$, $K_4(1,2,c,3,e,3)$ and $K_4(1,3,c,2,e,3)$. The chromaticity of the graphs $K_4(2,3,4,d,e,f)$ and $K_4(1,4,4,d,e,f)$ were solved by Karim et al. [8,9]. In this paper, we investigate the chromaticity of another type $K_4(1,2,6,d,e,f)$.

In [5], the following problem was posed:

**Problem A** Study the chromaticity of $K_4$-homeomorphs with exactly one path of length 1 (Page 123).

The results in this paper give a partial solution to Problem A and leaving the general case undecided as well as to complete the study of the chromaticity of $K_4$-homeomorph with girth 9.

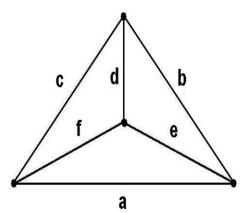


FIGURE 1. $K_4(a,b,c,d,e,f)$

## 2. **Preliminary results**

In this section, we give some known results used in this paper.

**Lemma 2.1.** *Assume that $G$ and $H$ are $\chi-$equivalent. Then*

(1) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$ ([10]);

(2) *$G$ and $H$ has the same girth and same number of cycles with length equal to their girth* ([21]);

(3) *If $G$ is a $K_4$-homeomorph, then $H$ must itself be a $K_4$-homeomorph* ([3]);

(4) *Let $G = K_4(a,b,c,d,e,f)$ and $H = K_4(a',b',c',d',e',f')$. Then*

(i) *min* $\{a, b, c, d, e, f\}$ = *min* $\{a', b', c', d', e', f'\}$ *and the number of times that this minimum occurs in the list* $\{a, b, c, d, e, f\}$ *is equal to the number of times that this minimum occurs in the list* $\{a', b', c', d', e', f'\}$ ([20]);

(ii) *if* $\{a, b, c, d, e, f\} = \{a', b', c', d', e', f'\}$ *as multisets, then* $H \cong G$ ([12]).

**Lemma 2.2.** (Catada-Ghimire et al. [2]) *Let* $K_4$*-homeomorphs* $K_4(1, 2, c, d, e, f)$ *and* $K_4(1, 2, c, d', e', f')$ *be non-isomorphic chromatically equivalent. Then*

$$K_4(1, 2, c, i, i + c + 1, i + 1) \sim K_4(1, 2, c, i + 2, i, i + c),$$
$$K_4(1, 2, c, i, i + 1, i + c + 1) \sim K_4(1, 2, c, i + c, i, i + 2),$$
$$K_4(1, 2, c, i, i + 1, i + 3) \sim K_4(1, 2, c, i + 2, i + 2, i),$$

*where* $i \geq 1$.

**Lemma 2.3.** (Karim et al. [9]) *Let* $K_4$*-homeomorphs* $K_4(1, 2, 6, d, e, f)$ *and* $K_4(1, 4, 4, d', e', f')$ *be chromatically equivalent. Then*

$$K_4(1, 2, 6, 4, 4, 4) \sim K_4(1, 4, 4, 2, 3, 7).$$

**Lemma 2.4.** (Hasni [7]) *Let* $K_4$*-homeomorphs* $K_4(1, 2, 6, d, e, f)$ *and* $K_4(2, 3, 4, d', e', f')$ *be chromatically equivalent. Then*

$$K_4(1, 2, 6, 4, s, 4) \sim K_4(2, 3, 4, 1, 7, s),$$
$$K_4(1, 2, 6, 6, 3, 4) \sim K_4(2, 3, 4, 7, 1, 5),$$
$$K_4(1, 2, 6, 6, 4, 4) \sim K_4(2, 3, 4, 1, 5, 8),$$
$$K_4(1, 2, 6, 9, 3, 5) \sim K_4(2, 3, 4, 10, 6, 1),$$
$$K_4(1, 2, 6, 5, 5, 5) \sim K_4(2, 3, 4, 6, 6, 1),$$

*where* $s \geq 4$.

**Lemma 2.5.** (Ren [17]) *Let* $G = K_4(a, b, c, d, e, f)$*, where exactly three of* $a, b, c, d, e, f$ *are the same. Then* $G$ *is not chromatically unique if and only if* $G$ *is isomorphic to* $K_4(s, s, s - 2, 1, 2, s)$ *or* $K_4(s, s - 2, s, 2s - 2, 1, s)$ *or* $K_4(t, t, 1, 2t, t + 2, t)$ *or* $K_4(t, t, 1, 2t, t - 1, t)$ *or* $K_4(t, t + 1, t, 2t + 1, 1, t)$ *or* $K_4(1, t, 1, t + 1, 3, 1)$ *or* $K_4(1, 1, t, 2, t + 2, 1)$*, where* $s \geq 3$, $t \geq 2$.

**Lemma 2.6.** (Catada-Ghimire and Hasni [1]) *A* $K_4$*-homeomorphic graph with exactly two paths of length two is* $\chi$*-unique if and only if it is not isomorphic to* $K_4(1, 2, 2, 4, 1, 1)$ *or* $K_4(4, 1, 2, 1, 2, 4)$ *or* $K_4(1, s + 2, 2, 1, 2, s)$ *or* $K_4(1, 2, 2, t + 2, t + 2, t)$ *or* $K_4(1, 2, 2, t, t + 1, t + 3)$ *or* $K_4(3, 2, 2, r, 1, 5)$ *or* $K_4(1, r, 2, 4, 2, 4)$ *or* $K_4(3, 2, 2, r, 1, r + 3)$ *or* $K_4(r + 2, 2, 2, 1, 4, r)$ *or* $K_4(r + 3, 2, 2, r, 1, 3)$ *or* $K_4(4, 2, 2, 1, r + 2, r)$ *or* $K_4(3, 4, 2, 4, 2, 6)$ *or* $K_4(3, 4, 2, 4, 2, 8)$ *or* $K_4(3, 4, 2, 8, 2, 4)$ *or* $K_4(7, 2, 2, 3, 4, 5)$ *or* $K_4(5, 2, 2, 3, 4, 7)$ *or* $K_4(8, 2, 2, 3, 4, 6)$ *or* $K_4(5, 2, 2, 9, 3, 4)$ *or* $K_4(5, 2, 2, 5, 3, 4)$*, where* $r \geq 3$, $s \geq 3$, $t \geq 3$.

## 3. Main result

In this section, we present our main results. In the following, we only consider graphs of girth 9 with at most one path of length 1.

We now study the chromaticity of $K_4(1,2,6,d,e,f)$. First, we prove the following lemma.

**Lemma 3.1.** *Let $K_4$-homeomorphs $K_4(1,2,6,d,e,f)$ and $K_4(1,3,5,d',e',f')$ be chromatically equivalent. Then*

$$\begin{aligned}&K_4(1,2,6,4,5,8) \sim K_4(1,3,5,2,6,9),\\&K_4(1,2,6,4,7,5) \sim K_4(1,3,5,2,8,6),\\&K_4(1,2,6,3,4,10) \sim K_4(1,3,5,9,2,6),\\&K_4(1,2,6,3,4,6) \sim K_4(1,3,5,5,6,2),\\&K_4(1,2,6,5,3,8) \sim K_4(1,3,5,7,2,7),\\&K_4(1,2,6,5,9,3) \sim K_4(1,3,5,7,8,2),\\&K_4(1,2,6,f+2,4,f) \sim K_4(1,3,5,2,f,f+4),\end{aligned}$$

where $f \geq 4$.

*Proof.* Let $G$ and $H$ be two graphs such that $G \cong K_4(1,2,6,d,e,f)$ and $H \cong K_4(1,3,5,d',e',f')$. Let

$$\begin{aligned}Q(K_4(a,b,c,d,e,f)) \quad = \quad & -(s+1)(s^a+s^b+s^c+s^d+s^e+s^f)+s^{a+d}+\\ & s^{b+f}+s^{c+e}+s^{a+b+e}+s^{b+d+c}+s^{a+c+f}+s^{d+e+f}.\end{aligned}$$

Let $s = 1-\lambda$ and let $x$ be the number of edges in $G$. From [20], we have the chromatic polynomial of $K_4$-homeomorphs $K_4(a,b,c,d,e,f)$ is as follows:

$$\begin{aligned}P(K_4(a,b,c,d,e,f) \quad = \quad & (-1)^{x-1}\frac{s}{(s-1)^2}\Big[(s^2+3s+2)+\\ & Q(K_4(a,b,c,d,e,f))-s^{x-1}\Big].\end{aligned}$$

Hence $P(G) = P(H)$ if and only if $Q(G) = Q(H)$. We solve the equation $Q(G) = Q(H)$ to get all solutions. Let the lowest remaining power and the highest remaining power to be denoted by l.r.p. and h.r.p., respectively.

As $G \cong K_4(1,2,6,d,e,f)$ and $H \cong K_4(1,3,5,d',e',f')$, we have

$$\begin{aligned}Q(G) \quad = \quad & -(s+1)(s+s^2+s^6+s^d+s^e+s^f)+s^{d+1}+s^{f+2}+\\ & s^{e+6}+s^{e+3}+s^{d+8}+s^{f+7}+s^{d+e+f}.\\ Q(H) \quad = \quad & -(s+1)(s+s^3+s^5+s^{d'}+s^{e'}+s^{f'})+s^{d'+1}+s^{f'+3}+\\ & s^{e'+5}+s^{e'+4}+s^{d'+8}+s^{f'+6}+s^{d'+e'+f'}.\end{aligned}$$

From Lemma 2.1 (1),

$$d+e+f = d'+e'+f' \tag{3.1}$$

$Q(G) = Q(H)$ yields

$$\begin{aligned} Q_1(G) &= -s^2 - s^7 - s^d - s^e - s^{e+1} - s^f - s^{f+1} + \\ &\quad s^{d+8} + s^{e+3} + s^{e+6} + s^{f+2} + s^{f+7}. \\ Q_1(H) &= -s^4 - s^5 - s^{d'} - s^{e'} - s^{e'+1} - s^{f'} - s^{f'+1} + \\ &\quad s^{d'+8} + s^{e'+4} + s^{e'+5} + s^{f'+3} + s^{f'+6}. \end{aligned}$$

By considering the l.r.p in $Q_1(G)$ and the l.r.p in $Q_1(H)$, we have three cases to consider, that is, $d' = 2$ or $e' = 2$ or $f' = 2$. Note that we consider $G$ with at most one path of length 1, then the l.r.p in $Q_1(G)$ cannot occur when $d = 1$ or $e = 1$.

**Case A** $d' = 2$. By cancelling the equal terms in $Q_1(G)$ and $Q_1(H)$, we obtain the following.

$Q_2(G) = -s^7 - s^d - s^e - s^{e+1} - s^f - s^{f+1} + s^{d+8} + s^{e+3} + s^{e+6} + s^{f+2} + s^{f+7}$,

$Q_2(H) = -s^4 - s^5 - s^{e'} - s^{e'+1} - s^{f'} - s^{f'+1} + s^{10} + s^{e'+4} + s^{e'+5} + s^{f'+3} + s^{f'+6}$.

Considering the l.r.p in $Q_2(G)$ and the l.r.p in $Q_2(H)$, we have $d = 4$ or $e = 4$ or $f = 4$ or $e = 3$ or $f = 3$.

**Case 1** $d = 4$. Since $G$ is of girth 9 and $d = 4$, then $e \geq 3$ and $e + f \geq 8$, so $f \geq 5$. We obtain the following after simplification.

$Q_3(G) = -s^7 - s^e - s^{e+1} - s^f - s^{f+1} + s^{12} + s^{e+3} + s^{e+6} + s^{f+2} + s^{f+7}$,

$Q_3(H) = -s^5 - s^{e'} - s^{e'+1} - s^{f'} - s^{f'+1} + s^{10} + s^{e'+4} + s^{e'+5} + s^{f'+3} + s^{f'+6}$.

Considering the l.r.p in $Q_3(G)$ and the l.r.p in $Q_3(H)$ and $f \geq 5$, we have $e = 5$ or $f = 5$ or $e = 4$.

**Case 1.1** $e = 5$. We obtain the following after simplification.

$Q_4(G) = -s^6 - s^7 - s^f - s^{f+1} + s^8 + s^{11} + s^{12} + s^{f+2} + s^{f+7}$,

$Q_4(H) = -s^{e'} - s^{e'+1} - s^{f'} - s^{f'+1} + s^{10} + s^{e'+4} + s^{e'+5} + s^{f'+3} + s^{f'+6}$.

Considering the h.r.p in $Q_4(G)$ and the h.r.p in $Q_4(H)$, we have $e' + 5 = 12$ or $f' + 6 = 12$ or $f + 7 = e' + 5$ or $f + 7 = f' + 6$.

**Case 1.1.1** $e' + 5 = 12$. So $e' = 7$. By Equation (3.1), $f = f'$. We obtain the following after simplification.

$Q_5(G) = -s^6 - s^8 + s^{f+2} + s^{f+7}$, $Q_5(H) = -s^8 + s^{10} + s^{f'+3} + s^{f'+6}$.

Thus, we obtain $Q_5(G) \neq Q_5(H)$, a contradiction.

**Case 1.1.2** $f' + 6 = 12$. So $f' = 6$. By Equation (3.1), $f + 1 = e'$. After simplifying, we obtain $Q_5(G) \neq Q_5(H)$, a contradiction.

**Case 1.1.3** $f + 7 = e' + 5$. So $f + 2 = e'$. By Equation (3.1), $f' = 5$. $Q_5(G) \neq Q_5(H)$, a contradiction.

**Case 1.1.4** $f + 7 = f' + 6$. So $f + 1 = f'$. By Equation (3.1), $e' = 6$. Simplifying $Q_4(G)$ and $Q_4(H)$, we obtain $f = 8$. So $f' = 9$. Therefore, $G \cong K_4(1,2,6,4,5,8)$ and $H \cong K_4(1,3,5,2,6,9)$. Thus, $K_4(1,2,6,4,5,8) \sim K_4(1,3,5,2,6,9)$.

**Case 1.2** $f = 5$. We obtain the following after simplification.

$Q_6(G) = -s^6 - s^e - s^{e+1} + 2s^{12} + s^{e+3} + s^{e+6}$,
$Q_6(H) = -s^{e'} - s^{e'+1} - s^{f'} - s^{f'+1} + s^{10} + s^{e'+4} + s^{e'+5} + s^{f'+3} + s^{f'+6}$.

Consider the h.r.p in $Q_6(G)$ and the h.r.p in $Q_6(H)$. We have $e' + 5 = 12$ or $f' + 6 = 12$ or $e + 6 = e' + 5$ or $e + 6 = f' + 6$.

**Case 1.2.1** $e' + 5 = 12$. So $e' = 7$. By Equation (3.1), $e = f'$. We obtain $Q_6(G) \neq Q_6(H)$, a contradiction.

**Case 1.2.2** $f'+6 = 12$. So $f' = 6$. By Equation (3.1), $e+1 = e'$. Simplifying $Q_6(G)$ and $Q_6(H)$, we obtain

$Q_7(G) = -s^e + s^{12} + s^{e+3}$, $Q_7(H) = -s^7 - s^{e+2} + s^9 + s^{10} + s^{e+5}$.

Then $e = 7$ and $e' = 8$. Therefore, $G \cong K_4(1, 2, 6, 4, 7, 5)$ and $H \cong K_4(1, 3, 5, 2, 8, 6)$. Hence $K_4(1, 2, 6, 4, 7, 5) \sim K_4(1, 3, 5, 2, 8, 6)$.

**Case 1.2.3** $e+6 = e'+5$. So $e+1 = e'$. By Equation (3.1), $f' = 6$. Similar to Case 1.2.2, we have $K_4(1, 2, 6, 4, 7, 5) \sim K_4(1, 3, 5, 2, 8, 6)$.

**Case 1.2.4** $e+6 = f'+6$. So $e = f'$. By Equation (3.1), $e' = 7$. We obtain $Q_6(G) \neq Q_6(H)$, a contradiction.

**Case 1.3** $e = 4$. We obtain the following after simplification.

$Q_8(G) = -s^4 - s^f - s^{f+1} + s^{12} + s^{f+2} + s^{f+7}$,
$Q_8(H) = -s^{e'} - s^{e'+1} - s^{f'} - s^{f'+1} + s^{e'+4} + s^{e'+5} + s^{f'+3} + s^{f'+6}$.

Considering the h.r.p in $Q_8(G)$ and the h.r.p in $Q_8(H)$, we have $e' + 5 = 12$ or $f' + 6 = 12$ or $f + 7 = e' + 5$ or $f + 7 = f' + 6$.

**Case 1.3.1** $e' + 5 = 12$. So $e' = 7$. By Equation (3.1), $f = f' + 1$. After simplifying, we have $Q_8(G) \neq Q_8(H)$, a contradiction.

**Case 1.3.2** $f' + 6 = 12$. So $f' = 6$. By Equation (3.1), $f = e'$. After simplifying, we have $Q_8(G) \neq Q_8(H)$, a contradiction.

**Case 1.3.3** $f + 7 = e' + 5$. So $f + 2 = e'$. By Equation (3.1), $f' = 4$. After simplifying, we have $Q_8(G) \neq Q_8(H)$, a contradiction.

**Case 1.3.4** $f + 7 = f' + 6$. So $f + 1 = f'$. By Equation (3.1), $e' = 5$. After simplifying, we have $Q_8(G) \neq Q_8(H)$, a contradiction.

**Case 1.4** $f = 4$. We already know that $e \geq 3$. But if $e = 3$, there is a cycle of girth 8, a contradiction. Thus we assume $e \geq 4$. We obtain the following after simplification.

$Q_9(G) = -s^4 - s^7 - s^e - s^{e+1} + s^6 + s^{11} + s^{12} + s^{e+3} + s^{e+6}$,
$Q_9(H) = -s^{e'} - s^{e'+1} - s^{f'} - s^{f'+1} + s^{10} + s^{e'+4} + s^{e'+5} + s^{f'+3} + s^{f'+6}$.

Comparing the h.r.p in $Q_9(G)$ and the h.r.p in $Q_9(H)$, we have $e' + 5 = 12$ or $f' + 6 = 12$ or $e + 6 = e' + 5$ or $e + 6 = f' + 6$.

**Case 1.4.1** $e' + 5 = 12$. So $e' = 7$. By Equation (3.1), $e = f' + 1$. After simplifying, we have $Q_9(G) \neq Q_9(H)$, a contradiction.

**Case 1.4.2** $f' + 6 = 12$. So $f' = 6$. By Equation (3.1), $e = e'$. After simplifying, we have $Q_9(G) \neq Q_9(H)$, a contradiction.

**Case 1.4.3** $e + 6 = e' + 5$. So $e + 1 = e'$. By Equation (3.1), $f' = 5$. After simplifying, we have $Q_9(G) \neq Q_9(H)$, a contradiction.

**Case 1.4.4** $e+6=f'+6$. So $e=f'$. By Equation (3.1), $e'=6$. After simplifying, we have $Q_9(G)\neq Q_9(H)$, a contradiction.

**Case 2** $e=4$. We obtain the following after simplification.

$Q_{10}(G)=-s^d-s^f-s^{f+1}+s^{d+8}+s^{f+2}+s^{f+7}$,

$Q_{10}(H)=-s^{e'}-s^{e'+1}-s^{f'}-s^{f'+1}+s^{e'+4}+s^{e'+5}+s^{f'+3}+s^{f'+6}$.

Considering the h.r.p in $Q_{10}(G)$ and the h.r.p in $Q_{10}(H)$, we have $d+8=e'+5$ or $d+8=f'+6$ or $f+7=e'+5$ or $f+7=f'+6$.

**Case 2.1** $d+8=e'+5$. So $d+3=e'$. By Equation (3.1), $f=f'+1$. We obtain the following after simplification.

$Q_{11}(G)=-s^d-s^{f+1}+s^{f+7}$, $Q_{11}(H)=-s^{d+3}-s^{d+4}-s^{f-1}+s^{d+7}+s^{f+5}$.

We obtain $Q_{11}(G)\neq Q_{11}(H)$, a contradiction.

**Case 2.2** $d+8=f'+6$. So $d+2=f'$. By Equation (3.1), $f=e'$. We obtain the following after simplification.

$Q_{12}(G)=-s^d+s^{f+2}+s^{f+7}$, $Q_{12}(H)=-s^{d+2}-s^{d+3}+s^{d+5}+s^{f+4}+s^{f+5}$.

Then we obtain $d=f+2$ and from $d+2=f'$, we have $f'=f+4$. Therefore, $G\cong K_4(1,2,6,f+2,4,f)$ and $H\cong K_4(1,3,5,2,f,f+4)$. Thus $K_4(1,2,6,f+2,4,f)\sim K_4(1,3,5,2,f,f+4)$.

**Case 2.3** $f+7=e'+5$. So $f+2=e'$. By Equation (3.1), $d=f'$. We obtain the following after simplification.

$Q_{13}(G)=-s^f-s^{f+1}+s^{d+8}+s^{f+2}$,

$Q_{13}(H)=-s^{d+1}-s^{f+2}-s^{f+3}+s^{d+3}+s^{d+6}+s^{f+6}$.

Comparing the h.r.p in $Q_{13}(G)$ and the h.r.p in $Q_{13}(H)$, we have $d+6=f+2$ or $d+8=f+6$.

**Case 2.3.1** $d+6=f+2$. So $d+4=f$. After simplification, we obtain $Q_{13}(G)\neq Q_{13}(H)$, a contradiction.

**Case 2.3.2** $d+8=f+6$. So $d+2=f$. After simplification, we obtain $Q_{13}(G)\neq Q_{13}(H)$, a contradiction.

**Case 2.4** $f+7=f'+6$. So $f+1=f'$. By Equation (3.1), $d+1=e'$. After simplification, similar to above cases, we obtain a contradiction.

**Case 3** $f=4$. Note that $e\geq 4$ since $G$ is of girth 9 and $d\geq 2$. We know that $e\geq 5$ when $d=2$. We obtain the following after simplification.

$Q_{14}(G)=-s^7-s^d-s^e-s^{e+1}+s^6+s^{11}+s^{d+8}+s^{e+3}+s^{e+6}$,

$Q_{14}(H)=-s^{e'}-s^{e'+1}-s^{f'}-s^{f'+1}+s^{10}+s^{e'+4}+s^{e'+5}+s^{f'+3}+s^{f'+6}$.

Comparing the h.r.p in $Q_{14}(G)$ and the h.r.p in $Q_{14}(H)$, we have $e'+5=11$ or $f'+6=11$ or $d+8=e'+5$ or $d+8=f'+6$ or $e+6=e'+5$ or $e+6=f'+6$.

**Case 3.1** $e'+5=11$. So $e'=6$. We obtain the following after simplification.

$Q_{15}(G)=-s^d-s^e-s^{e+1}+s^6+s^{d+8}+s^{e+3}+s^{e+6}$,

$Q_{15}(H)=-s^6-s^{f'}-s^{f'+1}+2s^{10}+s^{f'+3}+s^{f'+6}$.

The h.r.p in $Q_{15}(H)$ is 10 or $f'+6$.

**Case 3.1.1** $10\geq f'+6$. Considering the h.r.p in $Q_{15}(G)$, we have $d+8=10$ or $e+6=10$.

**Case 3.1.1.1** $d+8=10$. So $d=2$. By Equation (3.1), $e=f'+2$. After simplification, we obtain $Q_{15}(G)\neq Q_{15}(H)$, a contradiction.

**Case 3.1.1.2** $e+6=10$. So $e=4$. By Equation (3.1), $d=f'$. After simplification, we obtain $Q_{15}(G)\neq Q_{15}(H)$, a contradiction.

**Case 3.1.2** $f'+6>10$. Considering the h.r.p in $Q_{15}(G)$, we have $d+8=f'+6$ or $e+6=f'+6$.

**Case 3.1.2.1** $d+8=f'+6$. So $d+2=f'$. By Equation (3.1), $e=6$. After simplification, we obtain $Q_{15}(G)\neq Q_{15}(H)$.

**Case 3.1.2.2** $e+6=f'+6$. So $e=f'$. By Equation (3.1), $d=4$. After simplification, we obtain $Q_{15}(G)\neq Q_{15}(H)$.

**Case 3.2** $f'+6=11$. So $f'=5$. Note that $d\geq 3$. We obtain the following after simplification.

$Q_{16}(G)=-s^7-s^d-s^e-s^{e+1}+s^6+s^{d+8}+s^{e+3}+s^{e+6}$,

$Q_{16}(H)=-s^5-s^6-s^{e'}-s^{e'+1}+s^8+s^{10}+s^{e'+4}+s^{e'+5}$.

By comparing the h.r.p in $Q_{16}$ and the h.r.p in $Q_{16}$, we have $d+8=e'+5$ or $e+6=10$ or $e+6=e'+5$.

**Case 3.2.1** $d+8=e'+5$. So $d+3=e'$. By Equation (3.1), $e=6$. We obtain the following after simplification.

$Q_{17}(G)=-2s^7-s^d+s^6+s^9+s^{12}$, $Q_{17}(H)=-s^5-s^{d+3}-s^{d+4}+s^8+s^{10}+s^{d+7}$.

Note that there exists the term $-2s^7$ in $Q_{17}(G)$ but not in $Q_{17}(H)$, a contradiction.

**Case 3.2.2** $e+6=10$. So $e=4$. By Equation (3.1), $d+1=e'$. After simplifying, we have $Q_{16}(G)\neq Q_{16}(H)$, a contradiction.

**Case 3.2.3** $e+6=e'+5$. So $e+1=e'$. By Equation (3.1), $d=4$. After simplifying, we have $Q_{16}(G)\neq Q_{16}(H)$, a contradiction.

**Case 3.3** $d+8=e'+5$. So $d+3=e'$. By Equation (3.1), $d+3=e'$. After simplifying, we obtain a contradiction.

**Case 3.4** $d+8=f'+6$. So $d+2=f'$. By Equation (3.1), $e'=4$. After simplifying, we obtain $G\cong K_4(1,2,6,6,4,4)$ and $H\cong K_4(1,3,5,2,4,8)$. Hence $K_4(1,2,6,6,4,4)\sim K_4(1,3,5,2,4,8)$.

**Case 3.5** $e+6=e'+5$. So $e+1=e'$. By Equation (3.1), $d+1=f'$. We obtain the following after simplification.

$Q_{18}(G)=-s^7-s^d-s^e+s^6+s^{11}+s^{d+8}+s^{e+3}$,

$Q_{18}(H)=-s^{e+2}-s^{d+1}-s^{d+2}+s^{10}+s^{d+4}+s^{d+7}+s^{e+5}$.

Considering the h.r.p in $Q_{18}(G)$ and the h.r.p in $Q_{18}(H)$, we have $e+5=11$ or $d+7=11$ or $d+8=e+5$ or $d+7=e+3$.

**Case 3.5.1** $e+5=11$. So $e=6$. After simplification, we obtain $Q_{18}(G)\neq Q_{18}(H)$ and hence a contradiction.

**Case 3.5.2** $d+7=11$. So $d=4$. After simplification, we obtain $Q_{18}(G)\neq Q_{18}(H)$, a contradiction.

**Case 3.5.3** $d+8=e+5$. So $d+3=e$. After simplification, we obtain $Q_{18}(G) \neq Q_{18}(H)$, a contradiction.

**Case 3.5.4** $d+7=e+3$. So $d+4=e$. After simplification, we obtain $Q_{18}(G) \neq Q_{18}(H)$, a contradiction.

**Case 3.6** $e+6=f'+6$. So $e=f'$. By Equation (3.1), $d+2=e'$. After simplifying, we obtain a contradiction.

**Case 4** $e=3$. Note that $d \geq 4$ and $f \geq 5$. We obtain the following after simplification.

$Q_{19}(G) = -s^3 - s^7 - s^d - s^f - s^{f+1} + s^6 + s^9 + s^{d+8} + s^{f+2} + s^{f+7}$,

$Q_{19}(H) = -s^5 - s^{e'} - s^{e'+1} - s^{f'} - s^{f'+1} + s^{10} + s^{e'+4} + s^{e'+5} + s^{f'+3} + s^{f'+6}$.

Note that $e' \geq 4$ and $f' \geq 3$ since girth of $H$ is 9. By comparing the l.r.p in $Q_{19}(G)$ and the l.r.p in $Q_{17}(H)$, we have $f'=3$. We obtain the following after simplification.

$Q_{20}(G) = -s^7 - s^d - s^f - s^{f+1} + s^{d+8} + s^{f+2} + s^{f+7}$,

$Q_{20}(H) = -s^4 - s^5 - s^{e'} - s^{e'+1} + s^{10} + s^{e'+4} + s^{e'+5}$.

Since $f \geq 5$, the term $s^d$ in $Q_{20}(G)$ is equal to the term $s^4$ in $Q_{20}(H)$, that is, $d=4$. By Equation (3.1), $f+2=e'$. After simplification, we obtain $Q_{20}(G) \neq Q_{20}(H)$, a contradiction.

**Case 5** $f=3$. We obtain the following after simplification.

$Q_{21}(G) = -s^3 - s^7 - s^d - s^e - s^{e+1} + s^5 + s^{d+8} + s^{e+3} + s^{e+6}$,

$Q_{21}(H) = -s^5 - s^{e'} - s^{e'+1} - s^{f'} - s^{f'+1} + s^{e'+4} + s^{e'+5} + s^{f'+3} + s^{f'+6}$.

Note that $e' \geq 4$ since girth of $H$ is 9. By comparing the l.r.p in $Q_{21}(G)$ and the l.r.p in $Q_{21}(H)$, we have $f'=3$. We obtain the following after simplification.

$Q_{22}(G) = -s^7 - s^d - s^e - s^{e+1} + s^5 + s^{d+8} + s^{e+3} + s^{e+6}$,

$Q_{22}(H) = -s^4 - s^5 - s^{e'} - s^{e'+1} + s^6 + s^9 + s^{e'+4} + s^{e'+5}$.

Since $e \geq 5$, the term $s^d$ in $Q_{22}(G)$ is equal to the term $s^4$ in $Q_{22}(H)$, that is, $d=4$. By Equation (3.1), $e+2=e'$. After simplification, we obtain $Q_{22}(G) \neq Q_{22}(H)$, a contradiction.

**Case B** $e'=2$. We obtain the following after simplification.

$Q_{23}(G) = -s^7 - s^d - s^e - s^{e+1} - s^f - s^{f+1} + s^{d+8} + s^{e+3} + s^{e+6} + s^{f+2} + s^{f+7}$,

$Q_{23}(H) = -s^3 - s^4 - s^5 - s^{d'} - s^{f'} - s^{f'+1} + s^6 + s^7 + s^{d'+8} + s^{f'+3} + s^{f'+6}$.

Considering the l.r.p in $Q_{23}(G)$ and the l.r.p in $Q_{23}(H)$, we have $d=3$ or $e=3$ or $f=3$.

**Case 1** $d=3$. Note that $e \geq 4$ and $f \geq 3$. We obtain the following after simplification.

$Q_{24}(G) = -s^7 - s^e - s^{e+1} - s^f - s^{f+1} + s^{11} + s^{e+3} + s^{e+6} + s^{f+2} + s^{f+7}$,

$Q_{24}(H) = -s^4 - s^5 - s^{d'} - s^{f'} - s^{f'+1} + s^6 + s^7 + s^{d'+8} + s^{f'+3} + s^{f'+6}$.

Comparing the l.r.p in $Q_{24}(G)$ and the l.r.p in $Q_{24}(H)$, we have $e=4$ or $f=4$ or $f=3$.

**Case 1.1** $e=4$. We obtain the following after simplification.

$Q_{25}(G) = -s^7 - s^f - s^{f+1} + s^{10} + s^{11} + s^{f+2} + s^{f+7}$,

$Q_{25}(H) = -s^{d'} - s^{f'} - s^{f'+1} + s^6 + s^{d'+8} + s^{f'+3} + s^{f'+6}$.

Comparing the h.r.p in $Q_{25}(G)$ and the h.r.p in $Q_{25}(H)$, we have $f+7 = d'+8$ or $f + 7 = f' + 6$.

**Case 1.1.1** $f + 7 = d' + 8$. So $f = d' + 1$. By Equation (3.1), $f' = 6$. After simplification, we have $f = 10$ and $d' = 9$. Thus, $G \cong K_4(1, 2, 6, 3, 4, 10)$ and $H \cong K_4(1, 3, 5, 9, 2, 6)$. Hence $K_4(1, 2, 6, 3, 4, 10) \sim K_4(1, 3, 5, 9, 2, 6)$.

**Case 1.1.2** $f + 7 = f' + 6$. So $f + 1 = f'$. By Equation (3.1), $d' = 4$. After simplification, we obtain $Q_{25}(G) \neq Q_{25}(H)$, a contradiction.

**Case 1.2** $f = 4$. We obtain the following after simplification.

$Q_{26}(G) = -s^7 - s^e - s^{e+1} + 2s^{11} + s^{e+3} + s^{e+6}$,

$Q_{26}(H) = -s^{d'} - s^{f'} - s^{f'+1} + s^7 + s^{d'+8} + s^{f'+3} + s^{f'+6}$.

Note that the h.r.p. in $Q_{26}(G)$ is 11 or $e + 6$.

**Case 1.2.1** $11 \geq e + 6$. Consider h.r.p. in $Q_{26}(H)$ is $d' + 8$ or $f' + 6$. Then, $d' + 8 = 11$ or $f' + 6 = 11$.

**Case 1.2.1.1** $d' + 8 = 11$. So $d' = 3$. But, $d' \geq 4$, a contradiction.

**Case 1.2.1.2** $f' + 6 = 11$. So $f' = 5$. But $f' \geq 6$, a contradiction.

**Case 1.2.2** $11 < e + 6$. Consider the h.r.p. in $Q_{26}(H)$ is $d' + 8$ or $f' + 6$.

**Case 1.2.2.1** $e+6 = d'+8$. So $e = d'+2$. By Equation (3.1), $f' = 7$. After simplification, we obtain $2s^{11}$ is in $Q_{26}(G)$ but not in $Q_{26}(H)$, a contradiction.

**Case 1.2.2.2** $e + 6 = f' + 6$. So $e = f'$. By Equation (3.1), $d' = 5$. After simplification, we obtain $Q_{26}(G) \neq Q_{26}(H)$, a contradiction.

**Case 1.3** $f = 3$. We obtain the following after simplification.

$Q_{27}(G) = -s^3 - s^7 - s^e - s^{e+1} + s^5 + s^{10} + s^{11} + s^{e+3} + s^{e+6}$,

$Q_{27}(H) = -s^5 - s^{d'} - s^{f'} - s^{f'+1} + s^6 + s^7 + s^{d'+8} + s^{f'+3} + s^{f'+6}$.

Since $d' \geq 4$ and $f' \geq 6$, by comparing the l.r.p. in $Q_{27}(G)$ and the l.r.p. in $Q_{27}(H)$, we know that the term $-s^3$ is in $Q_{27}(G)$ but not in $Q_{27}(H)$, a contradiction.

**Case 2** $e = 3$. Note that $d \geq 4$ and $f \geq 5$. We obtain the following after simplification.

$Q_{28}(G) = -s^7 - s^d - s^f - s^{f+1} + s^9 + s^{d+8} + s^{f+2} + s^{f+7}$,

$Q_{28}(H) = -s^5 - s^{d'} - s^{f'} - s^{f'+1} + s^7 + s^{d'+8} + s^{f'+3} + s^{f'+6}$.

Comparing the l.r.p. in $Q_{28}(G)$ and the l.r.p. in $Q_{28}(H)$, we have $d = 5$ or $f = 5$.

**Case 2.1** $d = 5$. We obtain the following after simplification.

$Q_{29}(G) = -s^7 - s^f - s^{f+1} + s^9 + s^{13} + s^{f+2} + s^{f+7}$,

$Q_{29}(H) = -s^{d'} - s^{f'} - s^{f'+1} + s^7 + s^{d'+8} + s^{f'+3} + s^{f'+6}$.

Consider the h.r.p. in $Q_{29}(G)$ and the h.r.p. in $Q_{29}(H)$. Then we have $d' + 8 = 13$ or $f' + 6 = 13$ or $f + 7 = d' + 8$ or $f + 7 = f' + 6$.

**Case 2.1.1** $d' + 8 = 13$. Then $d' = 5$. By Equation (3.1), $f + 1 = f'$. Simplifying $Q_{29}(G)$ and $Q_{29}(H)$, we obtain $f = 5$ and $f' = 6$. Therefore, $G \cong K_4(1, 2, 6, 5, 3, 5)$ and $H \cong K_4(1, 3, 5, 5, 2, 6)$. Hence, $G \cong H$.

**Case 2.1.2** $f'+6=13$. Then $f'=7$. By Equation (3.1), $f=d'+1$. Simplifying $Q_{29}(G)$ and $Q_{29}(H)$, we obtain $f=8$ and $d'=7$. Therefore, $G\cong K_4(1,2,6,5,3,8)$ and $H\cong K_4(1,3,5,7,2,7)$. Hence, $K_4(1,2,6,5,3,8)\sim K_4(1,3,5,7,2,7)$.

**Case 2.1.3** $f+7=d'+8$. Then $f=d'+1$. By Equation (3.1), $f'=7$. After simplifying, we obtain $K_4(1,2,6,5,3,8)\sim K_4(1,3,5,7,2,7)$.

**Case 2.1.4** $f+7=f'+6$. Then $f+1=f'$. By Equation (3.1), $d'=5$. After simplifying, we obtain $G\cong H$.

**Case 2.2** $f=5$. We obtain the following after simplification.

$Q_{30}(G)=-s^6-s^7-s^d+s^9+s^{12}+s^{d+8}$,

$Q_{30}(H)=-s^{d'}-s^{f'}-s^{f'+1}+s^{d'+8}+s^{f'+3}+s^{f'+6}$.

Since $d'\geq 4$ and $f'\geq 6$, by comparing the l.r.p. in $Q_{30}(G)$ and the l.r.p. in $Q_{30}(H)$, we have $d'=6$ or $f'=6$.

**Case 2.2.1** $d'=6$. By Equation (3.1), $d=f'$. Simplifying $Q_{30}(G)$ and $Q_{30}(H)$, we obtain $d=6$. Then $G\cong H$.

**Case 2.2.2** $f'=6$. By Equation (3.1), $d=d'$. Simplifying $Q_{30}(G)$ and $Q_{30}(H)$, we obtain $G\cong H$.

**Case 3** $f=3$. Note that $d\geq 3$ and $e\geq 5$. We obtain the following after simplification.

$Q_{31}(G)=-s^7-s^d-s^e-s^{e+1}+s^5+s^{10}+s^{d+8}+s^{e+3}+s^{e+6}$,

$Q_{31}(H)=-s^5-s^{d'}-s^{f'}-s^{f'+1}+s^6+s^7+s^{d'+8}+s^{f'+3}+s^{f'+6}$.

Considering the l.r.p. in $Q_{31}(G)$ and the l.r.p. in $Q_{31}(H)$, we have $d=5$ or $e=5$.

**Case 3.1** $d=5$. Cancelling the equal terms in $Q_{31}(G)$ and $Q_{31}(H)$, we obtain the following.

$Q_{32}(G)=-s^7-s^e-s^{e+1}+s^5+s^{10}+s^{13}+s^{e+3}+s^{e+6}$,

$Q_{32}(H)=-s^{d'}-s^{f'}-s^{f'+1}+s^6+s^7+s^{d'+8}+s^{f'+3}+s^{f'+6}$.

Compare the h.r.p. in $Q_{32}(G)$ and the h.r.p. in $Q_{32}(H)$. We have $d'+8=13$ or $f'+6=13$ or $e+6=d'+8$ or $e+6=f'+6$.

**Case 3.1.1** $d'+8=13$. So $d'=5$. By Equation (3.1), $e+1=f'$. Then we obtain $Q_{32}(G)\neq Q_{32}(H)$, a contradiction.

**Case 3.1.2** $f'+6=13$. So $f'=7$. By Equation (3.1), $e=d'+1$. Then we obtain $Q_{32}(G)\neq Q_{32}(H)$, a contradiction.

**Case 3.1.3** $e+6=d'+8$. So $e=d'+2$. By Equation (3.1), $f'=8$. Then we obtain $Q_{32}(G)\neq Q_{32}(H)$, a contradiction.

**Case 3.1.4** $e+6=f'+6$. So $e=f'$. By Equation (3.1), $d'=6$. Then we obtain $Q_{32}(G)\neq Q_{32}(H)$, a contradiction.

**Case 3.2** $e=5$. Cancelling the equal terms in $Q_{31}(G)$ and $Q_{31}(H)$, we obtain the following.

$Q_{33}(G)=-s^6-s^7-s^d+s^5+s^8+s^{10}+s^{11}+s^{d+8}$,

$Q_{33}(H)=-s^{d'}-s^{f'}-s^{f'+1}+s^6+s^7+s^{d'+8}+s^{f'+3}+s^{f'+6}$.

Compare the h.r.p. in $Q_{33}(G)$ and the h.r.p. in $Q_{33}(H)$, we have $d+8 = d'+8$ or $d+8 = f'+6$.

**Case 3.2.1** $d+8 = d'+8$. So $d = d'$. By Equation (3.1), $f' = 6$. Then we obtain $Q_{33}(G) \neq Q_{33}(H)$, a contradiction.

**Case 3.2.2** $d+8 = f'+6$. So $d+2 = f'$. By Equation (3.1), $d' = 4$. Then we obtain $Q_{33}(G) \neq Q_{33}(H)$, a contradiction.

**Case C** $f' = 2$. We obtain the following after simplification.

$Q_{34}(G) = -s^7 - s^d - s^e - s^{e+1} - s^f - s^{f+1} + s^{d+8} + s^{e+3} + s^{e+6} + s^{f+2} + s^{f+7}$,

$Q_{34}(H) = -s^3 - s^4 - s^{d'} - s^{e'} - s^{e'+1} + s^8 + s^{d'+8} + s^{e'+4} + s^{e'+5}$.

Considering the l.r.p. in $Q_{34}(G)$ and the l.r.p. in $Q_{34}(H)$, we have $d = 3$ or $e = 3$ or $f = 3$.

**Case 1** $d = 3$. We obtain the following after simplification.

$Q_{35}(G) = -s^7 - s^e - s^{e+1} - s^f - s^{f+1} + s^{11} + s^{e+3} + s^{e+6} + s^{f+2} + s^{f+7}$,

$Q_{35}(H) = -s^4 - s^{d'} - s^{e'} - s^{e'+1} + s^8 + s^{d'+8} + s^{e'+4} + s^{e'+5}$.

Since $e \geq 4$ and $f \geq 3$, by comparing the l.r.p. in $Q_{35}(G)$ and the l.r.p. in $Q_{35}(H)$, we have $e = 4$ or $f = 4$ or $f = 3$.

**Case 1.1** $e = 4$. By simplifying $Q_{35}(G)$ and $Q_{35}(H)$, we obtain the following.

$Q_{36}(G) = -s^5 - s^f - s^{f+1} + s^{10} + s^{11} + s^{f+2} + s^{f+7}$,

$Q_{36}(H) = -s^{d'} - s^{e'} - s^{e'+1} + s^8 + s^{d'+8} + s^{e'+4} + s^{e'+5}$.

Since $e' \geq 6$, by considering the l.r.p. in $Q_{36}(G)$ and the l.r.p in $Q_{36}(H)$, we have $d' = 5$. After simplification, we obtain $f = e' = 6$. Therefore, $G \cong K_4(1,2,6,3,4,6)$ and $H \cong K_4(1,3,5,5,6,2)$. Hence, $K_4(1,2,6,3,4,6) \sim K_4(1,3,5,5,6,2)$.

**Case 1.2** $f = 4$. By simplifying $Q_{35}(G)$ and $Q_{35}(H)$, we obtain the following.

$Q_{37}(G) = -s^5 - s^7 - s^e - s^{e+1} + s^6 + 2s^{11} + s^{e+3} + s^{e+6}$,

$Q_{37}(H) = -s^{d'} - s^{e'} - s^{e'+1} + s^8 + s^{d'+8} + s^{e'+4} + s^{e'+5}$.

Since $e' \geq 6$, by considering the l.r.p. in $Q_{37}(G)$ and the l.r.p, in $Q_{37}(H)$, we have $d' = 5$. After simplification, we obtain $2s^{11}$ is in $Q_{37}(G)$ but not in $Q_{37}(H)$, a contradiction.

**Case 1.3** $f = 3$. By simplifying $Q_{35}(G)$ and $Q_{35}(H)$, we obtain the following.

$Q_{38}(G) = -s^3 - s^7 - s^e - s^{e+1} + s^5 + s^{10} + s^{11} + s^{e+3} + s^{e+6}$,

$Q_{38}(H) = -s^{d'} - s^{e'} - s^{e'+1} + s^8 + s^{d'+8} + s^{e'+4} + s^{e'+5}$.

Since $e' \geq 6$, by considering the l.r.p. in $Q_{38}(G)$ and the l.r.p. in $Q_{38}(H)$, we have $d' = 3$. After simplification, we obtain $e = 5$ and $e' = 6$. Therefore, $G \cong K_4(1,2,6,3,5,3)$ and $H \cong K_4(1,3,5,3,6,2)$. Hence, $G \cong H$.

**Case 2** $e = 3$. We obtain the following after simplification.

$Q_{39}(G) = -s^7 - s^d - s^f - s^{f+1} + s^6 + s^9 + s^{d+8} + s^{f+2} + s^{f+7}$,

$Q_{39}(H) = -s^{d'} - s^{e'} - s^{e'+1} + s^8 + s^{d'+8} + s^{e'+4} + s^{e'+5}$.

Consider the h.r.p. in $Q_{39}(G)$ and the h.r.p. in $Q_{39}(H)$. We have $d+8 = d'+8$ or $d+8 = e'+5$ or $f+7 = d'+8$ or $f+7 = e'+5$.

**Case 2.1** $d+8 = d'+8$. Then $d = d'$. By Equation (3.1), $f = e'$. We obtain $Q_{39}(G) \neq Q_{39}(H)$, a contradiction.

**Case 2.2** $d+8 = e'+5$. Then $d+3 = e'$. By Equation (3.1), $f = d'+2$. We obtain $Q_{39}(G) \neq Q_{39}(H)$, a contradiction.

**Case 2.3** $f+7 = d'+8$. Then $f = d'+1$. By Equation (3.1), $d+2 = e'$. We obtain $Q_{39}(G) \neq Q_{39}(H)$, a contradiction.

**Case 2.4** $f+7 = e'+5$. Then $f+2 = e'$. By Equation (3.1), $d = d'+1$. We obtain the following after simplification.

$Q_{40}(G) = -s^7 - s^d - s^f - s^{f+1} + s^6 + s^9 + s^{d+8} + s^{f+2}$,

$Q_{40}(H) = -s^{d-1} - s^{f+2} - s^{f+3} + s^8 + s^{d+7} + s^{f+6}$.

Compare the h.r.p. in $Q_{40}(G)$ and the h.r.p. in $Q_{40}(H)$. We have $d+8 = f+6$ or $d+7 = f+2$.

If $d+8 = f+6$, then $d+2 = f$. We obtain $Q_{40}(G) \neq Q_{40}(H)$, a contradiction.

If $d+7 = f+2$, then $d+5 = f$. We obtain $Q_{40}(G) \neq Q_{40}(H)$, a contradiction.

**Case 3** $f = 3$. We obtain the following after simplification.

$Q_{41}(G) = -s^7 - s^d - s^e - s^{e+1} + s^5 + s^{10} + s^{d+8} + s^{e+3} + s^{e+6}$,

$Q_{41}(H) = -s^{d'} - s^{e'} - s^{e'+1} + s^8 + s^{d'+8} + s^{e'+4} + s^{e'+5}$.

Comparing the h.r.p. in $Q_{41}(G)$ and the h.r.p. in $Q_{41}(H)$, we have $d+8 = d'+8$ or $d+8 = e'+5$ or $e+6 = d'+8$ or $e+6 = e'+5$.

**Case 3.1** $d+8 = d'+8$. Then $d = d'$. By Equation (3.1), $e+1 = e'$. By simplifying $Q_{41}(G)$ and $Q_{41}(H)$, we obtain $e = 5$ and $e' = 6$. Therefore, $G \cong K_4(1,2,6,d,5,3)$ and $H \cong K_4(1,3,5,d,6,2)$. Thus, $G \cong H$.

**Case 3.2** $d+8 = e'+5$. Then $d+3 = e'$. By Equation (3.1), $e = d'+2$. By simplifying $Q_{41}(G)$ and $Q_{41}(H)$, we obtain $d = 5$, $e = 9$, $d' = 7$ and $e' = 8$. Therefore, $G \cong K_4(1,2,6,5,9,3)$ and $H \cong K_4(1,3,5,7,8,2)$. Hence, $K_4(1,2,6,5,9,3) \sim K_4(1,3,5,7,8,2)$.

**Case 3.3** $e+6 = d'+8$. Then $e = d'+2$. By Equation (3.1), $d+3 = e'$. After simplification, we obtain $K_4(1,2,6,5,9,3) \sim K_4(1,3,5,7,8,2)$.

**Case 3.4** $e+6 = e'+5$. Then $e+1 = e'$. By Equation (3.1), $d = d'$. After simplification, we obtain $G \cong H$.

At this point, from Subcases 1.1.4, 1.2.2, 1.2.3, 2.2 and 3.4 of Case A, Subcases 1.1.1, 2.1.2, and 2.1.3 of Case B and Subcases 1.1, 3.2 and 3.3 of Case C, we obtain the following solutions.

$$\begin{aligned}
K_4(1,2,6,3,4,6) &\sim K_4(1,3,5,5,6,2),\\
K_4(1,2,6,3,4,10) &\sim K_4(1,3,5,9,2,6),\\
K_4(1,2,6,4,5,8) &\sim K_4(1,3,5,2,6,9),\\
K_4(1,2,6,4,7,5) &\sim K_4(1,3,5,2,8,6),\\
K_4(1,2,6,5,3,8) &\sim K_4(1,3,5,7,2,7),\\
K_4(1,2,6,5,9,3) &\sim K_4(1,3,5,7,8,2),\\
K_4(1,2,6,f+2,4,f) &\sim K_4(1,3,5,2,f,f+4),
\end{aligned}$$

where $f \geq 4$.
This completes the proof.

□

**Lemma 3.2.** *If $K_4(1,2,6,d,e,f)$ and $K_4(1,2,6,d',e',f')$ are chromatically equivalent, then*

$$\begin{aligned}
K_4(1,2,6,i,i+7,i+1) &\sim K_4(1,2,6,i+2,i,i+6),\\
K_4(1,2,6,i,i+1,i+7) &\sim K_4(1,2,6,i+6,i,i+2),\\
K_4(1,2,6,i,i+1,i+3) &\sim K_4(1,2,6,i+2,i+2,i),
\end{aligned}$$

*where $i \geq 1$.*

*Proof.* It follows directly from Lemma 2.2. □

**Lemma 3.3.** *$K_4(1,2,6,d,e,f)$ and $K_4(2,2,5,d',e',f')$ are not chromatically equivalent.*

*Proof.* If $H$ is of type of $K_4(2,2,5,d',e',f')$, then from Lemma 2.6, we know that $H$ is chromatically unique. Since $G \sim H$, we have $G \cong H$. But it is obvious that $G$ is not isomorphic to $H$. This is a contradiction. □

It follows that

**Lemma 3.4.** *$K_4(1,2,6,d,e,f,)$ and $K_4(1,2,c',2,e',4)$ are not chromatically equivalent.*

**Lemma 3.5.** *$K_4(1,2,6,d,e,f)$ and $K_4(1,2,c',4,e',2)$ are not chromatically equivalent.*

**Lemma 3.6.** *$K_4(1,2,6,d,e,f)$ and $K_4(1,3,c',2,e',3)$ are not chromatically equivalent.*

*Proof.* Let $G$ and $H$ be two graphs such that $G \cong K_4(1,2,6,d,e,f)$ and $H \cong K_4(1,3,c',2,e',3)$. Then

$$Q(G) = -(s+1)(s+s^2+s^6+s^d+s^e+s^f)+s^{d+1}+s^{f+2}+ s^{e+6}+s^{e+3}+s^{d+8}+s^{f+7}+s^{d+e+f}.$$
$$Q(H) = -(s+1)(s+s^2+2s^3+s^{c'}+s^{e'})+s^3+s^6+ s^{c'+4}+s^{c'+5}+s^{e'+4}+s^{e'+5}+s^{c'+e'}.$$

From $Q(G)=Q(H)$, we have

$$Q_1(G) = -s^6-s^7-s^d-s^e-s^f-s^{e+1}-s^{f+1}+ s^{d+8}+s^{e+3}+s^{e+6}+s^{f+2}+s^{f+7}.$$
$$Q_1(H) = -s^3-2s^4-s^{c'}-s^{c'+1}-s^{e'}-s^{e'+1}+ s^6+s^{c'+4}+s^{c'+5}+s^{e'+4}+s^{e'+5}.$$

Consider the terms $-s^3$ and $-2s^4$ in $Q_1(H)$. Due to $Q_1(G)=Q_1(H)$, there are terms in $Q_1(G)$ which are equal to $-s^3$ and $-2s^4$, so 1 of $d,e,f$ is equal to 3, and the other two is equal to 4. Thus we have $d=3$, $e=f=4$ or $e=3$, $d=f=4$ or $f=3$, $d=e=4$.

If $d=3$, $e=f=4$, simplifying $Q_1(G)$ and $Q_1(H)$, we obtain the following.

$Q_2(G)=-2s^5-s^6+s^{10}+2s^{11}$, $Q_2(H)=-s^{c'}-s^{c'+1}-s^{e'}-s^{e'+1}+s^{c\prime+4}+s^{c'+5}+s^{e'+4}+s^{e'+5}$.

Comparing the l.r.p in $Q_2(G)$ and the l.r.p in $Q_2(H)$, we obtain $c'=e'=5$. It can be easily checked that $Q_2(G)\neq Q_2(H)$, a contradiction.

If $e=3$, $d=f=4$, since the girth of $G$ and $H$ is 9, which needs $f+e\geq 8$, we obtain a contradiction.

If $f=3$, $d=e=4$, since the girth of $G$ and $H$ is 9, which needs $f+e\geq 8$, we obtain a contradiction.

This completes the proof. □

Similarly, we can prove the following result.

**Lemma 3.7.** *$K_4(1,2,6,d,e,f)$ and $K_4(1,2,c',3,e',3)$ are not chromatically equivalent.*

Now, the chromaticity of $K_4(1,2,6,d,e,f)$ is given as follows.

**Theorem 3.8.** *$K_4$-homeomorphs $K_4(1,2,6,d,e,f)$ with girth 9 is not $\chi$-unique if and only if it is isomorphic to $K_4(1,2,6,6,3,4)$, $K_4(1,2,6,9,3,5)$, $K_4(1,2,6,5,5,5)$, $K_4(1,2,6,4,5,8)$, $K_4(1,2,6,3,4,10)$, $K_4(1,2,6,5,3,8)$, $K_4(1,2,6,4,s,4)$, $K_4(1,2,6,f+2,4,f)$, $K_4(1,2,6,i,i+7,i+1)$, $K_4(1,2,6,i+2,i,i+6)$, $K_4(1,2,6,i,i+1,i+3)$ or $K_4(1,2,6,i+2,i+2,i)$, where $i\geq 1$, $s\geq 4$, $f\geq 4$.*

*Proof.* Let $G$ and $H$ be two graphs such that $G\cong K_4(1,2,6,d,e,f)$ and $G\sim H$. Since the girth of $G$ is 9, at most one among $d,e,f$ are 1. Moreover, by

Lemma 2.1(2)(3), it follows that $H$ is a $K_4$-homeomorph with girth 9. Then $H$ must be one of the following 10 types.

Type 1: $K_4(1,2,6,d',e',f')$, where $d'+e' \geq 7, d'+f' \geq 6, e'+f' \geq 8$;

Type 2: $K_4(1,3,5,d',e',f')$, where $d'+e' \geq 6, d'+f' \geq 5, e'+f' \geq 8$;

Type 3: $K_4(1,4,4,d',e',f')$, where $d'+e' \geq 5, d'+f' \geq 5, e'+f' \geq 8$;

Type 4: $K_4(2,3,4,d',e',f')$, where $d'+e' \geq 6, d'+f' \geq 5, e'+f' \geq 7$;

Type 5: $K_4(2,2,5,d',e',f')$, where $d'+e' \geq 7, d'+f' \geq 5, e'+f' \geq 7$;

Type 6: $K_4(1,2,c',2,e',4)$, where $c' \geq 6$, $e' \geq 5$;

Type 7: $K_4(1,2,c',4,e',2)$, where $c' \geq 6$, $e' \geq 6$;

Type 8: $K_4(1,2,c',3,e',3)$, where $c' \geq 6$, $e' \geq 5$;

Type 9: $K_4(1,3,c',2,e',3)$, where $c' \geq 5$, $e' \geq 5$;

Type 10: $K_4(2,2,c',2,e',3)$, where $c' \geq 5$, $e' \geq 5$.

If $H$ has Type 1, then from Lemma 2.2, we know that the solutions of the equation $P(G)=P(H)$ are

$$\begin{aligned}
K_4(1,2,6,i,i+7,i+1) &\sim K_4(1,2,6,i+2,i,i+6),\\
K_4(1,2,6,i,i+1,i+7) &\sim K_4(1,2,6,i+6,i,i+2),\\
K_4(1,2,6,i,i+1,i+3) &\sim K_4(1,2,6,i+2,i+2,i),
\end{aligned}$$

where $i \geq 1$.

If $H$ has Type 2, then from Lemma 3.1, we know that the solutions of the equation $P(G)=P(H)$ are

$$\begin{aligned}
K_4(1,2,6,3,4,6) &\sim K_4(1,3,5,5,6,2),\\
K_4(1,2,6,3,4,10) &\sim K_4(1,3,5,9,2,6),\\
K_4(1,2,6,4,5,8) &\sim K_4(1,3,5,2,6,9),\\
K_4(1,2,6,4,7,5) &\sim K_4(1,3,5,2,8,6),\\
K_4(1,2,6,5,3,8) &\sim K_4(1,3,5,7,2,7),
\end{aligned}$$

$$K_4(1,2,6,5,9,3) \sim K_4(1,3,5,7,8,2),$$
$$K_4(1,2,6,f+2,4,f) \sim K_4(1,3,5,2,f,f+4),$$

where $f \geq 4$.

If $H$ has Type 3, then from Lemma 2.3, we know that the solution of the equation $P(G) = P(H)$ is

$$K_4(1,2,6,4,4,4) \sim K_4(1,4,4,2,3,7).$$

If $H$ has Type 4, then from Lemma 2.4, we know that the solutions of the equation $P(G) = P(H)$ are

$$K_4(1,2,6,4,s,4) \sim K_4(2,3,4,1,7,s),$$
$$K_4(1,2,6,6,3,4) \sim K_4(2,3,4,7,1,5),$$
$$K_4(1,2,6,6,4,4) \sim K_4(2,3,4,1,5,8),$$
$$K_4(1,2,6,9,3,5) \sim K_4(2,3,4,10,6,1),$$
$$K_4(1,2,6,5,5,5) \sim K_4(2,3,4,6,6,1),$$

where $s \geq 4$.

If $H$ has Types 5–9, then from Lemmas 3.3–3.7, we know that there is no solution of the equation $P(G) = P(H)$, i.e., a contradiction.

If $H$ has Type 10, then from Lemma 2.5, we know that $H$ is chromatically unique. Since $G \sim H$, we have $G \cong H$. But it is obvious that $G$ is not isomorphic to $H$. This is a contradiction.

This completes the proof of Theorem 3.8.

□

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