ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 43 (2017), No. 2, pp. 349–371

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 2, pp. 349–371
Online ISSN: 1735-8515

STRONG CONVERGENCE THEOREM FOR SOLVING SPLIT EQUALITY FIXED POINT PROBLEM WHICH DOES NOT INVOLVE THE PRIOR KNOWLEDGE OF OPERATOR NORMS

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(Communicated by Madjid Eshaghi Gordji)

Abstract. Our contribution in this paper is to propose an iterative algorithm which does not require prior knowledge of operator norm and prove a strong convergence theorem for approximating a solution of split equality fixed point problem for quasi-nonexpansive mappings in a real Hilbert space. So many have used algorithms involving the operator norm for solving split equality fixed point problem, but as widely known the computation of these algorithms may be difficult and for this reason, some researchers have recently started constructing iterative algorithms with a way of selecting the step-sizes such that the implementation of the algorithm does not require the calculation or estimation of the operator norm. To the best of our knowledge most of the works in literature that do not involve the calculation or estimation of the operator norm only obtained weak convergence results. In this paper, by appropriately modifying the simultaneous iterative algorithm introduced by Zhao, we state and prove a strong convergence result for solving split equality problem. We present some applications of our result and then give some numerical example to study its efficiency and implementation at the end of the paper.
Keywords: Strong convergence, split equality fixed point problem, quasi-nonexpansive mappings, simultaneous iterative algorithm, Hilbert spaces.
MSC(2010): Primary: 47H06; Secondary: 47H09, 47J05.

1. **Introduction**

Let H be a real Hilbert space with inner product and norm as $\langle \cdot, \cdot \rangle$ and $|| \cdot ||$, respectively. Let C be a nonempty, closed and convex subset of H. A mapping $T : C \to C$ is called *nonexpansive* if

$$||Tx - Ty|| \leq ||x - y||, \ \forall x, y \in C. \tag{1.1}$$

Article electronically published on 30 April, 2017.
Received: 16 July 2014, Accepted: 24 November 2015.
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A mapping $T : C \to C$ is called *firmly nonexpansive* if

(1.2) $||Tx - Ty||^2 \le ||x - y||^2 - ||(x - y) - (Tx - Ty)||^2, \ \forall x, y \in C.$

A point $p \in C$ is called *a fixed point* of T, if $Tp = p$. The set of fixed points of T is denoted by $F(T) := \{x \in C : Tx = x\}$. A mapping $T : C \to C$ is called *quasi-nonexpansive* if

(1.3) $$||Tx - Tp|| \le ||x - p||, \ \forall (x, p) \in C \times F(T).$$

It is easily observed that every nonexpansive mapping with a nonempty fixed point set is quasi-nonexpansive. We give an example of a quasi-nonexpansive mapping which is not nonexpansive.

Example 1.1. Let $C := \{x \in \ell_2 : ||x||_{\ell_2} \le 1\}$. Define $T : C \to C$ by $Tx = (0, x_1^2, x_2^2, x_3^2, \dots)$ for $x = (x_1, x_2, x_3, \dots)$ in C. Then it is clear that $Tp = p$ if and only if $p = 0_{\ell_2}$. Furthermore,

$$\begin{aligned} ||Tx - Tp||_{\ell_2} &= ||Tx||_{\ell_2} = ||(0, x_1^2, x_2^2, x_3^2, \dots)||_{\ell_2} \\ &\le ||(x_1, x_2, x_3, \dots)||_{\ell_2} = ||x||_{\ell_2} = ||x - p||_{\ell_2}, \end{aligned}$$

for all $x \in C$. Therefore, T is quasi-nonexpansive. However, T is not nonexpansive, for if $x = (\frac{3}{4}, 0, 0, 0, \dots)$ and $y = (\frac{1}{2}, 0, 0, 0, \dots)$, it is clear that x and y belong to C. Furthermore, $||x - y||_{\ell_2} = ||(\frac{1}{4}, 0, 0, 0, \dots)||_{\ell_2} = \frac{1}{4}$, and $||Tx - Ty||_{\ell_2} = ||(0, \frac{5}{16}, 0, 0, 0, \dots)||_{\ell_2} = \frac{5}{16} > \frac{1}{4} = ||x - y||_{\ell_2}$.

A mapping $T : C \to C$ is called *firmly quasi-nonexpansive* if

(1.4) $||Tx - Tp||^2 \le ||x - p||^2 - ||x - Tp||^2, \ \forall (x, p) \in C \times F(T).$

Let H_1, H_2 and H_3 be real Hilbert spaces. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators, and let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be two nonlinear operators such that $F(S) \ne \emptyset$ and $F(T) \ne \emptyset$. The split equality fixed point problem (SEFP) (see [1] and some of the references therein) is to find

(1.5) $$x \in F(S), y \in F(T), \text{ such that } Ax = By.$$

We shall denote by Γ the solution set of SEFP (1.5). The split equality fixed point problem (1.5) which allows asymmetric and partial relations between the variables x and y was introduced by Moudafi [19].

If $H_2 = H_3$ and $B = I$, then SEFP (1.5) reduces to the following split common fixed-point problem (SCFP) introduced by Censor and Segal [8]:

(1.6) $$\text{find } x \in F(S), \text{ such that } Ax \in F(T).$$

Thus the SEFP generalizes the SCFP which is at the core of the modelling of many inverse problems in various areas of physical sciences and has been used to model significant real world inverse problem in sensor networks, in radiation therapy treatment planning, in resolution enhancement, in watermarking, in data compression, in magnetic resonance imaging, in holography, in colour imaging, in optics and neural networks and in graph matching (for more details, see, for example, [7]). SEFP also has some other important applications in different areas of applied mathematics, such as; fully discretized models of inverse problems which arise from phase retrievals and in medical image reconstruction (see, for example, [3–5, 9, 33]).

Let C be a nonempty, closed and convex subset of a real Hilbert space H_1, Q a nonempty, closed and convex subset of a real Hilbert space H_2, and let $A : H_1 \to H_2$ be a bounded linear operator. A split feasibility problem (SFP) (see, [10, 12–14, 22, 26–28, 31]) is to find a point x satisfying

$$x \in C, Ax \in Q.$$

The convex feasibility problem (CFP), as an important optimization problem is to find a common element of the intersection of finitely many convex sets (see, for example, [15, 17] for more details). The split common fixed point problem (SCFP) (e.g., see [11, 24]) is a generalization of the split feasibility problem (SFP) and the convex feasibility problem (CFP).
To solve (1.6), Censor and Segal [8] proposed and proved, in finite-dimensional spaces, the convergence of the sequence $\{x_n\}$ generated by the following algorithm:

$$x_{n+1} = U(x_n + \gamma A^t(T - I)Ax_n), \ n \in \mathbb{N}.$$

where $\gamma \in (0, \frac{2}{\lambda})$, with λ being the largest eigenvalue of the matrix A^tA (A^t stands for matrix transposition).

To solve the SEFP (1.5), Moudafi [19] introduced the following algorithm:

$$\left\{ \begin{array}{l} x_{n+1} = U(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = T(y_n + \gamma_n B^*(Ax_{n+1} - By_n)), \end{array} \right. \tag{1.7}$$

and proved convergence result for the sequence generated by the algorithm (1.7) for firmly quasi-nonexpansive operators U and T, where non-decreasing sequence $\gamma_n \in (\epsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \epsilon)$ and λ_A, λ_B are the spectral radii of A^*A and B^*B, respectively.

Byrne and Moudafi [6] studied the approximate split equality problem (ASEP), which can be regarded as obtaining the consistent case and the inconsistent case of the split equality problem (SEP):

$$x \in C, y \in Q, \text{ such that } Ax = By, \tag{1.8}$$

where $C \subset H_1, Q \subset H_2$ are two nonempty, closed and convex sets and proposed the following simultaneous iterative algorithm:

$$\begin{cases} x_{n+1} = P_C(x_n - \gamma_n A^T(Ax_n - By_n)), \\ y_{n+1} = P_Q(y_n + \gamma_n B^T(Ax_n - By_n)), \end{cases}$$

where $\epsilon \leq \gamma_n \leq \frac{2}{\lambda_G} - \epsilon$, λ_G stands for the spectral radius of G^TG and $G = [A - B]$.

Furthermore, Moudafi and Al-Shemas [1] recently introduced the following simultaneous iterative method to solve SEFP (1.5)

$$\text{(1.9)} \qquad \begin{cases} x_{n+1} = U(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = T(y_n + \gamma_n B^*(Ax_n - By_n)), \end{cases}$$

for firmly quasi-nonexpansive operators U and T, where $\gamma_n \in (\epsilon, \frac{2}{\lambda_A, +\lambda_B} - \epsilon)$, λ_A and λ_B stand for the spectral radius of A^*A and B^*B, respectively. We note that in the algorithms (1.7) and (1.9) mentioned above, the determination of the step-size $\{\gamma_n\}$ depends on the operator (matrix) norms $||A||$ and $||B||$ (or the largest eigenvalues of A^*A and B^*B). In order to implement the alternating algorithm (1.7) and (1.9) for solving SEFP (1.5), the computation (or, at least, estimation) of the operator norms of A and B, which is in general not an easy task in practice will be required.

To overcome this difficulty, Lopez *et al.* [16] and Zhao and Yang [30] introduced a method for estimating the step-sizes which does not require any knowledge of the operator norms for solving the split feasibility problems and multiple-set split feasibility problems. Inspired by them, Zhao [32] recently introduced a new choice of the step-size sequence $\{\gamma_n\}$ for the simultaneous Mann iterative algorithm to solve SEFP (1.5) for quasi-nonexpansive operators as follows:
Choose an initial guess $x_1 \in H_1, y_1 \in H_2$ arbitrarily. Let $\alpha_n \in [0,1]$ and $\beta_n \in [0,1]$. Assume that the nth iterate $x_n \in H_1, y_n \in H_2$ has been constructed. Then, we calculate the $\{(n+1)\}th$ iterate (x_{n+1}, y_{n+1}) via the formula:

$$\text{(1.10)} \qquad \begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)U(u_n), \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = \beta_n v_n + (1-\beta_n)Tv_n, \end{cases}$$

Using the iterative scheme (1.10), Zhao [32] proved the following *weak convergence* theorem in a real Hilbert space.

Theorem 1.2. *Let $U : H_1 \to H_1, T : H_2 \to H_2$ be quasi-nonexpansive mappings. Assume that $U-I$, $T-I$ are demi-closed at the origin. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators and the solution set Γ of (1.5) is not empty. Let the sequence $\{(x_n, y_n)\}$ be generated by Algorithm (1.10).*

Assume for small enough $\epsilon > 0$,

(1.11) $\gamma_n \in \Big(0, \frac{2||Ax_n - By_n||^2}{||A^*(Ax_n - By_n)||^2 + ||B^*(Ax_n - By_n)||^2}\Big),\ n \in \Omega,$

otherwise $\gamma_n = \gamma$ *(*γ *being any non-negative value), where the index set* $\Omega = \{n : Ax_n - By_n \neq 0\}$*. Then,* $\{(x_n, y_n)\}$ *weakly converges to a solution* (x^*, y^*) *of* (1.5) *provided that* $\{\alpha_n\} \subset (\delta, 1-\delta)$ *and* $\{\beta_n\} \subset (\sigma, 1-\sigma)$ *for small enough* $\delta, \sigma > 0$*.*

Motivated by the result of Zhao [32] and other important recent results in this direction, we propose an iterative scheme which does not require any prior knowledge of operator norm and prove strong convergence of the sequence generated by our scheme for approximating a solution of split equality fixed point problem (1.5) for quasi-nonexpansive mappings in a real Hilbert space. Our result complements the result of Zhao [32] and other recent results in the literature.

2. Preliminaries

We first give a definition of demi-closedness.

Definition 2.1. Let $T : H \to H$ be a nonlinear mapping. Then T is said to be demi-closed at $y \in H$, if $x_n \rightharpoonup x \in H$ and $Tx_n \to y$, then $y = Tx$.

We next give some technical lemmas which will be used in the sequel.

Lemma 2.2 ([18])**.** *Let* H *be a real Hilbert space, and let* $T : H \to H$ *be a quasi-nonexpansive mapping. Set* $T_\alpha = \alpha I + (1-\alpha)T$ *for* $\alpha \in [0, 1)$*. Then the following holds, for all* $(x, p) \in H \times F(T)$*:*
(i) $||T_\alpha x - p||^2 \leq ||x - p||^2 - \alpha(1-\alpha)||Tx - x||^2$.
(ii) $F(T_\alpha) = F(T)$.

Lemma 2.3. *Let* H *be a real Hibert space. Then the following identities are obtained:*

$$2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x - y||^2 = ||x + y||^2 - ||x||^2 - ||y||^2.$$

Lemma 2.4. *Let* H *be a real Hilbert space. Then the following result holds*

$$||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle\ \forall\ x, y \in H.$$

Lemma 2.5 ([25])**.** *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n,\ n \geq 0,$$

where $\{\gamma_n\}$ *is a sequence in (0,1) and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*
(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$*;*
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$.
Then $\lim_{n\to\infty} a_n = 0$.

In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

3. Main results

Theorem 3.1. *Let H_1, H_2 and H_3 be real Hilbert spaces. Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be quasi-nonexpansive mappings such that $S - I$ and $T - I$ are demi-closed at 0 and $F(S) \neq \emptyset$, $F(T) \neq \emptyset$ and $\Gamma \neq \emptyset$. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ and $\{t_n\}$ be a sequence in $(0, 1 - a)$ for some $a > 0$. Let the step size γ_n be chosen in such a way that for some $\epsilon > 0$,*

$$\gamma_n \in \Big(\epsilon, \frac{2||Ax_n - By_n||^2}{||A^*(Ax_n - By_n)||^2 + ||B^*(Ax_n - By_n)||^2} - \epsilon\Big),\ n \in \Omega,$$

otherwise $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set $\Omega = \{n : Ax_n - By_n \neq 0\}$.
Let $u, x_1 \in H_1$ and $v, y_1 \in H_2$ be arbitrary and the sequences $\{x_n\}$ and $\{y_n\}$ be iteratively generated by

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = (1 - \alpha_n - t_n)u_n + \alpha_n S u_n + t_n u, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = (1 - \alpha_n - t_n)v_n + \alpha_n T v_n + t_n v, \end{cases} \tag{3.1}$$

with the conditions
(i) $\lim_{n\to\infty} t_n = 0$ *and* $\sum_{n=0}^{\infty} t_n = \infty$,
(ii) $\alpha_n + t_n < 1$,
(iii)$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$.
Then $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$ in the solution set Γ of (1.5).

Proof. Let $(x^*, y^*) \in \Gamma$. Then by convexity of $||.||^2$, we have

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= ||(1 - \alpha_n - t_n)u_n + \alpha_n S u_n + t_n u - x^*||^2 \\ &= ||(1 - \alpha_n - t_n)(u_n - x^*) + \alpha_n (S u_n - x^*) + t_n (u - x^*)||^2 \\ &\le (1 - \alpha_n - t_n)||u_n - x^*||^2 + \alpha_n ||S u_n - x^*||^2 + t_n ||u - x^*||^2 \\ &\le (1 - \alpha_n - t_n)||u_n - x^*||^2 + \alpha_n ||u_n - x^*||^2 + t_n ||u - x^*||^2 \\ &= (1 - t_n)||u_n - x^*||^2 + t_n ||u - x^*||^2. \end{aligned} \tag{3.2}$$

Following similar process as in (3.2), we obtain

$$||y_{n+1} - y^*||^2 \le (1 - t_n)||v_n - y^*||^2 + t_n ||v - y^*||^2. \tag{3.3}$$

Adding (3.2) and(3.3), we have

$$\begin{aligned} ||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2 \le\ & (1 - t_n)(||u_n - x^*||^2 + ||v_n - y^*||^2) \\ & + t_n(||u - x^*||^2 + ||v - y^*||^2). \end{aligned} \tag{3.4}$$

Using (3.1) and Lemma 2.3, we have

$$\begin{aligned}||u_n - x^*||^2 &= ||x_n - \gamma_n A^*(Ax_n - By_n) - x^*||^2\\ &= ||x_n - x^*||^2 + \gamma_n^2||A^*(Ax_n - By_n)||^2 - 2\gamma_n\langle x_n - x^*, A^*(Ax_n - By_n)\rangle\\ &= ||x_n - x^*||^2 + \gamma_n^2||A^*(Ax_n - By_n)||^2 - 2\gamma_n\langle Ax_n - Ax^*, (Ax_n - By_n)\rangle\\ &= ||x_n - x^*||^2 + \gamma_n^2||A^*(Ax_n - By_n)||^2 - \gamma_n||Ax_n - Ax^*||^2\\ &\quad -\gamma_n||Ax_n - By_n||^2 + \gamma_n||By_n - Ax^*||^2. \end{aligned} \tag{3.5}$$

By similar steps as in (3.5), we have

$$\begin{aligned}||v_n - y^*||^2 &= ||y_n - y^*||^2 + \gamma_n^2||B^*(Ax_n - By_n)||^2 - \gamma_n||By_n - By^*||^2\\ &\quad -\gamma_n||Ax_n - By_n||^2 + \gamma_n||Ax_n - By^*||^2. \end{aligned} \tag{3.6}$$

Adding (3.5) and (3.6), $Ax^* = By^*$, and noting the assumption on γ_n, we arrive at

$$\begin{aligned}||u_n - x^*||^2 + ||v_n - y^*||^2 &= ||x_n - x^*||^2 + ||y_n - y^*||^2 - \gamma_n[2||Ax_n - By_n||^2\\ &\quad -\gamma_n(||A^*(Ax_n - By_n)||^2 + ||B^*(Ax_n - By_n)||^2)]\\ &\leq ||x_n - x^*||^2 + ||y_n - y^*||^2. \end{aligned} \tag{3.7}$$

Inserting (3.7) into (3.4) yields

$$\begin{aligned}&||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2\\ &\leq (1 - t_n)(||x_n - x^*||^2 + ||y_n - y^*||^2)\\ &\quad + t_n(||u - x^*||^2 + ||v - y^*||^2)\\ &\leq \max\{||x_n - x^*||^2 + ||y_n - y^*||^2, ||u - x^*||^2 + ||v - y^*||^2\}\\ &\ \ \vdots\\ &\leq \max\{||x_1 - x^*||^2 + ||y_1 - y^*||^2, ||u - x^*||^2 + ||v - y^*||^2\}.\end{aligned}$$

Therefore, $\{||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2\}$ is bounded. Hence, $\{x_n\}$ and $\{y_n\}$ are bounded. Consequently $\{u_n\}$, $\{v_n\}$, $\{Su_n\}$ and $\{Tv_n\}$ are all bounded. Therefore,

$$\begin{aligned}||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2 &\leq (1 - t_n)(||x_n - x^*||^2 + ||y_n - y^*||^2)\\ &\quad -(1 - t_n)\gamma_n[2||Ax_n - By_n||^2 - \gamma_n(||A^*(Ax_n - By_n)||^2\\ &\quad +||B^*(Ax_n - By_n||^2)] + t_n(||u - x^*||^2 + ||v - y^*||^2). \end{aligned} \tag{3.8}$$

Now we divide the rest of the proof into two cases.

<u>Case 1.</u> Assume that $\{||x_n - x^*||^2 + ||y_n - y^*||^2\}$ is monotonically decreasing. Obviously $\{||x_n - x^*||^2 + ||y_n - y^*||^2\}$ is convergent and $(||x_n - x^*||^2 + ||y_n - y^*||^2) - (||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2) \to 0$ as $n \to \infty$.

From (3.8),

$$
\begin{aligned}
(1-t_n)\gamma_n^2(||A^*(Ax_n-By_n)||^2+||B^*(Ax_n-By_n)||^2) & \\
\le (1-t_n)(||x_n-x^*||^2+||y_n-y^*||^2) & \\
-(||x_{n+1}-x^*||^2+||y_{n+1}-y^*||^2) & \\
+t_n(||u-x^*||^2+||v-y^*||^2). &
\end{aligned}
$$

That is,

$$
\begin{aligned}
\gamma_n^2(||A^*(Ax_n-By_n)||^2+||B^*(Ax_n-By_n)||^2) & \\
\le \quad & (||x_n-x^*||^2+||y_n-y^*||^2) \\
& -\frac{1}{(1-t_n)}(||x_{n+1}-x^*||^2+||y_{n+1}-y^*||^2) \\
& +\frac{t_n}{(1-t_n)}(||u-x^*||^2+||v-y^*||^2). \qquad (3.9)
\end{aligned}
$$

From the condition $t_n \to 0$ as $n \to \infty$, we have

$$\gamma_n^2(||A^*(Ax_n-By_n)||^2+||B^*(Ax_n-By_n)||^2) \to 0.$$

By the condition

$$\gamma_n \in \Big(\epsilon, \frac{2||Ax_n-By_n||^2}{||A^*(Ax_n-By_n)||^2+||B^*(Ax_n-By_n)||^2}-\epsilon\Big),\ n \in \Omega,$$

we conclude that

$$\lim_{n\to\infty}(||A^*(Ax_n-By_n)||^2+||B^*(Ax_n-By_n)||^2)=0. \qquad (3.10)$$

Note that $Ax_n - By_n = 0$ if $n \notin \Omega$. Thus,

$$\lim_{n\to\infty}||A^*(Ax_n-By_n)||=\lim_{n\to\infty}||B^*(Ax_n-By_n)||=0. \qquad (3.11)$$

Again, by Lemma 2.3 and Lemma 2.2(i)

$$
\begin{aligned}
||x_{n+1}-x^*||^2 &= ||(1-\alpha_n)(u_n-x^*)+\alpha_n(Su_n-x^*)+t_n(u-u_n)||^2 \\
&= ||(1-\alpha_n)(u_n-x^*)+\alpha_n(Su_n-x^*)||^2+t_n^2||u_n-u||^2 \\
&\quad +2t_n\langle u-u_n,(1-\alpha_n)(u_n-x^*)+\alpha_n(Su_n-x^*)\rangle \\
&\le ||u_n-x^*||^2-\alpha_n(1-\alpha_n)||Su_n-u_n||^2 \\
&\quad +t_n^2||u_n-u||^2+2t_n\langle u-u_n,(1-\alpha_n)(u_n-x^*)+\alpha_n(Su_n-x^*)\rangle. \qquad (3.12)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
||y_{n+1}-y^*||^2 &\le ||v_n-y^*||^2-\alpha_n(1-\alpha_n)||Tv_n-v_n||^2 \\
&\quad +t_n^2||v_n-v||^2+2t_n\langle v-v_n,(1-\alpha_n)(v_n-y^*)+\alpha_n(Tv_n-y^*)\rangle. \qquad (3.13)
\end{aligned}
$$

Therefore, from (3.7) we obtain

$$\begin{aligned} & ||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2 \\ \leq\ & ||u_n - x^*||^2 + ||v_n - y^*||^2 - \alpha_n(1-\alpha_n)(||Su_n - u_n||^2 + ||Tv_n - v_n||^2) \\ & + t_n^2(||u_n - u||^2 + ||v_n - v||^2) \\ & + 2t_n(\langle u - u_n, (1-\alpha_n)(u_n - x^*) + \alpha_n(Su_n - x^*)\rangle \\ & + \langle v - v_n, (1-\alpha_n)(v_n - y^*) + \alpha_n(Tv_n - y^*)\rangle) \\ \leq\ & ||x_n - x^*||^2 + ||y_n - y^*||^2 - \alpha_n(1-\alpha_n)(||Su_n - u_n||^2 + ||Tv_n - v_n||^2) \\ & + t_n^2(||u_n - u||^2 + ||v_n - v||^2) \\ & + 2t_n(\langle u - u_n, (1-\alpha_n)(u_n - x^*) + \alpha_n(Su_n - x^*)\rangle \\ & + \langle v - v_n, (1-\alpha_n)(v_n - y^*) + \alpha_n(Tv_n - y^*)\rangle). \end{aligned} \tag{3.14}$$

Since $\{u_n\}$, $\{v_n\}$, $\{Su_n\}$ and $\{Tv_n\}$ are all bounded, there exists $M > 0$ such that

$$\begin{aligned} & [t_n(||u_n - u||^2 + ||v_n - v||^2) + 2(\langle u - u_n, (1-\alpha_n)(u_n - x^*) + \alpha_n(Su_n - x^*)\rangle \\ & + \langle v - v_n, (1-\alpha_n)(v_n - y^*) + \alpha_n(Tv_n - y^*)\rangle)] \\ & \leq M. \end{aligned}$$

Thus, from (3.14) we obtain that

$$\begin{aligned} & \alpha_n(1-\alpha_n)(||Su_n - u_n||^2 + ||Tv_n - v_n||^2) \\ & \quad \leq\ ||x_n - x^*||^2 + ||y_n - y^*||^2 \\ & \qquad -(||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2) + t_n M \to 0. \end{aligned} \tag{3.15}$$

Therefore, by condition (iii), we get

$$||Su_n - u_n||^2 + ||Tv_n - v_n||^2 \to 0.$$

Hence,

$$\lim_{n\to\infty} ||Su_n - u_n|| = \lim_{n\to\infty} ||Tv_n - v_n|| = 0. \tag{3.16}$$

Using (3.11), we obtain that

$$||u_n - x_n|| = \gamma_n ||A^*(Ax_n - By_n)|| \to 0, n \to \infty. \tag{3.17}$$

Hence,

$$||Su_n - x_n|| \leq ||Su_n - u_n|| + ||u_n - x_n|| \to 0, n \to \infty. \tag{3.18}$$

From 3.1, we obtain

$$\begin{aligned} ||x_{n+1} - x_n|| &= ||(1 - t_n - \alpha_n)(u_n - x_n) + \alpha_n(Su_n - x_n) + t_n(u - x_n)|| \\ &\leq (1 - t_n - \alpha_n)||u_n - x_n|| + \alpha_n||Su_n - x_n|| + t_n||u - x_n|| \to 0. \end{aligned} \tag{3.19}$$

Following the same line of arguments of (3.17), (3.18) and (3.19), we can show that

$$\lim_{n\to\infty} ||v_n - y_n|| = \lim_{n\to\infty} ||y_{n+1} - y_n|| = 0. \tag{3.20}$$

Since $\{x_n\}$ is bounded, there exists $\bar{x} \in H_1$, such that $x_n \rightharpoonup \bar{x}$. By (3.17), we have $u_n \rightharpoonup \bar{x}$ and by the demi-closedness of $S - I$ and (3.16), we get $\bar{x} \in F(S)$. In the same way, by the boundedness of $\{y_n\}$, there exists $\bar{y} \in H_2$, such that $y_n \rightharpoonup \bar{y}$. Similarly, by (3.17), we have $v_n \rightharpoonup \bar{x}$ and by the demi-closedness of $T - I$ and (3.16), we get $\bar{y} \in F(T)$. On the other hand since A and B are bounded linear operators, we have $Ax_n \rightharpoonup A\bar{x}$ and $By_n \rightharpoonup B\bar{y}$. Also by the weakly semi continuity of the norm, we have

$$||A\bar{x} - B\bar{y}|| \leq \liminf_{n\to\infty} ||Ax_n - By_n|| = 0.$$

Hence $(\bar{x}, \bar{y}) \in \Gamma$.

Next we prove that $\{x_n\}$ converges strongly to $\bar{x}$ and $\{y_n\}$ converges strongly to $\bar{y}$. Now, from (3.1), we have

$$\begin{aligned} ||x_{n+1} - \bar{x}||^2 &= ||(1-\alpha_n - t_n)u_n + \alpha_n S u_n + t_n u - \bar{x}||^2 \\ &= ||(1-\alpha_n - t_n)(u_n - \bar{x}) + \alpha_n(Su_n - \bar{x}) + t_n(u - \bar{x})||^2 \\ &\leq ||(1-\alpha_n - t_n)(u_n - \bar{x}) + \alpha_n(Su_n - \bar{x})||^2 + 2t_n\langle x_{n+1} - \bar{x}, u - \bar{x}\rangle \\ &\leq [(1-\alpha_n - t_n)||u_n - \bar{x}|| + \alpha_n||Su_n - \bar{x}||]^2 + 2t_n\langle x_{n+1} - \bar{x}, u - \bar{x}\rangle \\ &\leq [(1-\alpha_n - t_n)||u_n - \bar{x}|| + \alpha_n||u_n - \bar{x}||]^2 + 2t_n\langle x_{n+1} - \bar{x}, u - \bar{x}\rangle \\ &\leq (1-t_n)^2||u_n - \bar{x}||^2 + 2t_n\langle x_{n+1} - \bar{x}, u - \bar{x}\rangle. \end{aligned} \tag{3.21}$$

Similarly, from (3.1), we can show that

$$||y_{n+1} - \bar{y}||^2 \leq (1-t_n)^2||v_n - \bar{y}||^2 + 2t_n\langle y_{n+1} - \bar{y}, v - \bar{y}\rangle. \tag{3.22}$$

Adding (3.21) and (3.22) gives,

$$\begin{aligned} ||x_{n+1} - \bar{x}||^2 + ||y_{n+1} - \bar{y}||^2 &\leq (1-t_n)^2(||u_n - \bar{x}||^2 + ||v_n - \bar{y}||^2) \\ &\quad + 2t_n(\langle x_{n+1} - \bar{x}, u - \bar{x}\rangle + \langle y_{n+1} - \bar{y}, v - \bar{y}\rangle) \\ &\leq (1-t_n)(||u_n - \bar{x}||^2 + ||v_n - \bar{y}||^2) \\ &\quad + 2t_n(\langle x_{n+1} - \bar{x}, u - \bar{x}\rangle + \langle y_{n+1} - \bar{y}, v - \bar{y}\rangle) \\ &\leq (1-t_n)(||x_n - \bar{x}||^2 + ||y_n - \bar{y}||^2) \\ &\quad + 2t_n(\langle x_{n+1} - \bar{x}, u - \bar{x}\rangle + \langle y_{n+1} - \bar{y}, v - \bar{y}\rangle). \end{aligned} \tag{3.23}$$

Since $x_n \rightharpoonup \bar{x}$ and $y_n \rightharpoonup \bar{y}$, then $\langle x_{n+1} - \bar{x}, u - \bar{x}\rangle + \langle y_{n+1} - \bar{y}, v - \bar{y}\rangle \to 0$. Using Lemma 2.5 in (3.23), we obtain

$$||x_n - \bar{x}||^2 + ||y_n - \bar{y}||^2 \to 0.$$

Hence,

$$\lim_{n\to\infty} ||x_n - \bar{x}|| = \lim_{n\to\infty} ||y_n - \bar{y}|| = 0.$$

Therefore, $(x_n, y_n) \to (\bar{x}, \bar{y}) \in \Gamma$.

Case 2. Assume that $\{||x_n - x^*||^2 + ||y_n - y^*||^2\}$ is not a monotonically decreasing sequence. Set $\Gamma_n = ||x_n - x^*||^2 + ||y_n - y^*||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some large enough n_0) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Obviously, $\{\tau(n)\}$ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \text{ for } n \geq n_0.$$

It follows from (3.9) that

$$\begin{aligned}\gamma^2_{\tau(n)}(||A^*(Ax_{\tau(n)} - By_{\tau(n)})||^2 + ||B^*(Ax_{\tau(n)} - By_{\tau(n)})||^2)\\ \leq (||x_{\tau(n)} - x^*||^2 + ||y_{\tau(n)} - y^*||^2)\\ -\frac{1}{(1-t_{\tau(n)})}(||x_{\tau(n)+1} - x^*||^2 + ||y_{\tau(n)+1} - y^*||^2)\\ +\frac{t_{\tau(n)}}{(1-t_{\tau(n)})}(||u - x^*||^2 + ||v - y^*||^2).\end{aligned}$$

Hence,

$$\gamma^2_{\tau(n)}(||A^*(Ax_{\tau(n)} - By_{\tau(n)})||^2 + ||B^*(Ax_{\tau(n)} - By_{\tau(n)})||^2) \to 0.$$

By the condition

$$\gamma_{\tau(n)} \in \Big(\epsilon, \frac{2||Ax_{\tau(n)} - By_{\tau(n)}||^2}{||A^*(Ax_{\tau(n)} - By_{\tau(n)})||^2 + ||B^*(Ax_{\tau(n)} - By_{\tau(n)})||^2} - \epsilon\Big), \ \tau(n) \in \Omega,$$

we can conclude that

$$\lim_{n\to\infty}(||A^*(Ax_{\tau(n)} - By_{\tau(n)})||^2 + ||B^*(Ax_{\tau(n)} - By_{\tau(n)})||^2) = 0.$$

Note that $Ax_{\tau(n)} - By_{\tau(n)} = 0$ if $\tau(n) \notin \Omega$. Thus,

$$\lim_{n\to\infty} ||A^*(Ax_{\tau(n)} - By_{\tau(n)})|| = \lim_{n\to\infty} ||B^*(Ax_{\tau(n)} - By_{\tau(n)})|| = 0.$$

By following the same line of arguments as in case 1, we can show that

$$\lim_{n\to\infty} ||Su_{\tau(n)} - u_{\tau(n)}|| = \lim_{n\to\infty} ||Tv_{\tau(n)} - v_{\tau(n)}|| = 0$$

and $\{(x_{\tau(n)}, y_{\tau(n)})\}$ converges weakly to $(\bar{x}, \bar{y}) \in \Gamma$. Now for all $n \geq n_0$, we have from (3.23) that

$$\begin{aligned}0 &\leq ||x_{\tau(n)+1} - \bar{x}||^2 + ||y_{\tau(n)+1} - \bar{y}||^2 - (||x_{\tau(n)} - \bar{x}||^2 + ||y_{\tau(n)} - \bar{y}||^2)\\ &\leq (1 - t_{\tau(n)})^2(||u_{\tau(n)} - \bar{x}||^2 + ||v_{\tau(n)} - \bar{y}||^2) + 2t_{\tau(n)}(\langle x_{\tau(n)+1} - \bar{x}, u - \bar{x}\rangle\\ &\quad +\langle y_{\tau(n)+1} - \bar{y}, v - \bar{y}\rangle) - (||x_{\tau(n)} - \bar{x}||^2 + ||y_{\tau(n)} - \bar{y}||^2)\\ &\leq (1 - t_{\tau(n)})(||x_{\tau(n)} - \bar{x}||^2 + ||y_{\tau(n)} - \bar{y}||^2) + 2t_{\tau(n)}(\langle x_{\tau(n)+1} - \bar{x}, u - \bar{x}\rangle\\ &\quad +\langle y_{\tau(n)+1} - \bar{y}, v - \bar{y}\rangle) - (||x_{\tau(n)} - \bar{x}||^2 + ||y_{\tau(n)} - \bar{y}||^2)\\ &= t_{\tau(n)}[2(\langle x_{\tau(n)+1} - \bar{x}, u - \bar{x}\rangle + \langle y_{\tau(n)+1} - \bar{y}, v - \bar{y}\rangle)\\ &\quad -(||x_{\tau(n)} - \bar{x}||^2 + ||y_{\tau(n)} - \bar{y}||^2)].\end{aligned}$$

Thus,

$$||x_{\tau(n)}-\bar{x}||^2+||y_{\tau(n)}-\bar{y}||^2 \leq 2(\langle x_{\tau(n)+1}-\bar{x}, u-\bar{x}\rangle + \langle y_{\tau(n)+1}-\bar{y}, v-\bar{y}\rangle) \to 0.$$

Hence,

$$\lim_{n\to\infty} ||x_{\tau(n)}-\bar{x}||^2 = \lim_{n\to\infty} ||y_{\tau(n)}-\bar{y}||^2 = 0.$$

Therefore,

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$$

Furthermore, for $n \geq n_0$, it is easily observed that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$) because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n$. Consequently for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

So, $\lim_{n\to\infty} \Gamma_n = 0$. That is, $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y}) \in \Gamma$. □

Corollary 3.2. *Let H_1,H_2 and H_3 be real Hilbert spaces. Let $S: H_1 \to H_1$ and $T: H_2 \to H_2$ be nonexpansive mappings such that $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$. Let $A: H_1 \to H_3$ and $B: H_2 \to H_3$ be bounded linear operators. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ and $\{t_n\}$ be a sequence in $(0, 1-a)$ for some $a > 0$. Let the step size γ_n be chosen in such a way that*

$$\gamma_n \in \Big(\epsilon, \frac{2||Ax_n - By_n||^2}{||A^*(Ax_n - By_n)||^2 + ||B^*(Ax_n - By_n)||^2} - \epsilon\Big), \; n \in \Omega,$$

otherwise $\gamma_n = \gamma$(γ being any non-negative value), where the index set $\Omega = \{n : Ax_n - By_n \neq 0\}$. Let $x_1 \in H_1$ and $y_1 \in H_2$ be arbitrary and the sequences $\{x_n\}$ and $\{y_n\}$ iteratively generated by

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = (1-\alpha_n - t_n)u_n + \alpha_n S u_n + t_n u, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = (1-\alpha_n - t_n)v_n + \alpha_n T v_n + t_n v, \end{cases} \tag{3.24}$$

with the conditions
(i) $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=0}^{\infty} t_n = \infty$,
(ii) $\alpha_n + t_n < 1$,
(iii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.
Then $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y}) \in \Gamma$.

Proof. Since every nonexpansive mapping with nonempty fixed point set is quasi-nonexpansive, we have our desired conclusion, by using Theorem 3.1. □

Corollary 3.3. *Let H_1,H_2 and H_3 be real Hilbert spaces. Let $S: H_1 \to H_1$ and $T: H_2 \to H_2$ be k_1, k_2- demi-contractive mappings such that $S - I$ and $T - I$ are demi-closed at 0 and $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$. Let $A: H_1 \to H_3$ and $B: H_2 \to H_3$ be bounded linear operators. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ and*

$\{t_n\}$ be a sequence in $(0,1-a)$ for some $a>0$. Let the step size γ_n be chosen in such a way that

$$\gamma_n \in \left(\epsilon, \frac{2||Ax_n - By_n||^2}{||A^*(Ax_n - By_n)||^2 + ||B^*(Ax_n - By_n)||^2} - \epsilon\right),\ n \in \Omega,$$

otherwise $\gamma_n = \gamma$ (γ being any non-negative value), where the index set $\Omega = \{n : Ax_n - By_n \neq 0\}$. Let $u, x_1 \in H_1$ and $v, y_1 \in H_2$ be arbitrary and the sequences $\{x_n\}$ and $\{y_n\}$ iteratively generated by

$$(3.25)\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = (1-\alpha_n - t_n)u_n + \alpha_n[(1-k_1)u_n + k_1 S u_n] + t_n u, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = (1-\alpha_n - t_n)v_n + \alpha_n[(1-k_2)v_n + k_2 T v_n] + t_n v, \end{cases}$$

with the conditions
(i) $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=0}^{\infty} t_n = \infty$,
(ii) $\alpha_n + t_n < 1$,
(iii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.
Then $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y}) \in \Gamma$.

Proof. If S and T are demi-contractive mappings, then it is easy to see that the mappings $V := (1-k_1)I + k_1 S$ and $Q := (1-k_2)I + k_2 T$ are quasi-nonexpansive mappings and $F(V) = F(S)$ and $F(Q) = F(T)$. Hence, by Theorem 3.1, we have our desired conclusion. □

4. Some applications

Relying on some convex and nonlinear analysis notions (for example, see [23]), we present some of the applications of our result.

4.1. Split equality problem. Let H_1, H_2 and H_3 be real Hilbert spaces and $C \subset H_1, Q \subset H_2$ be nonempty, closed and convex sets. Suppose that $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators. Taking $S = P_C$ and $T = P_Q$ in (3.1), we have the following simultaneous iterative algorithm for solving split equality problems (SEP) (1.8). Let $x_1, u \in H_1, y_1, v \in H_2$;

$$(4.1)\quad \begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = (1-\alpha_n - t_n)u_n + \alpha_n P_C u_n + t_n u, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = (1-\alpha_n - t_n)v_n + \alpha_n P_Q v_n + t_n v, \end{cases}$$

where the step-size γ_n is chosen as in Theorem 3.1. Using algorithm 4.1 and following the method of proof of Theorem 3.1, we can prove the following strong theorem for solving SEP 1.8.

Theorem 4.1. *Let H_1, H_2 and H_3 be real Hilbert spaces. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators. Let $C \subset H_1, Q \subset H_2$ be nonempty, closed and convex sets and suppose that the solution set of problem (1.8) is* *denoted by* $\Gamma \neq \emptyset$*. Let* $\{\alpha_n\}$ *be a sequence in* $(0,1)$ *and* $\{t_n\}$ *be a sequence in* $(0, 1-a)$ *for some* $a > 0$*. Let the step size* γ_n *be chosen in such a way that for some* $\epsilon > 0$*,*

$$\gamma_n \in \Big(\epsilon, \frac{2||Ax_n - By_n||^2}{||A^*(Ax_n - By_n)||^2 + ||B^*(Ax_n - By_n)||^2} - \epsilon\Big), \ n \in \Omega,$$

otherwise $\gamma_n = \gamma$ *(*γ *being any nonnegative value), where the index set* $\Omega = \{n : Ax_n - By_n \neq 0\}$*. For arbitrary but fixed* $u \in H_1, v \in H_2$*, let the sequences* $\{x_n\}$ *and* $\{y_n\}$ *be iteratively generated by (4.1) with the conditions:*
(i) $\lim_{n\to\infty} t_n = 0$ *and* $\sum_{n=0}^{\infty} t_n = \infty$*,*
(ii) $\alpha_n + t_n < 1$*,*
(iii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$*.*
Then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y})$ *in the solution set* Γ *of (1.8).*

4.2. **The variational inclusion problem.** Suppose that $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators. Let $M : H_1 \to 2^{H_1}$ and $N : H_2 \to 2^{H_2}$ be maximal monotone operators. Let us consider the following problem:

$$\text{find } x^* \in M^{-1}(0), y^* \in N^{-1}(0) \text{ such that } Ax^* = By^*. \tag{4.2}$$

Given a maximal monotone operator $M : H_1 \to 2^{H_1}$, it is well known that the associated resolvent mapping, $J_\mu^M(x) := (I + \mu M)^{-1}x, x \in H_1$ is firmly nonexpansive (hence quasi-nonexpansive) and $0 \in M(x)$ if and only if x is a fixed point of $J_\mu^M, \mu > 0$. Taking $S = J_\mu^M$, $T = J_\nu^N, \nu > 0$ in (3.1), we have the following simultaneous iterative algorithm for solving problem (4.2). Let $x_1, u \in H_1, y_1, v \in H_2$;

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = (1 - \alpha_n - t_n)u_n + \alpha_n J_\mu^M u_n + t_n u, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = (1 - \alpha_n - t_n)v_n + \alpha_n J_\nu^N v_n + t_n v, \end{cases} \tag{4.3}$$

where the step-size γ_n is chosen as in Theorem 3.1. Using the iterative scheme 4.3 and following the method of proof of Theorem 3.1, we have the following strong convergence theorem for solving problem 4.2.

Theorem 4.2. *Let* H_1, H_2 *and* H_3 *be real Hilbert spaces. Let* $A : H_1 \to H_3$ *and* $B : H_2 \to H_3$ *be bounded linear operators. Let* $M : H_1 \to 2^{H_1}$ *and* $N : H_2 \to 2^{H_2}$ *be maximal monotone operators and suppose that the solution set of problem 4.2 is denoted by* $\Gamma \neq \emptyset$*. Let* $\{\alpha_n\}$ *be a sequence in* $(0,1)$ *and* $\{t_n\}$ *be a sequence in* $(0, 1-a)$ *for some* $a > 0$*. Let the step size* γ_n *be chosen in such a way that for some* $\epsilon > 0$*,*

$$\gamma_n \in \Big(\epsilon, \frac{2||Ax_n - By_n||^2}{||A^*(Ax_n - By_n)||^2 + ||B^*(Ax_n - By_n)||^2} - \epsilon\Big), \ n \in \Omega,$$

otherwise $\gamma_n = \gamma$ *(*γ *being any nonnegative value), where the index set* $\Omega = \{n : Ax_n - By_n \neq 0\}$*. For arbitrary but fixed* $u \in H_1, v \in H_2$*, let the sequences*

$\{x_n\}$ and $\{y_n\}$ be iteratively generated by (4.3) with the conditions:
(i) $\lim_{n\to\infty} t_n = 0$ *and* $\sum_{n=0}^{\infty} t_n = \infty$,
(ii) $\alpha_n + t_n < 1$,
(iii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.
Then $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$ in the solution set Γ of (4.2).

4.3. **Equilibrium problem.** Let H_1, H_2 and H_3 be real Hilbert spaces and $C \subset H_1, Q \subset H_2$ be nonempty, closed and convex sets. Suppose that $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators. Let $F : C \times C \to \mathbb{R}, H : Q \times Q \to \mathbb{R}$ be bi-functions. Let us consider the following problem:

(4.4)
$$\text{find } x^* \in C, y^* \in Q \text{ such that } F(x^*, z) \geq 0, H(y^*, u) \geq 0, \forall z \in C, \forall u \in Q \text{ and } Ax^* = By^*.$$

To solve problem 4.4, we assume the following conditions on F and H:
(A1) $F(x, x) = 0$ for all $x \in C$ and $H(x, x) = 0$ for all $x \in Q$;
(A2) F and H are monotone, that is, $F(x, y) + F(y, x) \leq 0$, for all $x, y \in C$ and $H(x, y) + H(y, x) \leq 0$, for all $x, y \in Q$;
(A3) for each $x, y, z \in C$; $\limsup_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$ and for each $x, y, z \in Q$; $\limsup_{t\downarrow 0} H(tz + (1-t)x, y) \leq H(x, y)$;
(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous and for each $x \in Q$, $y \mapsto H(x, y)$ is convex and lower semicontinuous.
Let us define the resolvent mappings $T_\mu^F, \mu > 0$ and $T_\nu^H, \nu > 0$ as

$$T_\mu^F(x) = \{z \in C : F(z, y) + \frac{1}{\mu}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}.$$

and

$$T_\nu^H(x) = \{z \in Q : H(z, y) + \frac{1}{\nu}\langle y - z, z - x\rangle \geq 0, \forall y \in Q\}.$$

It is well known that the resolvent mappings T_μ^F and T_ν^H are firmly-nonexpansive mappings and hence quasi-nonexpansive mappings. Furthermore, it is known that if x^*, y^* solves problem 4.4, then $x^* = T_\mu^F x^*$ and $y^* = T_\nu^H y^*$.

By setting $S = T_\mu^F$ and $T = T_\nu^H$, then problem 1.5 becomes problem 4.4 and hence the algorithm (3.1) becomes

$$\begin{cases} \forall x_1, u \in H_1, y_1, v \in H_2 \\ u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = (1 - \alpha_n - t_n)u_n + \alpha_n T_\mu^F u_n + t_n u, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = (1 - \alpha_n - t_n)v_n + \alpha_n T_\nu^H v_n + t_n v, \end{cases} \tag{4.5}$$

where the step-size γ_n is chosen as in Theorem 3.1. Using the iterative scheme 4.5 and following the method of proof of Theorem 3.1, we have the following strong convergence theorem for solving problem 4.4.

Theorem 4.3. *Let H_1, H_2 and H_3 be real Hilbert spaces and $C \subset H_1, Q \subset H_2$ be nonempty, closed and convex sets. Suppose that $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators. Let $F : C \times C \to \mathbb{R}, H : Q \times Q \to \mathbb{R}$ be bi-functions. Suppose that the solution set of problem 4.4 is denoted by $\Gamma \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ and $\{t_n\}$ be a sequence in $(0, 1-a)$ for some $a > 0$. Let the step size γ_n be chosen in such a way that for some $\epsilon > 0$,*

$$\gamma_n \in \Big(\epsilon, \frac{2||Ax_n - By_n||^2}{||A^*(Ax_n - By_n)||^2 + ||B^*(Ax_n - By_n)||^2} - \epsilon\Big),\ n \in \Omega,$$

otherwise $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set $\Omega = \{n : Ax_n - By_n \neq 0\}$. For arbitrary but fixed $u \in H_1, v \in H_2$, let the sequences $\{x_n\}$ and $\{y_n\}$ be iteratively generated by 4.5 with the conditions:
(i) $\lim_{n\to\infty} t_n = 0$ *and* $\sum_{n=0}^{\infty} t_n = \infty$,
(ii) $\alpha_n + t_n < 1$,
(iii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.
Then $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$ in the solution set Γ of 4.4.

4.4. **Proximal split feasibility problem.** Let $f : H_1 \to \mathbb{R}\cup\{+\infty\}, g : H_2 \to \mathbb{R}\cup\{+\infty\}$ two proper, convex, lower semicontinuous functions and $A : H_1 \to H_3$ and $B : H_2 \to H_3$ bounded linear operators. Let $argmin f := \{\bar{x} \in H_1 : f(\bar{x}) \leq f(x), \forall x \in H_1\}$ and $argmin\ g := \{\bar{y} \in H_2 : g(\bar{y}) \leq g(y), \forall y \in H_2\}$. Let us consider the following problem:

(4.6) find $x^* \in argmin f, y^* \in argmin\ g$ such that $Ax^* = By^*$.

We define the proximal mapping of g as $prox_{\lambda g} := argmin_{u\in H_2}\{g(u) + \frac{1}{2\lambda}||u - y||^2\}, \lambda > 0$. We recall that minimizers of any function are exactly fixed-points of its proximal mapping and proximal mapping is firmly-nonexpansive. For more details, please see Moudafi and Thakur [21]. Using the algorithm in Theorem 3.1, we propose the following algorithm for solving (4.6). Let $x_1, u \in H_1, y_1, v \in H_2$;

$$(4.7)\qquad \begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n),\\ x_{n+1} = (1-\alpha_n - t_n)u_n + \alpha_n prox_{\lambda f} u_n + t_n u,\\ v_n = y_n + \gamma_n B^*(Ax_n - By_n),\\ y_{n+1} = (1-\alpha_n - t_n)v_n + \alpha_n prox_{\lambda g} v_n + t_n v, \end{cases}$$

where the step-size γ_n is chosen as in Theorem 3.1. Using the iterative scheme (4.3) and following the method of proof of Theorem 3.1, we have the following strong convergence theorem for solving problem 4.2.

Theorem 4.4. *Let H_1, H_2 and H_3 be real Hilbert spaces. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators. Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}, g : H_2 \to$*

$\mathbb{R}\cup\{+\infty\}$ *two proper, convex, lower semicontinuous functions and suppose that the solution set of problem* (4.6) *is denoted by* $\Gamma \neq \emptyset$. *Let* $\{\alpha_n\}$ *be a sequence in* $(0,1)$ *and* $\{t_n\}$ *be a sequence in* $(0, 1-a)$ *for some* $a > 0$. *Let the step size* γ_n *be chosen in such a way that for some* $\epsilon > 0$,

$$\gamma_n \in \Big(\epsilon, \frac{2||Ax_n - By_n||^2}{||A^*(Ax_n - By_n)||^2 + ||B^*(Ax_n - By_n)||^2} - \epsilon\Big),\ n \in \Omega,$$

otherwise $\gamma_n = \gamma$ *(*γ *being any nonnegative value), where the index set* $\Omega = \{n : Ax_n - By_n \neq 0\}$. *For arbitrary but fixed* $u \in H_1, v \in H_2$, *let the sequences* $\{x_n\}$ *and* $\{y_n\}$ *be iteratively generated by* 4.3 *with the conditions:*
(i) $\lim_{n\to\infty} t_n = 0$ *and* $\sum_{n=0}^{\infty} t_n = \infty$,
(ii) $\alpha_n + t_n < 1$,
(iii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.
Then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y})$ *in the solution set* Γ *of* (4.2).

Remark 4.5. Here we make the following Observation
1. Zhao [32] employed the scheme,

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)U(u_n), \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = \beta_n v_n + (1-\beta_n)Tv_n, \end{cases} \tag{4.8}$$

to get a weak convergence while we used the modified iterative

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = (1-\alpha_n - t_n)u_n + \alpha_n S u_n + t_n u, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = (1-\alpha_n - t_n)v_n + \alpha_n T v_n + t_n v, \end{cases} \tag{4.9}$$

to get a strong convergence result which is more reliable.
2. Prototypes of sequences $\{\alpha_n\}$ and $\{t_n\}$ in this paper are:

$$\alpha_n = \frac{n+1}{2(n+2)},\ t_n = \frac{1}{2(n+2)},\ n \geq 0.$$

5. **Numerical example**

We present here a numerical example for our result in an infinite dimensional Hilbert space.

Example 5.1. Let $H_1 = H_2 = H_3 = L^2[0,1]$. Let $A, B : L^2[0,1] \to L^2[0,1]$ be defined as

$$(Ax)(t) = (Bx)(t) := f(t)x(t)$$

where f is a continuous function on $[0,1]$. Then A and B are bounded linear operators on $L^2[0,1]$ and $A^* = A$ and $B^* = B$. Here we shall let $f(t) := 3t^4$.

Define $T : L^2[0,1] \to L^2[0,1]$, by $T(x) := \begin{cases} (\frac{x}{2})sin\frac{1}{||x||}, \ x \neq 0_{L^2[0,1]} \\ 0_{L^2[0,1]}, \ otherwise \end{cases}$, and $S : L^2[0,1] \to L^2[0,1]$ by $S(x) := \beta x, \ |\beta| \leq 1$. Then T and S are quasi-nonexpansive.
Obviously $0_{L^2[0,1]} \in F(T), 0_{L^2[0,1]} \in F(S)$ and $A0_{L^2[0,1]} = B0_{L^2[0,1]} = 0_{L^2[0,1]}$. Therefore the solution set Γ of problem 1.5, which is to find

$$x \in F(S), y \in F(T), \text{ such that } Ax = By. \tag{5.1}$$

is nonempty, since $(0_{L^2[0,1]}, 0_{L^2[0,1]}) \in \Gamma$.
Let α_n and t_n be as in the remark above and the step size γ_n be chosen in such a way that for some $\epsilon > 0$,

$$\gamma_n \in \Big(\epsilon, \frac{2||Ax_n - By_n||^2}{||A^*(Ax_n - By_n)||^2 + ||B^*(Ax_n - By_n)||^2} - \epsilon\Big), \ n \in \Omega,$$

otherwise $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set $\Omega = \{n : Ax_n - By_n \neq 0\}$.
Let $u, x_1 \in H_1$ and $v, y_1 \in H_2$ be arbitrary. Then our iterative scheme (3.1) becomes

$$(5.2)\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = (1 - \frac{n+1}{2(n+2)} - \frac{1}{2(n+2)})u_n + \frac{n+1}{2(n+2)}Su_n + \frac{1}{2(n+2)}u, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = (1 - \frac{n+1}{2(n+2)} - \frac{1}{2(n+2)})v_n + \frac{n+1}{2(n+2)}Tv_n + \frac{1}{2(n+2)}v, \ n \geq 0. \end{cases}$$

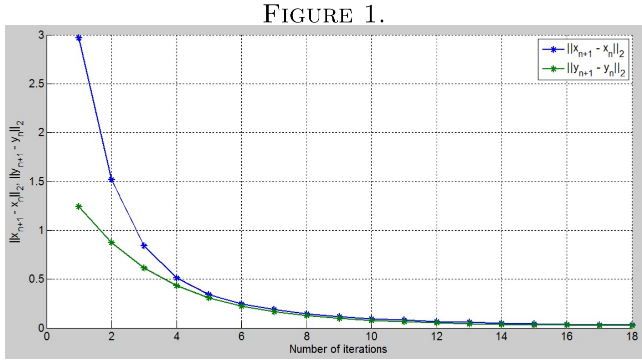
We make different choices of u, v, x_1 and y_1 and use $\frac{\max\{||x_{n+1}-x_n||, ||y_{n+1}-y_n||\}}{\max\{||x_2-x_1||, ||y_2-y_1||\}} < 10^{-2}$, for stopping criterion.

Case I: Take $u = \cos(t)$, $v = \sqrt[3]{t}$, $x_1 = 2t^6 + \frac{t^7}{3}$ and $y_1 = t^2$.
Subcase I: $(\beta, \gamma_n) = (0.1, 0.01)$
Using these parameters for Case I, the numerical result is displayed in Table 1 and the graph is given in Figure 1.
Subcase II: $(\beta, \gamma_n) = (-0.9, 0.01)$
Using these parameters for Case I, the numerical result is displayed in Table 2 and the graph is given in Figure 2.
Case II: Take $u = \sqrt{t}$, $v = \frac{t}{2}$, $x_1 = 2t$ and $y_1 = t^2 + sin(t)$.
Subcase I: $(\beta, \gamma_n) = (0.1, 0.01)$
Using these parameters for Case I, the numerical result is displayed in Table 3 and the graph is given in Figure 3.
Subcase II: $(\beta, \gamma_n) = (-0.9, 0.01)$
Using these parameters for Case I, the numerical result is displayed in Table 4 and the graph is given in Figure 4.

TABLE 1.

Time Taken	No. of Iterations	$\|\|x_{n+1} - x_n\|\|_2$	$\|\|y_{n+1} - y_n\|\|_2$
0.0059	1	2.9669	1.2413
	2	1.5211	0.8728
	3	0.8415	0.6142
	4	0.5109	0.4318
	5	0.3414	0.3068
	6	0.2465	0.2222
	7	0.1873	0.1648
	8	0.1469	0.1255
	9	0.1178	0.0980
	10	0.0961	0.0783
	11	0.0796	0.0640
	12	0.0668	0.0533
	13	0.0568	0.0451
	14	0.0488	0.0387
	15	0.0423	0.0336
	16	0.0371	0.0294
	17	0.0328	0.0260
	18	0.0292	0.0232

FIGURE 1.



Remark 5.2. The choice of γ_n, as long as it is in the range, does not have any significant effect on both the number of iterations and the cpu time. While we notice that different choices of β have effect on both the number of iterations and the cpu time as can be seen from the Tables and Figures presented.

Remark 5.3. We emphasize here that the main reason of our numerical example is not to compare our algorithm to that of Zhao [32] but rather to show how simple and effective is the implementation of our iterative scheme (3.1). The work of Zhao [32] and the work presented in this paper are comparable in terms of the type of convergence in the sense that weak convergence was proved by Zhao and we prove strong convergence for the scheme in (3.1).

TABLE 2.

Time Taken	No. of Iterations	$\|\|x_{n+1} - x_n\|\|_2$	$\|\|y_{n+1} - y_n\|\|_2$
0.0020	1	4.9728	1.2413
	2	1.1239	0.9249
	3	0.3702	0.6279
	4	0.2076	0.4254
	5	0.1375	0.2965
	6	0.0983	0.2135
	7	0.0740	0.1587
	8	0.0578	0.1213
	9	0.0465	0.0952
	10	0.0382	0.0765
	11	0.0320	0.0628
	12	0.0272	0.0524
	13	0.0234	0.0444

FIGURE 2.

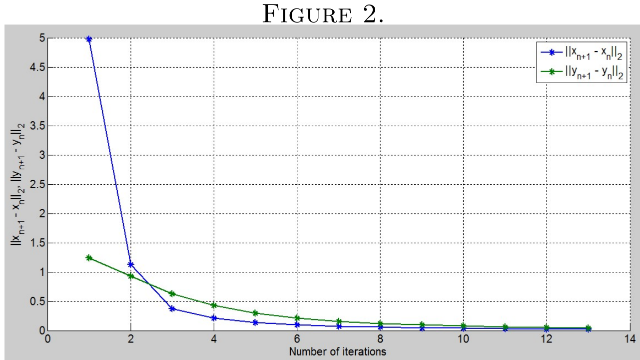


TABLE 3.

Time Taken	No. of Iterations	$\|\|x_{n+1} - x_n\|\|_2$	$\|\|y_{n+1} - y_n\|\|_2$
0.0030	1	4.2927	4.3383
	2	2.5495	2.2866
	3	1.5299	1.2178
	4	0.9338	0.6603
	5	0.5836	0.3676
	6	0.3759	0.2120
	7	0.2509	0.1278
	8	0.1741	0.0813
	9	0.1259	0.0548
	10	0.0946	0.0391
	11	0.0737	0.0294
	12	0.0592	0.0231
	13	0.0488	0.0188
	14	0.0411	0.0157

FIGURE 3.

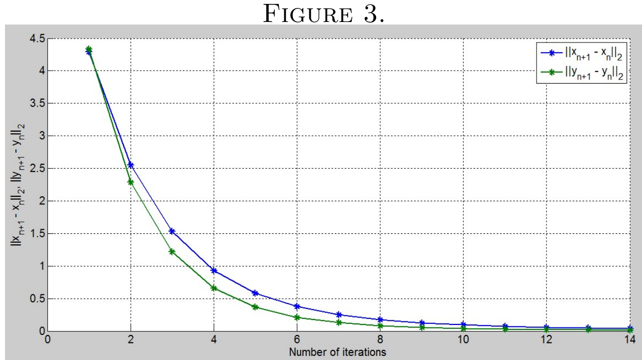
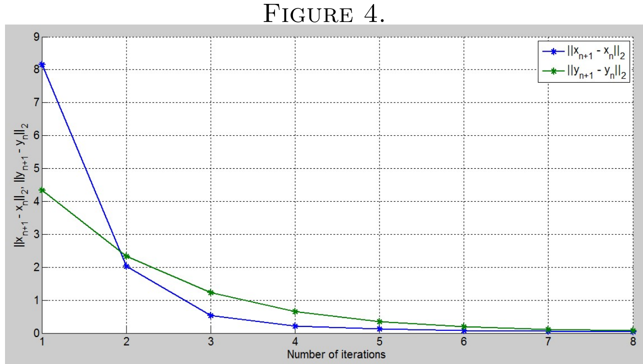


TABLE 4.

Time Taken	No. of Iterations	$\|\|x_{n+1} - x_n\|\|_2$	$\|\|y_{n+1} - y_n\|\|_2$
0.0015	1	8.1563	4.3383
	2	2.0352	2.3399
	3	0.5423	1.2346
	4	0.2108	0.6544
	5	0.1200	0.3550
	6	0.0826	0.2000
	7	0.0616	0.1185
	8	0.0480	0.0746

FIGURE 4.



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