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TRIPLE POSITIVE SOLUTIONS OF m-POINT BOUNDARY VALUE PROBLEM ON TIME SCALES WITH p-LAPLACIAN

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ABSTRACT. In this paper, we consider the multipoint boundary value problem for one-dimensional p-Laplacian dynamic equation on time scales. We prove the existence at least three positive solutions of the boundary value problem by using the Avery and Peterson fixed point theorem. The interesting point is that the non-linear term f involves a first-order derivative explicitly. Our results are new for the special cases of difference equations and differential equations as well as in the general time scale setting.
Keywords: Time scales, boundary value problem, p-Laplacian, positive solutions, fixed point theorem
MSC(2010): Primary: 34B15; Secondary: 34B16, 34B18, 39A10.

1. Introduction

The theory of dynamic equation on time scales was initiated by Stefan Hilger in his Ph.D. thesis in 1988 [13] as a means of unifying structure for the study of differential equations in the continuous case and study of finite difference equations in the discrete case. In recent years, it has found a considerable amount of interest and attracted the attention of many researchers. It is still a new area, and research in this area is rapidly growing. The study of time scales has led to several important applications, e.g., in the study of insect population models, heat transfer, neural networks, phytoremediation of metals, wound healing, and epidemic models [5, 14, 19, 21].

In this paper, we study the existence of at least three positive solutions to the following p-Laplacian multipoint boundary value problem (BVP) on time scales

$$(\varphi_p(u^{\Delta}(t)))^{\nabla} + q(t)f(t, u(t), u^{\Delta}(t)) = 0, \qquad t \in [0, T]_{\mathbb{T}}, \tag{1.1}$$

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$$u(0)=\sum_{i=1}^{m-2}\alpha_i u(\xi_i),\qquad \varphi_p(u^\Delta(T))=\sum_{i=1}^{m-2}\beta_i\varphi_p(u^\Delta(\xi_i)), \tag{1.2}$$

where $\varphi_p(u)$ is the p-Laplacian operator, i.e., $\varphi_p(u)=|u|^{p-2}u$, for $p>1$, with $(\varphi_p)^{-1}=\varphi_q$ and $\frac{1}{p}+\frac{1}{q}=1$. The usual notation and terminology for time scales as can be found in [4, 5], will be used here. Throughout the paper, we will suppose that the following conditions are satisfied:

(H1) $f:[0,T]_{\mathbb{T}}\times\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}^+$ is continuous, and does not vanish identically on any closed subinterval of $[0,T]_{\mathbb{T}}$, where $\mathbb{R}^+$ denotes the nonnegative real numbers;

(H2) $q:\mathbb{T}\to\mathbb{R}^+$ is left dense continuous (i.e., $q\in C_{ld}(\mathbb{T},\mathbb{R}^+)$), and does not vanish identically on any closed subinterval of $[0,T]_{\mathbb{T}}$, where $C_{ld}(\mathbb{T},\mathbb{R}^+)$ denotes the set of all left dense continuous functionals from $\mathbb{T}$ to $\mathbb{R}^+$;

(H3) If $\xi_{m-2}>0$, let $\xi_{m-2}\le\eta$, and if $\xi_{m-2}=0$, let $\eta\ge\min\left\{t\in\mathbb{T}:t\ge\frac{T}{2}\right\}$, and there exists $r\in\mathbb{T}$ such that $\eta<r<T$ holds.

Recently, there has been much current attention focused on the study of multipoint positive solutions of BVPs on time scales. When the nonlinear term f does not depend on the first order derivative, many researchers have studied multipoint boundary conditions on time scales; see [1, 3, 7–9, 11, 12, 15–18, 22]. However, there are few papers dealing with the existence of triple positive solutions for boundary value problems on time scales, when the nonlinear term f is involved in the first-order derivative explicitly; see [6, 20].

All the above works about positive solutions were done under the assumption that f is allowed to depend just on u, while the first order derivative u^Δ is not involved explicitly in the nonlinear term f.

Motivated by all the above works, our main results will depend on an application of the Avery and Peterson fixed point theorem. Here, the emphasis is that the nonlinear term is involved explicitly with the first order derivative. As we know, when the nonlinear term f is involved in the first-order derivative, difficulties arise immediately. In this work, we use a fixed point theorem due to Avery and Peterson to overcome the difficulties. We shall prove that the BVP (1.1) and (1.2) has at least three positive solutions.

2. Preliminaries

In this section, we provide some background material from the theory of cones in Banach spaces [10].

Definition 2.1. Let E be a real Banach space. A nonempty, closed, convex set $P\subset E$ is a cone if it satisfies the following two conditions:

(i) $x\in P,\quad \lambda\ge 0$ imply $\quad \lambda x\in P$;

(ii) $x\in P,\quad -x\in P$ imply $\quad x=0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. Then for positive real numbers a, b, c and d, we define the following sets:

$$\begin{aligned}
P(\gamma, d) &= \{x \in P : \gamma(x) < d\}, \\
P(\gamma, \alpha, b, d) &= \{x \in P : b \leq \alpha(x), \gamma(x) \leq d\}, \\
P(\gamma, \theta, a, b, c, d) &= \{x \in P : b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq d\}, \\
R(\gamma, \psi, a, d) &= \{x \in P : a \leq \psi(x), \gamma(x) \leq d\}.
\end{aligned}$$

To prove our results, we need the following fixed point theorem due to Avery and Peterson [2].

Theorem 2.2 ([2]). *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda u) \leq \lambda \psi(u)$ for $0 \leq \lambda \leq 1$ such that for some positive numbers h and d,*

$$\alpha(u) \leq \psi(u) \quad \text{and} \quad \|u\| \leq h\gamma(u)$$

for all $u \in \overline{P(\gamma, d)}$. Suppose $F : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers a, b and c with $a < b$ such that

(S1) $\{u \in P(\gamma, \theta, \alpha, b, c, d) : \alpha(u) > b\} \neq \emptyset$, *and* $\alpha(Fu) > b$ *for* $u \in P(\gamma, \theta, \alpha, b, c, d)$;

(S2) $\alpha(Fu) > b$, *for* $u \in P(\gamma, \alpha, b, d)$, *with* $\theta(Fu) > c$;

(S3) $0 \notin R(\gamma, \psi, a, d)$ *and* $\psi(Fu) < a$ *for* $u \in R(\gamma, \psi, a, d)$, *with* $\psi(u) = a$.

Then F has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\gamma, d)}$ such that

$$\gamma(u_i) \leq d \quad \text{for} \quad i = 1, 2, 3, \; b < \alpha(u_1), \; a < \psi(u_2), \; \text{with } \alpha(u_2) < b, \; \psi(u_3) < a.$$

Let the Banach space $E = C^1_{ld}([0, \sigma(T)]_{\mathbb{T}} \to \mathbb{R})$ with the norm

$$\|u\| = \max\left\{ \sup_{t \in [0, \sigma(T)]_{\mathbb{T}}} |u(t)|, \sup_{t \in [0, T]_{\mathbb{T}}} |u^{\Delta}(t)| \right\},$$

and define the cone $P \subset E$ by

$$P = \left\{ u \in E : u(t) \geq 0, \text{ for } t \in [0, \sigma(T)]_{\mathbb{T}} \text{ and } u^{\Delta\nabla}(t) \leq 0, \; u^{\Delta}(t) \geq 0, \text{ for } t \in [0, T]_{\mathbb{T}} \right\}.$$

We note that $u(t)$ is a solution of (1.1) and (1.2), if and only if

$$\begin{aligned} u(t) &= \int_0^t \varphi_q\Bigg(\int_s^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau \\ &\quad + \frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\Bigg)\Delta s \\ &\quad + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi_q\Bigg(\int_s^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau \\ &\quad + \frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\Bigg)\Delta s. \end{aligned}$$

Define a completely continuous operator $F: P\to E$ by

$$\begin{aligned} (Fu)(t) &= \int_0^t \varphi_q\Bigg(\int_s^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau \\ &\quad + \frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\Bigg)\Delta s \\ &\quad + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi_q\Bigg(\int_s^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau \\ &\quad + \frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\Bigg)\Delta s. \end{aligned}$$

Lemma 2.3. *The operator $F: P\to P$ is completely continuous.*

Proof. Firstly, we prove that F maps P into P. From (H1) and (H2), it is obvious that $(Fu)(t)\geq 0$ for $t\in[0,T]_{\mathbb{T}}\subset[0,\sigma(T)]_{\mathbb{T}}$ and

$$\begin{aligned} (Fu)^\Delta(t) &= \varphi_q\Bigg(\int_t^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau \\ &\quad + \frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\Bigg)\geq 0 \end{aligned}$$

is continuous and nonincreasing in $[0,T]_{\mathbb{T}}$

$$\begin{aligned} &\left(\int_t^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau + \frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right)^\nabla \\ &= -q(t)f(t,u,u^\Delta)\leq 0,\ \ t\in[0,T]_{\mathbb{T}}. \end{aligned}$$

In addition, $\varphi_q(u)$ is a monotone increasing continuously differentiable function for $u > 0$.

Secondly, we prove that F maps a bounded set into a bounded set. Assume that $c > 0$ is a constant and

$$u \in \overline{P_c} = \left\{ u \in P : \|u\| = \max\left\{ \sup_{t\in[0,\sigma(T)]_{\mathbb{T}}} |u(t)|, \sup_{t\in[0,T]_{\mathbb{T}}} |u^{\Delta}(t)| \right\} \le c \right\}.$$

By the continuity of f, there exists a constant $C > 0$ such that $f(t, u, u^{\Delta}) \le \varphi_p(C)$ for $(t, u, u^{\Delta}) \in [0, T]_{\mathbb{T}} \times [0, c] \times [0, c]$. So $t \in [0, T]_{\mathbb{T}}$,

$$\left|\varphi_q\left(\int_t^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau + \frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right)\right| < +\infty \tag{2.1}$$

and

$$\begin{aligned}\Bigg|\int_0^t \varphi_q\left(\int_s^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau + \frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right)\Delta s \\ +\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi_q\Bigg(\int_s^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau \\ + \frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\Bigg)\Delta s\Bigg| < +\infty.\end{aligned} \tag{2.2}$$

Consequently, F maps a bounded set into a bounded set.

Thirdly, if $t_1, t_2 \in [0, T]_{\mathbb{T}}$ and $t_1 < t_2$, then we have

$$\begin{aligned} & \Big|(Fu)(t_1) - (Fu)(t_2)\Big| \\ = & \left|\int_{t_1}^{t_2}\varphi_q\left(\int_s^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau + \frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right)\Delta s\right| \\ \le & C\left|\int_{t_1}^{t_2}\varphi_q\left(\int_s^T q(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right)\Delta s\right| \\ \le & C\big|t_1-t_2\big|\varphi_q\left(\int_0^T q(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2}\beta_i\int_0^T q(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right) \\ = & C\big|t_1-t_2\big|\varphi_q\left(\frac{\int_0^T q(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right) \to 0 \text{ as } t_1 \to t_2. \end{aligned}$$

By applying the Arzela-Ascoli theorem on time scales, we see that $F\overline{P_c}$ is relatively compact.

We next claim $F : \overline{P_c} \to P$ is continuous. Assume that $\{u_n\}_{n=1}^{\infty} \subset \overline{P_c}$ and $\lim\limits_{n\to\infty} \|u_n - u_0\| \to 0$. This means that $\lim\limits_{n\to\infty} |u_n - u_0| \to 0$ and $\lim\limits_{n\to\infty} |u_n^{\Delta} - u_0^{\Delta}| \to 0$. Since $\{(Fu_n)(t)\}_{n=1}^{\infty}$ is uniformly bounded on $[0,T]_{\mathbb{T}}$, there exists a uniformly convergent subsequence in $\{(Fu_n)(t)\}_{n=1}^{\infty}$. Let $\{(Fu_{n(m)})(t)\}_{m=1}^{\infty}$ be a subsequence which converges to $v(t)$ uniformly on $[0,T]_{\mathbb{T}}$. Examine that

$$
\begin{aligned}
(Fu_n)(t) \quad = \quad & \int_0^t \varphi_q\Bigg(\int_s^T q(\tau) f(\tau, u_n, u_n^{\Delta}) \nabla\tau \\
& + \frac{\sum_{i=1}^{m-2} \beta_i \int_{\xi_i}^T q(\tau) f(\tau, u_n, u_n^{\Delta}) \nabla\tau}{1 - \sum_{i=1}^{m-2} \beta_i} \Bigg) \Delta s \\
& + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \varphi_q\Bigg(\int_s^T q(\tau) f(\tau, u_n, u_n^{\Delta}) \nabla\tau \\
& + \frac{\sum_{i=1}^{m-2} \beta_i \int_{\xi_i}^T q(\tau) f(\tau, u_n, u_n^{\Delta}) \nabla\tau}{1 - \sum_{i=1}^{m-2} \beta_i} \Bigg) \Delta s.
\end{aligned}
$$

By using (2.1) and (2.2), inserting $u_{n(m)}$ into the above and then letting $m \to \infty$, we find

$$
\begin{aligned}
v(t) \quad = \quad & \int_0^t \varphi_q\Bigg(\int_s^T q(\tau) f(\tau, u_0, u_0^{\Delta}) \nabla\tau \\
& + \frac{\sum_{i=1}^{m-2} \beta_i \int_{\xi_i}^T q(\tau) f(\tau, u_0, u_0^{\Delta}) \nabla\tau}{1 - \sum_{i=1}^{m-2} \beta_i} \Bigg) \Delta s \\
& + \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \varphi_q\Bigg(\int_s^T q(\tau) f(\tau, u_0, u_0^{\Delta}) \nabla\tau \\
& + \frac{\sum_{i=1}^{m-2} \beta_i \int_{\xi_i}^T q(\tau) f(\tau, u_0, u_0^{\Delta}) \nabla\tau}{1 - \sum_{i=1}^{m-2} \beta_i} \Bigg) \Delta s,
\end{aligned}
$$

where we have used Lebesque's dominated convergence theorem on time scales. From the definition of F, we know that $v(t) = Fu_0(t)$ on $[0,T]_{\mathbb{T}}$. This shows that each subsequence of $\{(Fu_n)(t)\}_{n=1}^{\infty}$ uniformly converges to $(Fu_0)(t)$. So the sequence $\{(Fu_n)(t)\}_{n=1}^{\infty}$ uniformly converges to $(Fu_0)(t)$. This means that F is continuous at $u_0 \in \overline{P_c}$. Therefore F is continuous on $\overline{P_c}$, since u_0 is arbitrary. Thus F is completely continuous. The proof is complete. □

3. **Existence results**

We define the nonnegative continuous convex functionals γ and θ, nonnegative continuous concave functional α, and nonnegative continuous functional ψ, respectively on P by

$$\gamma(u)=\sup_{t\in[0,T]_{\mathbb{T}}}u^{\Delta}(t)=u^{\Delta}(0),\qquad \theta(u)=\sup_{t\in[r,T]_{\mathbb{T}}}u^{\Delta}(t)=u^{\Delta}(r),$$

$$\alpha(u)=\inf_{t\in[\eta,T]_{\mathbb{T}}}u(t)=u(\eta),\qquad \psi(u)=\inf_{t\in[\eta,T]_{\mathbb{T}}}u(t)=u(\eta).$$

Now for convenince we introduce the following notations:

$$\begin{aligned}
m &= \varphi_q\left(\frac{\int_0^T q(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right),\\
M &= \left(\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}T+\eta\right)\varphi_q\left(\frac{\int_\eta^T q(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right),\\
\lambda &= \left(\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}T+\eta\right)\varphi_q\left(\frac{\int_0^T q(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right).
\end{aligned}$$

Theorem 3.1. *Let $0<a<b<\dfrac{\eta M}{Tm}d$, $M\eta>m$, and suppose that f satisfies the following conditions:*

(A1) $f(t,u,u^{\Delta})\le\varphi_p\left(\dfrac{d}{m}\right)$ *for* $(t,u,u^{\Delta})\in[0,T]_{\mathbb{T}}\times[0,d]\times[-d,d]$;

(A2) $f(t,u,u^{\Delta})>\varphi_p\left(\dfrac{b}{M}\right)$ *for* $(t,u,u^{\Delta})\in[\eta,T]_{\mathbb{T}}\times[b,d]\times[-d,d]$;

(A3) $f(t,u,u^{\Delta})<\varphi_p\left(\dfrac{a}{\lambda}\right)$ *for* $(t,u,u^{\Delta})\in[0,T]_{\mathbb{T}}\times[0,a]\times[-d,d]$.

Then the BVP (1.1) *and* (1.2) *has at least three positive solutions u_1,u_2 and u_3, such that*

$$\|u_i\|\le d \text{ for } i=1,2,3,\ \ b<u_1(\eta),\ \ a<u_2(\eta) \text{ and } u_2(\eta)<b \text{ with } u_3(\eta)<a. \tag{3.1}$$

Proof. The BVP (1.1) and (1.2) has a solution $u=u(t)$ if and only if u solves the operator equation $u=Fu$. Thus we set out to verify that the operator F satisfies Avery and Peterson's fixed point theorem which will prove the existence of three fixed points of F which satisfy the conclusion of the theorem.

Firstly, we will show that F maps $\overline{P(\gamma,d)}$ into $\overline{P(\gamma,d)}$. For any $u\in\overline{P(\gamma,d)}$, we have $\gamma(u)=\sup_{t\in[0,T]_{\mathbb{T}}}u^{\Delta}(t)\le d$. Assumption (A1) implies

$f(t,u,u^{\Delta}) \le \varphi_p\Big(\dfrac{d}{m}\Big)$; then

$$\begin{aligned}
\gamma(Fu) &= \sup_{t\in[0,T]_{\mathbb{T}}} (Fu)^{\Delta}(t) = (Fu)^{\Delta}(0)\\
&= \varphi_q\left(\int_0^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau + \frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right)\\
&\le \frac{d}{m}\varphi_q\left(\int_0^T q(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right)\\
&\le \frac{d}{m}\varphi_q\left(\int_0^T q(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2}\beta_i\int_0^T q(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right)\\
&= \frac{d}{m}\varphi_q\left(\frac{\int_0^T q(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right)\\
&= d.
\end{aligned}$$

Secondly, we show that condition (S1) in Theorem 2.2 holds. Let $u = \dfrac{Mb}{m}t - \dfrac{Mb}{m}\eta + 2b$. It is easy to see that $\alpha(u) = 2b > b$. $\theta(u) = \dfrac{Mb}{m} \le \dfrac{Mb}{m}$ and $\gamma(u) = \dfrac{Mb}{m} < d$. Thus $\left\{u \in P(\gamma,\theta,\alpha,b,\dfrac{Mb}{m},d) : \alpha(u) > b\right\} \ne \emptyset$. For any $u \in P(\gamma,\theta,\alpha,b,\frac{Mb}{m},d)$, by condition (A2) of this theorem, one has $f(t,u,u^{\Delta}) > \varphi_p\Big(\dfrac{b}{M}\Big)$ for $t \in [\eta,T]_{\mathbb{T}}$, and

$$\begin{aligned}
\alpha(Fu) &= \inf_{t\in[\eta,T]_{\mathbb{T}}} (Fu)(t) = Fu(\eta)\\
&= \int_0^{\eta} \varphi_q\Bigg(\int_s^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau\\
&\quad + \frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\Bigg)\Delta s\\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi_q\Bigg(\int_s^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau\\
&\quad + \frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\Bigg)\Delta s
\end{aligned}$$

$$
\begin{aligned}
&> \left(\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}T+\eta\right)\varphi_q\left(\int_\eta^T q(\tau)\varphi_p\Big(\frac{b}{M}\Big)\nabla\tau\right.\\
&\qquad \left.+\frac{\sum_{i=1}^{m-2}\beta_i\int_\eta^T q(\tau)\varphi_p\Big(\frac{b}{M}\Big)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right)\\
&= \frac{b}{M}\left(\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}T+\eta\right)\varphi_q\left(\frac{\int_\eta^T q(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right)\\
&= b.
\end{aligned}
$$

Therefore we have $\alpha(u) > b$, for all $u \in P(\gamma,\theta,\alpha,b,\frac{Mb}{m},d)$. Consequently, condition (S1) in Theorem 2.2 is satisfied.

Thirdly, we prove that (S2) in Theorem 2.2 holds. For any $u \in P(\gamma,\alpha,b,d)$ with $\theta(Fu) > \frac{M}{m}b$, that is

$$
\begin{aligned}
\theta(Fu) &= (Fu)^\Delta(r)\\
&= \varphi_q\left(\int_r^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau+\frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right)\\
&> \frac{M}{m}b,
\end{aligned}
$$

one has

$$
\begin{aligned}
\alpha(Fu) &= \inf_{t\in[\eta,T]_{\mathbb{T}}}(Fu)(t)=Fu(\eta)\\
&= \int_0^\eta \varphi_q\left(\int_s^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau\right.\\
&\qquad \left.+\frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right)\Delta s\\
&\qquad +\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi_q\left(\int_s^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau\right.\\
&\qquad \left.+\frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right)\Delta s\\
&> \eta\varphi_q\left(\int_r^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau+\frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)f(\tau,u,u^\Delta)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right)\\
&> \eta\frac{M}{m}b\\
&> b.
\end{aligned}
$$

Hence, condition (S2) in Theorem 2.2 is satisfied.

Finally, we prove that (S3) in Theorem 2.2 is satisfied. Since $\psi(0) = 0 < a$, it follows that $0 \notin R(\gamma, \psi, a, d)$. Suppose that $u \in R(\gamma, \psi, a, d)$ with $\psi(u) = \inf_{t\in[\eta,T]_{\mathbb{T}}} u(t) = u(\eta) = a$. Then, by the condition (A3) of this theorem, we have

$$
\begin{aligned}
\alpha(Fu) &= \inf_{t\in[\eta,T]_{\mathbb{T}}} (Fu)(t) = Fu(\eta) \\
&= \int_0^{\eta} \varphi_q\Bigg(\int_s^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau \\
&\quad + \frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\Bigg)\Delta s \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\varphi_q\Bigg(\int_s^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau \\
&\quad + \frac{\sum_{i=1}^{m-2}\beta_i\int_{\xi_i}^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\Bigg)\Delta s \\
&\leq \int_0^{\eta} \varphi_q\Bigg(\int_0^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau \\
&\quad + \frac{\sum_{i=1}^{m-2}\beta_i\int_0^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\Bigg)\Delta s \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{T}\varphi_q\Bigg(\int_0^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau \\
&\quad + \frac{\sum_{i=1}^{m-2}\beta_i\int_0^T q(\tau)f(\tau,u,u^{\Delta})\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\Bigg)\Delta s \\
&< \int_0^{\eta} \varphi_q\Bigg(\int_0^T q(\tau)\varphi_p\Big(\frac{a}{\lambda}\Big)\nabla\tau \\
&\quad + \frac{\sum_{i=1}^{m-2}\beta_i\int_0^T q(\tau)\varphi_p\Big(\frac{a}{\lambda}\Big)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\Bigg)\Delta s \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{T}\varphi_q\Bigg(\int_0^T q(\tau)\varphi_p\Big(\frac{a}{\lambda}\Big)\nabla\tau \\
&\quad + \frac{\sum_{i=1}^{m-2}\beta_i\int_0^T q(\tau)\varphi_p\Big(\frac{a}{\lambda}\Big)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\Bigg)\Delta s \\
&= \left(\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}T+\eta\right)\varphi_q\left(\frac{\int_0^T q(\tau)\varphi_p\Big(\frac{a}{\lambda}\Big)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right)
\end{aligned}
$$

$$\begin{aligned} &= \frac{a}{\lambda}\left(\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}T+\eta\right)\varphi_q\left(\frac{\int_0^T q(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}\beta_i}\right) \\ &= a. \end{aligned}$$

Thus condition (S3) in Theorem 2.2 holds. As a result, all the conditions of Theorem 2.2 are satisfied. The proof is complete. □

Theorem 3.2. *Let* $i = 1, 2, \cdots, n$, $0 < a_1 < b_1 < \dfrac{\eta M}{Tm} d_1 < a_2 < b_2 < \dfrac{\eta M}{Tm} d_2 < a_3 < \ldots < a_n$, $n \in \mathbb{N}$, $M\eta > m$, *and suppose that* f *satisfies the following conditions:*

(B1) $f(t, u, u^\Delta) \le \varphi_p\left(\dfrac{d_i}{m}\right)$ *for* $(t, u, u^\Delta) \in [0, T]_{\mathbb{T}} \times [0, d_i] \times [-d_i, d_i]$;

(B2) $f(t, u, u^\Delta) > \varphi_p\left(\dfrac{b_i}{M}\right)$ *for* $(t, u, u^\Delta) \in [\eta, T]_{\mathbb{T}} \times [b_i, d_i] \times [-d_i, d_i]$;

(B3) $f(t, u, u^\Delta) < \varphi_p\left(\dfrac{a_i}{\lambda}\right)$ *for* $(t, u, u^\Delta) \in [0, T]_{\mathbb{T}} \times [0, a_i] \times [-d_i, d_i]$.

Then the BVP (1.1) *and* (1.2) *has at least* $2n + 1$ *positive solutions.*

Proof. When $i = 1$, it is clear that Theorem 3.2 holds. Then we can find at least three positive symmetric solutions u_1, u_2 and u_3 satisfying (3.1). Hence, we finish the proof by induction. □

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