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NEW CLASSES OF LYAPUNOV TYPE INEQUALITIES OF FRACTIONAL Δ-DIFFERENCE STURM-LIOUVILLE PROBLEMS WITH APPLICATIONS

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Abstract. In this paper, we consider a new study about fractional Δ-difference equations. We consider two special classes of Sturm-Liouville problems equipped with fractional Δ-difference operators. In couple of steps, the Lyapunov type inequalities for both classes will be obtained. As application, some qualitative behaviour of mentioned fractional problems such as stability, spectral, disconjugacy and some interesting results about zeros of (oscillatory) solutions will be concluded.
Keywords: Discrete fractional calculus, discrete fractional Sturm-Liouville problem, Lyapunov type inequalities, stability, Mittag-Leffler type functions.
MSC(2010): Primary: 34A08, Secondary: 39A12, 97H30.

1. Introduction

The main purpose of this paper is devoted to study discrete fractional differential equations and corresponding Lyapunov type inequalities. Indeed, we are interested to study those discrete fractional operators that have analogous structures to the fractional differential operators such as well known Riemann-Liouville operators instead of Grünwald-Letnikov approach. The key point in this way turns to the integral kernel $(t-s)^{\alpha-1}/\Gamma(\alpha)$ for $\alpha > 0$ in Riemann-Liouville integrals that is a continuous generalization of cauchy function $(t-s)^{n-1}/(n-1)!$, for nth order ordinary differential equations (see [18, 19]). So we are interested to apply discrete fractional operators that have kernels similar to fractional Riemann-Liouville operators. In the sequel we will show that via fractional falling functions we can reach these desired kernels. Also, the idea of using the fractional Δ-difference operators relies on this fact that, despite rich literature on fractional order differential equations during the

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last three centuries, there exist poor research on discrete fractional calculus, in particular fractional order difference equations. See demonstrative references [15, 22] for details.

On the other hand fortunately, in the last decade we can observe the invaluable developments of discrete approaches of fractional calculus. In this direction we refer an eager follower to the references [2–6, 11, 12]. As we stated above, in this paper we will study the Lyapunov type inequalities corresponding to the fractional Δ-difference equations. In fact continued investigations about Lyapunov type inequalities during past 130 years, have been made it the comprehensive theory not only for abstract investigations about the Lyapunov type inequalities but also for studying the qualitative behaviour of ordinary differential and difference equations.

An interesting and maybe unbelievable fact about this theory turns back to the simple inequality appeared in the study of stability of motion due to A. M. Lyapunov that can be stated as below:

If we consider the nontrivial solution $y(t)$ of the second order differential equation

$$\begin{cases} y^{''}(t) + p(t)y(t) = 0, \quad t \in (a, b), \\ y(a) = 0 = y(b), \end{cases} \tag{1.1}$$

then

$$\int_a^b |p(t)|dt > \frac{4}{b-a}. \tag{1.2}$$

Inequality (1.2) which is called Lyapunov inequality, provides an effective tool for studying behavior of solutions of ODE (1.1). Detailed discussions can be found in [17].

Despite the absence of research works about Lyapunov type inequalities related to the fractional difference equations, every interested researcher can find the great number of papers about Lyapunov inequalities and their applications in continuous differential equations such as [8, 9, 13, 14, 16, 20, 21, 23–25].

R. C. Ferreira in [9], proved that if $u(t)$ is a nontrivial solution of Caputo fractional boundary value problem

$$\begin{cases} \left({}^cD_{0^+}^{\alpha} u\right)(t) + q(t)u(t) = 0, \ a < t < b, \ 1 < \alpha \leq 2, \\ \qquad u(a) = 0, \ u(b) = 0, \end{cases} \tag{1.3}$$

then the Lyapunov type inequality

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha)\alpha^{\alpha}}{[(\alpha-1)(b-a)]^{\alpha-1}}, \tag{1.4}$$

holds, provided that $q(t)$ is a real continuous function. The author in [10], studied discrete fractional boundary value problem

$$\begin{cases} (\Delta^{\alpha} y)(t) + q(t+\alpha-1)y(t+\alpha-1) = 0, \quad t \in [0, b+1]_{\mathbb{N}_0}, \\ y(\alpha-2) = 0 = y(\alpha+b+1), \ or \ y(\alpha-2) = 0 = \Delta y(\alpha+b). \end{cases} \tag{1.5}$$

He proved that if the discrete fractional boundary value problem (1.5) has a nontrivial solution, then for the first boundary conditions, one can derive the following Lyapunov type inequalities

(1.6)
$$\sum_{s=0}^{b+1} |q(s+\alpha-1)| > 4\Gamma(\alpha)\frac{\Gamma(b+\alpha+2)\Gamma^2(\frac{b}{2}+2)}{(b+2\alpha)(b+2)\Gamma^2(\frac{b}{2}+\alpha)\Gamma(b+3)}, \quad b: \ even,$$
$$\sum_{s=0}^{b+1} |q(s+\alpha-1)| > \Gamma(\alpha)\frac{\Gamma(b+\alpha+2)\Gamma^2(\frac{b+3}{2})}{\Gamma^2(\frac{b+1}{2}+\alpha)\Gamma(b+3)}, \quad b: \ odd,$$

also for the second boundary conditions, corresponding Lyapunov type inequality is as below

$$\sum_{s=0}^{b+1} |q(s+\alpha-1)| > \frac{1}{(b+2)\Gamma(\alpha-1)}. \tag{1.7}$$

F. M. Atici and *P. W. Eloe* in [4] obtained existence results for positive solutions of the following two-point fractional difference equation

$$\begin{cases} -\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)), \quad t = 1, 2, ..., b+1, \\ \quad y(\nu-2) = 0, \ y(\nu+b+1) = 0, \end{cases} \tag{1.8}$$

where $1 < \nu \leq 2$ is a real number, $b \geq 2$ is an integer and Δ^{ν} denotes fractional Δ-difference operator of order $\nu > 0$. In addition $f : [\nu, \nu+b]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to \mathbb{R}$ is continuous.

Motivated by the fractional order problems mentioned above, in this paper we consider the following fractional Δ-difference generalized Sturm-Liouville problem

$$\begin{cases} \Delta^{\alpha}_{b_-}\left(p(t)\Delta^{\alpha}_{a^+}u(t)\right) + [q(t+\alpha-1)-\lambda]u(t+\alpha-1) = 0, \\ u(\alpha+a-1) = 0, \ u(\alpha+b) = 0, \end{cases} \tag{1.9}$$

where $t = a, a+1, ..., b$ and

(P_1)

$$\alpha \in (0.5, 1), \quad a, b \in \mathbb{Z}, \ a \geq 1, b \geq 3, \quad \lambda \in \mathbb{R}. \tag{1.10}$$

(P_2)

(1.11)
$$p : [a, b]_{\mathbb{N}_0} \to \mathbb{R}, \ p(t) = \infty, \ for \ t \in \ _{a-1}\mathbb{N}, \quad q : [\alpha+a-1, \alpha+b-1]_{\mathbb{N}_{\alpha-1}} \to \mathbb{R}.$$

(P_3) $\Delta^{\alpha}_{b_-}$ and $(\Delta^{\alpha}_{a^+})$ are right and left sided fractional Δ-difference operators, respectively.

Considering the discussion above, we organize the paper in the following manner:

- Obtaining Lyapunov type inequality for the fractional Δ-difference generalized Sturm-Liouville problem (1.9).

- Considering the canonical fractional Δ-difference Sturm-Liouville problem

$$\begin{cases} \Delta^{\alpha}_{a^+}u(t) + [q(t+\alpha-1) - \lambda]u(t+\alpha-1) = 0, \ 1 < \alpha \leq 2, \\ u(\alpha+a-2) = 0, \ u(\alpha+b+1) = 0, \end{cases} \tag{1.12}$$

  where $t = a, a+1, ..., b, b+1, \ a \in \mathbb{Z}_{\geq 1}, b \in \mathbb{Z}_{\geq 2}$, we find corresponding Lyapunov type inequality.
- In order to see the applicability of obtained Lyapunov type inequalities, we will establish some qualitative gestures for the both fractional Δ-difference problems (1.9) and (1.12).

2. **Technical requirements**

This section includes some essential definitions and lemmas that will construct foundations of the paper. So let us begin with fractional falling functions that are cornerstones of fractional Δ-difference equations.

Definition 2.1. Factorial falling polynomial $t^{\lfloor n \rfloor}$ is defined by

$$t^{\lfloor n \rfloor} = \prod_{j=0}^{n}(t-j) = \frac{\Gamma(t+1)}{\Gamma(t+1-n)}, \quad t \in \mathbb{R}\backslash\{..., n-3, n-2, n-1\}, \ n \in \mathbb{N}. \tag{2.1}$$

Considering the polynomial (2.1), arbitrary order generalization of factorial polynomial $t^{\lfloor n \rfloor}$ is as follows:

$$t^{\lfloor \alpha \rfloor} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}, \quad t \in \mathbb{R}\backslash\{..., \alpha-3, \alpha-2, \alpha-1\}, \ \alpha \in \mathbb{R}, \tag{2.2}$$

such that

(i) $t^{\lfloor \alpha \rfloor} = 0$, provided that $\{t+1-\alpha\} \in \mathbb{Z}_{\leq 0}, \ \alpha \in \mathbb{R}$,
(ii) $t^{\lfloor 0 \rfloor} = 1$.

Some properties of fractional falling functions will be represented in the next lemma, that detailed proofs can be found in [3, Theorem 2.1].

Lemma 2.2. *Assume that the following fractional falling functions are well defined. Then*

(C_1) $\Delta_t t^{\lfloor \alpha \rfloor} = \alpha t^{\lfloor \alpha-1 \rfloor}$,
(C_2) $(t-\alpha)t^{\lfloor \alpha \rfloor} = t^{\lfloor \alpha+1 \rfloor}$
(C_3) $\alpha^{\lfloor \alpha \rfloor} = \Gamma(\alpha+1)$,
(C_4) $t \leq r \Rightarrow t^{\lfloor \alpha \rfloor} \leq r^{\lfloor \alpha \rfloor}, \ \alpha > r$,
(C_5) $0 < \beta < 1 \Rightarrow t^{\lfloor \alpha\beta \rfloor} \geq \left(t^{\lfloor \alpha \rfloor}\right)^{\beta}$,
(C_6) $t^{\lfloor \alpha+\beta \rfloor} = (t-\beta)^{\lfloor \alpha \rfloor} t^{\lfloor \beta \rfloor}$,

where $\alpha, \beta \in \mathbb{R}$ and Δ_t denotes the forward difference operator with respect to the variable t.

Notation. Let $a, b \in \mathbb{R}$ and $c, d \in \mathbb{Z}$. For each $\nu \in \mathbb{R}$

$$\begin{aligned} &\mathbb{N}_a = \{a, a+1, a+2, ...\},\ {}_b\mathbb{N} = \{..., b-2, b-1, b\}, \\ &\mathbb{N}_{[c,d]} = \{c, c+1, ..., d-1, d\},\ [\nu, \nu+c]_{\mathbb{N}_{\nu-1}} = \{1, 2, ..., c+1\}. \end{aligned} \tag{2.3}$$

Definition 2.3 ([1])**.** The left and right sided $\alpha-$th fractional Δ-sums are defined as

$$\Delta^{-\alpha} f(t) = \begin{cases} \Delta_{a^+}^{-\alpha} f(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{s=a}^{t-\alpha} (t-\sigma(s))^{\lfloor \alpha-1 \rfloor} f(s), \\ \Delta_{b_-}^{-\alpha} f(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{s=t+\alpha}^{b} (s-\sigma(t)^{\lfloor \alpha-1 \rfloor} f(s), \end{cases} \tag{2.4}$$

where $\alpha > 0,\ \sigma(s) = s+1$.

Remark 2.4. The fractional left and right sided Δ-sums of order $\alpha > 0$, defined by (2.4) have the following properties:

(*i*) $\Delta_{a^+}^{-\alpha}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\alpha}$.

(*ii*) $\Delta_{b_-}^{-\alpha}$ maps functions defined on ${}_b\mathbb{N}$ to functions defined on ${}_{b-\alpha}\mathbb{N}$.

Definition 2.5 ([1])**.** The $\alpha-$th left and right sided fractional Δ-differences are given by

$$\Delta^{\alpha} f(t) = \begin{cases} \Delta_{a^+}^{\alpha} f(t) = \Delta_t^n \Delta_{a^+}^{-(n-\alpha)} f(t) \\ \qquad = \dfrac{1}{\Gamma(n-\alpha)} \Delta_t^n \left(\displaystyle\sum_{s=a}^{t-(n-\alpha)} (t-\sigma(s))^{\lfloor n-\alpha-1 \rfloor} f(s) \right), \\ \Delta_{b_-}^{\alpha} f(t) = (-1)^n \nabla_t^n \Delta_{b_-}^{-(n-\alpha)} f(t) \\ \qquad = \dfrac{(-1)^n}{\Gamma(n-\alpha)} \nabla_t^n \left(\displaystyle\sum_{s=t+(n-\alpha)}^{b} (s-\sigma(t)^{\lfloor n-\alpha-1 \rfloor} f(s) \right), \end{cases} \tag{2.5}$$

such that $\alpha > 0,\ n = [\alpha]+1$ and ∇_t denotes the backward difference operator with respect to the variable t.

Remark 2.6. The fractional left and right sided Δ-differences of order $\alpha > 0$, defined by (2.5) have the following properties:

(*i*) $\Delta_{a^+}^{\alpha}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+(n-\alpha)}$,

(*ii*) $\Delta_{b_-}^{\alpha}$ maps functions defined on ${}_b\mathbb{N}$ to functions defined on ${}_{b-(n-\alpha)}\mathbb{N}$,

where $n = [\alpha]+1$.

Naturally we can expect that the fractional Δ-sum and Δ-difference operators defined by (2.4),(2.5), satisfy in the basic properties of continuous fractional operators. In the following lemma we collect some of these properties for fractional Δ-difference operators. Corresponding proofs can be found in references [3, 6].

Lemma 2.7. *Assume that f is a real-valued function and $\mu > 0,\ 0 \le n-1 < \nu \le n$. Then*

(Q_1) $\Delta_{a^+}^{-\mu}\Delta_{a^+}^{-\nu} f(t) = \Delta_{a^+}^{-(\mu+\nu)} f(t) = \Delta_{a^+}^{-\nu}\Delta_{a^+}^{-\mu} f(t)$,

(Q_2) $\Delta_{a^+}^{-\nu}\ \Delta_{a^+}^{\nu} f(t) = f(t)+c_1(t-a)^{\lfloor \nu-1\rfloor}+c_2(t-a)^{\lfloor \nu-2\rfloor}+\ldots+c_n(t-a)^{\lfloor \nu-n\rfloor}$, $c_i \in \mathbb{R},\ i = 1, 2, \ldots, n$.

(Q_3) $\Delta_{a^+}^{\nu}\ \Delta_{a^+}^{-\nu} f(t) = f(t)$.

(Q_4) $\Delta_{a^+}^{-\nu}(t-a)^{\lfloor \mu\rfloor} = \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a)^{\lfloor \mu+\nu\rfloor},\ \mu+\nu+1 \notin \mathbb{Z}_{\le 0}$.

3. Main results

As we stated in the organization of the paper, in this section we are going to obtain corresponding Lyapunov type inequalities for the fractional Δ-difference Sturm-Liouville problems (1.9) and (1.12), respectively. So we begin this process as follows.

Theorem 3.1. *Assume that $p(t) > 0$ and $p(t), q(t)$ are real-valued functions defined by (1.11). If $u(t)$ defined on $[\alpha+a-1, \alpha+b+1]_{\mathbb{N}_{\alpha-1}}$ is a nontrivial solution of the fractional Sturm-Liouville problem*

$$\begin{cases} \Delta_{b_-}^{\alpha}\left(p(t)\Delta_{a^+}^{\alpha} u(t)\right) + [q(t+\alpha-1) - \lambda]u(t+\alpha-1) = 0, \\ u(\alpha+a-1) = 0,\ u(\alpha+b) = 0, \end{cases} \tag{3.1}$$

where $\alpha \in (0.5, 1)$ and $t = a, a+1, \ldots, b,\ a, b \in \mathbb{Z},\ \lambda \in \mathbb{R}$ such that $a \ge 1, b \ge 3$, then the following Lyapunov type inequality holds:

$$\sum_{s=a}^{b}\sum_{w=a}^{b}\left(\frac{|q(w+\alpha-1)-\lambda|}{p(s)}\right) \ge \frac{1}{2}. \tag{3.2}$$

Proof. In the first step of the proof, we transform the fractional Δ-difference equation

$$\Delta_{b_-}^{\alpha}\left(p(t)\Delta_{a^+}^{\alpha} u(t)\right) + [q(t+\alpha-1) - \lambda]u(t+\alpha-1) = 0, \tag{3.3}$$

to an equivalent fractional Δ-sum equation. Applying (Q_2) of Lemma 2.7 to (3.3), we obtain

$$\Delta_{a^+}^{\alpha} u(t) = c_1\frac{(b-t)^{\lfloor \alpha-1\rfloor}}{p(t)} - \frac{\Delta_{b_-}^{-\alpha}\left[[(q(t+\alpha-1)-\lambda]u(t+\alpha-1)\right]}{p(t)}. \tag{3.4}$$

Once again applying property (Q_2) of Lemma 2.7 to (3.4), we find

$$\begin{aligned} u(t) = c_2(t-a)^{\lfloor \alpha-1\rfloor} &+ c_1\Delta_{a^+}^{-\alpha}\frac{(b-t)^{\lfloor \alpha-1\rfloor}}{p(t)} \\ &- \Delta_{a^+}^{-\alpha}\frac{\Delta_{b_-}^{-\alpha}\left[[q(t+\alpha-1)-\lambda]u(t+\alpha-1)\right]}{p(t)}. \end{aligned} \tag{3.5}$$

Implementing the boundary conditions, will uniquely gives us the coefficients c_1, c_2. Thus the first boundary condition $u(\alpha+a-1)=0$ together with (1.11) and property (C_3) in Lemma 2.2, ensures that $c_2=0$. On the other hand imposing the second boundary condition, namely $u(\alpha+b)=0$, gives us the coefficient c_1 as follows:

$$c_1 = \frac{\displaystyle\sum_{s=a}^{b}(\alpha+b-s-1)^{\lfloor\alpha-1\rfloor}.\frac{\Delta_{b_-}^{-\alpha}\left[[q(s+\alpha-1)-\lambda]u(s+\alpha-1)\right]}{p(s)}}{\displaystyle\sum_{s=a}^{b}(\alpha+b-s-1)^{\lfloor\alpha-1\rfloor}\frac{(b-s)^{\lfloor\alpha-1\rfloor}}{p(s)}}. \tag{3.6}$$

Substituting c_1, c_2 into (3.5), we find

$$u(t) = \frac{G_1(\alpha+b,s)G_2(t,s)-G_1(t,s)G_2(\alpha+b,s)}{G_2(\alpha+b,s)}, \tag{3.7}$$

where

(3.8)

$$\begin{aligned} G_1(t,s) &= \Delta_{a^+}^{-\alpha}\frac{\Delta_{b_-}^{-\alpha}\left[[q(t+\alpha-1)-\lambda]u(t+\alpha-1)\right]}{p(t)} \\ &= \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-s-1)^{\lfloor\alpha-1\rfloor}.\frac{\Delta_{b_-}^{-\alpha}\left[[q(s+\alpha-1)-\lambda]u(s+\alpha-1)\right]}{p(s)}, \\ G_2(t,s) &= \Delta_{a^+}^{-\alpha}\frac{(b-t)^{\lfloor\alpha-1\rfloor}}{p(t)} = \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-s-1)^{\lfloor\alpha-1\rfloor}\frac{(b-s)^{\lfloor\alpha-1\rfloor}}{p(s)}. \end{aligned}$$

Using definition of the fractional falling functions (2.2), it is clear that for $\alpha\in(0.5,1)$

$$\Delta_t(t-\sigma(s))^{\lfloor\alpha-1\rfloor}\le 0, \quad a\le s\le t-\alpha,\ t\in[\alpha+a-1,\alpha+b-1]_{\mathbb{N}_{\alpha-1}}, \tag{3.9}$$

$$\Delta_s(s-\sigma(t))^{\lfloor\alpha-1\rfloor}\le 0, \quad t+\alpha\le s\le b,\ t\in[a-\alpha+1,b-\alpha+1]_{1-\alpha}\mathbb{N}. \tag{3.10}$$

Using property (C_3) of Lemma 2.2, implies that for $\alpha\in(0.5,1)$

$$\max(t-\sigma(s))^{\lfloor\alpha-1\rfloor} = ((s+\alpha)-s-1)^{\lfloor\alpha-1\rfloor} = (\alpha-1)^{\lfloor\alpha-1\rfloor} = \Gamma(\alpha), \tag{3.11}$$

for $a\le s\le t-\alpha,\ t\in[\alpha+a-1,\alpha+b-1]_{\mathbb{N}_{\alpha-1}}$ and

$$\max(s-\sigma(t))^{\lfloor\alpha-1\rfloor} = ((t+\alpha)-t-1)^{\lfloor\alpha-1\rfloor} = (\alpha-1)^{\lfloor\alpha-1\rfloor} = \Gamma(\alpha), \tag{3.12}$$

for $t+\alpha\le s\le b,\ t\in[a-\alpha+1,b-\alpha+1]_{1-\alpha}\mathbb{N}$, also

$$(t-\sigma(s))^{\lfloor\alpha-1\rfloor}\ge 0, \quad a\le s\le t-\alpha,\ t\in[\alpha+a-1,\alpha+b-1]_{\mathbb{N}_{\alpha-1}}. \tag{3.13}$$

By means of (3.13), the following inequality can be derived directly

$$\frac{\displaystyle\sum_{s=a}^{t-\alpha}(t-s-1)^{\lfloor\alpha-1\rfloor}\frac{(b-s)^{\lfloor\alpha-1\rfloor}}{p(s)}}{\displaystyle\sum_{s=a}^{b}(\alpha+b-s-1)^{\lfloor\alpha-1\rfloor}\frac{(b-s)^{\lfloor\alpha-1\rfloor}}{p(s)}}\leq 1. \tag{3.14}$$

Relevant Banach space that will be needed in the sequel is defined as below:

$$(E,\|.\|),\quad E=\{u|\ u:[a,b]_{\mathbb{N}_0}\to\mathbb{R}\},\ \|u\|=\max_{t\in[a,b]_{\mathbb{N}_0}}|u(t)|. \tag{3.15}$$

Considering (3.7),(3.14) and taking max −norm on both sides and then using relations (3.9)-(3.12),(3.13) we conclude that

(3.16)

$$\begin{aligned}\|u\|&\leq\frac{2\|u\|}{\Gamma^2(\alpha)}\sum_{s=a}^{b}\sum_{w=s+\alpha}^{b}(t-\sigma(s))^{\lfloor\alpha-1\rfloor}(w-\sigma(s))^{\lfloor\alpha-1\rfloor}.\frac{|q(w+\alpha-1)-\lambda|}{p(s)}\\&\leq\frac{2\Gamma^2(\alpha)\|u\|}{\Gamma^2(\alpha)}\sum_{s=a}^{b}\sum_{w=a}^{b}\frac{|q(w+\alpha-1)-\lambda|}{p(s)}.\end{aligned}$$

Therefore we deduce that

$$\sum_{s=a}^{b}\sum_{w=a}^{b}\left(\frac{|q(w+\alpha-1)-\lambda|}{p(s)}\right)\geq\frac{1}{2}.$$

This completes the proof. □

The remainder of the present section, will devote to obtain corresponding Lyapunov type inequality of the canonical fractional Δ-difference Sturm-Liouville problem (1.12) as below.

Theorem 3.2. *Suppose that $q(t)$ is a real-valued function defined by (1.11), for $1<\alpha\leq 2$. Assume that $u(t)$ defined on $[\alpha+a-2,\alpha+b+1]_{\mathbb{N}_{\alpha-2}}$ is a nontrivial solution of the fractional Δ-difference boundary value problem*

$$\begin{cases}\Delta^{\alpha}_{a^+}u(t)+[q(t+\alpha-1)-\lambda]u(t+\alpha-1)=0,\ 1<\alpha\leq 2,\\ u(\alpha+a-2)=0,\ u(\alpha+b+1)=0,\end{cases} \tag{3.17}$$

where $t=a,a+1,...,b,b+1,\ a,b\in\mathbb{Z},\ a\geq 1,b\geq 2$. Then the Lyapunov type inequalities

$$\sum_{a}^{b+1}|q(s+\alpha-1)-\lambda|\geq\Gamma(\alpha)\frac{b-a+2}{b-a+2\alpha}\frac{\Gamma(\alpha+b-a+2)\Gamma^2\left(\dfrac{b-a}{2}+1\right)}{\Gamma(b-a+2)\Gamma^2\left(\dfrac{b-a}{2}+\alpha\right)}, \tag{3.18}$$

if $a+b$ is even and

(3.19)
$$\sum_{a}^{b+1}|q(s+\alpha-1)-\lambda| \geq \Gamma(\alpha)\frac{b-a+3}{b-a+2\alpha+1}\frac{\Gamma(\alpha+b-a+2)\Gamma^2\left(\frac{b-a+1}{2}+1\right)}{\Gamma(b-a+2)\Gamma^2\left(\frac{b-a+1}{2}+\alpha\right)},$$

if $a+b$ is odd are satisfied.

Proof. We will divide proof process into three steps as follows:

(S_1) First we show that the fractional Δ-difference equation (3.17) can be reduced to the following fractional Δ-sum equation:

$$u(t)=\sum_{s=a}^{b+1}\mathcal{G}(t,s)[q(s+\alpha-1)-\lambda]u(s+\alpha-1), \tag{3.20}$$

where

(3.21)
$$\mathcal{G}(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}\dfrac{(t-a)^{\lfloor\alpha-1\rfloor}(\alpha+b-s)^{\lfloor\alpha-1\rfloor}}{(\alpha+b-a+1)^{\lfloor\alpha-1\rfloor}}-(t-\sigma(s))^{\lfloor\alpha-1\rfloor}, \\ a\leq s+\alpha-1\leq t\leq b+1, \\ \\ \dfrac{(t-a)^{\lfloor\alpha-1\rfloor}(\alpha+b-s)^{\lfloor\alpha-1\rfloor}}{(\alpha+b-a+1)^{\lfloor\alpha-1\rfloor}},\ a\leq t\leq s+\alpha-1\leq b+1.\end{cases}$$

We begin by using property (Q_2) in Lemma 2.7. So the fractional Δ-difference equation (3.17) transforms to the fractional Δ-sum equation

$$u(t)=-\Delta_{a^+}^{\alpha}[q(t+\alpha-1)-\lambda]u(t+\alpha-1)+c_1(t-a)^{\lfloor\alpha-1\rfloor}+c_2(t-a)^{\lfloor\alpha-2\rfloor}. \tag{3.22}$$

Implementing boundary condition $u(\alpha+a-2)=0$, using (1.11) and property (C_3) of Lemma 2.2, ensures that $c_2=0$. Also applying the boundary condition $u(\alpha+b+1)=0$ gives us coefficient c_1 as follows:

$$c_1=\frac{\Delta_{a^+}^{-\alpha}\left[[q(t+\alpha-1)-\lambda]u(t+\alpha-1)\right]\Big|_{t=\alpha+b+1}}{(\alpha+b-a+1)^{\lfloor\alpha-1\rfloor}}. \tag{3.23}$$

For simplicity assume that $h(t+\alpha-1)=[q(t+\alpha-1)-\lambda]u(t+\alpha-1)$. Substituting c_1, c_2 into (3.22), we conclude that

$$\begin{aligned}
u(t) &= \frac{-1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-\sigma(s))^{\lfloor\alpha-1\rfloor}h(s+\alpha-1)\\
&\quad+\frac{(t-a)^{\lfloor\alpha-1\rfloor}}{\Gamma(\alpha)(\alpha+b-a+1)^{\lfloor\alpha-1\rfloor}}\sum_{s=a}^{b+1}(\alpha+b-s)^{\lfloor\alpha-1\rfloor}h(s+\alpha-1)\\
&=\frac{-1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}\left[(t-\sigma)^{\lfloor\alpha-1\rfloor}-\frac{(t-a)^{\lfloor\alpha-1\rfloor}(\alpha+b-s)^{\lfloor\alpha-1\rfloor}}{(\alpha+b-a+1)^{\lfloor\alpha-1\rfloor}}\right]h(s+\alpha-1)\\
&\quad+\frac{(t-a)^{\lfloor\alpha-1\rfloor}}{\Gamma(\alpha)(\alpha+b-a+1)^{\lfloor\alpha-1\rfloor}}\sum_{s=t-\alpha+1}^{b+1}(\alpha+b-s)^{\lfloor\alpha-1\rfloor}h(s+\alpha-1)\\
&=\sum_{s=a}^{b+1}\mathcal{G}(t,s)[q(t+\alpha-1)-\lambda]u(t+\alpha-1).
\end{aligned}$$

(S_2) In this step we show that

$$\max_{t\in[\alpha+a-2,\alpha+b+1]_{\mathbb{N}_{\alpha-2}}}\mathcal{G}(t,s)=\mathcal{G}(s+\alpha-1,s),\quad s\in[a,b+1]_{\mathbb{N}_0}. \tag{3.24}$$

Using property (C_1) of Lemma 2.2 and simple calculation, show that

$$\begin{aligned}
&\Delta_t\mathcal{G}(t,s)<0, \qquad a\le s+\alpha-1\le t\le b+1,\\
&\Delta_t\mathcal{G}(t,s)>0, \qquad a\le t\le s+\alpha-1\le b+1.
\end{aligned}\tag{3.25}$$

Thus we conclude that not only $\mathcal{G}(t,s)>0$, for $t\in[\alpha+a-2,\alpha+b+1]_{\mathbb{N}_{\alpha-2}}$, $s\in[a,b+1]_{\mathbb{N}_0}$ but also

$$\max_{t\in[\alpha+a-2,\alpha+b]_{\mathbb{N}_{\alpha-1}}}\mathcal{G}(t,s)=\mathcal{G}(s+\alpha-1,s),\quad s\in[a,b+1]_{\mathbb{N}_0}.$$

Equivalently we deduce that

$$\max_{a\le s,t\le b+1}\mathcal{G}(t,s)=\max_{s\in[a,b+1]_{\mathbb{N}_0}}\frac{\mathcal{G}_2(s+\alpha-1,s)}{\Gamma(\alpha)}. \tag{3.26}$$

On the other hand taking into account that

$$\begin{aligned}
&\Delta\mathcal{G}_2(s+\alpha-1,s)\\
&\quad=(1-\alpha)\frac{\Gamma(s+\alpha-a)\Gamma(\alpha+b-s)}{(\alpha+b-a+1)^{\lfloor\alpha-1\rfloor}(s-a+2)!(b-s+2)!}[2s-(a+b)],
\end{aligned}$$

we find that $\mathcal{G}(s+\alpha-1,s)$ is increasing for $s<\frac{a+b}{2}$ and is decreasing for $s>\frac{a+b}{2}$. Therefore it follows that

$$\max_{s\in[a,b+1]_{\mathbb{N}_0}}\mathcal{G}(s+\alpha-1,s)=\frac{\mathcal{G}_2\left(\frac{a+b}{2}+\alpha-1,\frac{a+b}{2}\right)}{\Gamma(\alpha)} \tag{3.27}$$
$$=\frac{1}{\Gamma(\alpha)}\frac{b-a+2\alpha}{b-a+2}\frac{\Gamma(b-a+2)\Gamma^2\left(\frac{b-a}{2}+\alpha\right)}{\Gamma(\alpha+b-a+2)\Gamma^2\left(\frac{b-a}{2}+1\right)},$$

if $a+b$ is even and

$$\max_{s\in[a,b+1]_{\mathbb{N}_0}}\mathcal{G}(s+\alpha-1,s)=\frac{\mathcal{G}_2\left(\frac{a+b+1}{2}+\alpha-1,\frac{a+b+1}{2}\right)}{\Gamma(\alpha)} \tag{3.28}$$
$$=\frac{1}{\Gamma(\alpha)}\frac{b-a+2\alpha+1}{b-a+3}\frac{\Gamma(b-a+2)\Gamma^2\left(\frac{b-a+1}{2}+\alpha\right)}{\Gamma(\alpha+b-a+2)\Gamma^2\left(\frac{b-a+1}{2}+1\right)},$$

if $a+b$ is odd.

(S_3) Using (3.20) and (3.26), we conclude that

$$|u(t)|\leq\sum_{s=a}^{b+1}\mathcal{G}(s+\alpha-1,s)|q(s+\alpha-1)-\lambda||u(s+\alpha-1)|. \tag{3.29}$$

Taking max $-$norm on both sides of the inequality (3.29) and considering coupled maximization (3.27) and (3.28), give us

$$\sum_{a}^{b+1}|q(s+\alpha-1)-\lambda|\geq\Gamma(\alpha)\frac{b-a+2}{b-a+2\alpha}\frac{\Gamma(\alpha+b-a+2)\Gamma^2\left(\frac{b-a}{2}+1\right)}{\Gamma(b-a+2)\Gamma^2\left(\frac{b-a}{2}+\alpha\right)}, \tag{3.30}$$

if $a+b$ is even,

$$\sum_{a}^{b+1}|q(s+\alpha-1)-\lambda|\geq\Gamma(\alpha)\frac{b-a+3}{b-a+2\alpha+1}\frac{\Gamma(\alpha+b-a+2)\Gamma^2\left(\frac{b-a+1}{2}+1\right)}{\Gamma(b-a+2)\Gamma^2\left(\frac{b-a+1}{2}+\alpha\right)}, \tag{3.31}$$

if $a+b$ is odd.

Now the proof is completed. □

4. **Applications**

In the present section, we are seeking desirable applications for the fractional discrete Lyapunov type inequalities (3.2) and (3.18),(3.19). So, as first application we are going to study the stability of fractional Δ-difference Sturm-Liouville problem (3.1) as follows.

Definition 4.1. The fractional Δ-difference problem (3.1) is said to be

(i) Stable, if all solutions are bounded in $\mathbb{Z}_{\alpha-1}$.
(ii) Unstable, if all solutions are unbounded on $\mathbb{Z}_{\alpha-1}$.
(iii) Conditionally stable, if there exists at least one nontrivial solution that is bounded on $\mathbb{Z}_{\alpha-1}$.

Let us consider the following fractional Δ-difference Hamiltonian system

(4.1)
$$\begin{cases} \Delta^{\alpha}_{a^+} u(t) = e_1(t)u(t+\alpha-1) + e_2(t+\alpha-1)v(t), \\ \\ \Delta^{\alpha}_{b_-} v(t) = -e_3(t)u(t+\alpha-1) - e_4(t+\alpha-1)v(t), \end{cases} \qquad t \in [a,b]_{\mathbb{N}_0},$$

where

(4.2)
$$\begin{cases} \alpha \in (0.5, 1),\ a, b \in \mathbb{Z},\ a \geq 1,\ b \geq 3, \\ e_1, e_2, e_3, e_4 \textit{ are } \text{real-valued and T-periodic functions that is} \\ e_i(t+T) = e_i(t), \quad e_j(t+\alpha-1+T) = e_j(t+\alpha-1),\ i = 1, 3,\ j = 2, 4, \\ T \in \mathbb{N},\ t \in [a,b]_{\mathbb{N}_0}. \end{cases}$$

So we can rewrite Hamiltonian system (4.1) as below:

$$\Delta^{\alpha}\phi(t) = JH(t)\phi^{\alpha}(t),\ t \in [a,b]_{\mathbb{N}_0}, \tag{4.3}$$

where

$$\Delta^{\alpha} = \begin{bmatrix} \Delta^{\alpha}_{a^+} \\ \Delta^{\alpha}_{b_-} \end{bmatrix},\ J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ H(t) = \begin{bmatrix} e_3(t) & e_4(t+\alpha-1) \\ e_1(t) & e_2(t+\alpha-1) \end{bmatrix},$$
$$\phi(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix},\ \phi^{\alpha}(t) = \begin{bmatrix} u(t+\alpha-1) \\ v(t) \end{bmatrix}.$$

In the sequel we will review the so called Floquet theory. Indeed, in order to describe relationship between Lyapunov type inequalities and Floquet theory, one can consider the following theorem due to A. M. Lyapunov.

Theorem 4.2 ([17, Chapter III, Theorem II])**.** *Consider the second order differential equation with ω-periodic coefficient*

$$y'' + q(t)y = 0, \quad -\infty < t < \infty. \tag{4.4}$$

If the function q takes only positive or zero values (without being identically zero), and if further it satisfies the condition

$$\omega \int_0^{\omega} q(t) \leq 4, \tag{4.5}$$

then the roots of the characteristic equation corresponding to (4.4) will always be complex and their modulus are equal to 1.

Via Floquet theory equivalence of the result of the Theorem 4.2 and stability of the ODE (4.4), can be verified. Therefore in order to study the stability and instability of differential equations by Lyapunov type inequalities, we apply the Floquet theory.
Indeed we are going to find a nonzero complex number ρ and a solution of Hamiltonian system (4.3) such that

$$\phi(t+T) = \rho\phi(t), \quad t \in [a+\alpha-1, b+\alpha-1]_{\mathbb{N}_{\alpha-1}}. \tag{4.6}$$

Assume that

$$\Phi(t) = [\phi_1(t), \phi_2(t)] = \begin{bmatrix} u_1(t) & u_2(t) \\ v_1(t) & v_2(t) \end{bmatrix}, \quad \Phi(0) = I_2,\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.7}$$

is a fundamental matrix solution of (4.3). In this case we have

$$\begin{cases} \Delta^\alpha \Phi(t) = JH(t)\Phi^\alpha(t),\ t \in [a,b]_{\mathbb{N}_0}, \\ \Phi(0) = I_2, \end{cases} \tag{4.8}$$

where

$$\Phi^\alpha(t) = [\phi_1^\alpha(t), \phi_2^\alpha(t)] = \begin{bmatrix} u_1(t+\alpha-1) & u_2(t+\alpha-1) \\ v_1(t) & v_2(t) \end{bmatrix}.$$

So the general solution $\phi(t)$ of (4.3) is

$$\phi(t) = c_1\phi_1(t) + c_2\phi_2(t) = \Phi(t)c, \quad c_1, c_2 \in \mathbb{C},\ c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{4.9}$$

Now substituting (4.9) into (4.6) we can see that

$$\Phi(t+T) = \rho\Phi(t)c, \quad t \in [a+\alpha-1, b+\alpha-1]_{\mathbb{N}_{\alpha-1}}. \tag{4.10}$$

So we have $\Phi(T) = \rho c$. Hence ϕ defined by (4.9) is a nontrivial solution of the Hamiltonian system (4.3), if and only if ρ is an eigenvalue and c is the corresponding eigenvector of the so called *monodromy* matrix $\Phi(T)$ of the Hamiltonian system (4.1). As we know the mentioned eigenvalues ρ are roots of the characteristic equation

$$\det[\Phi(T) - \rho\ I] = 0. \tag{4.11}$$

Equivalently one can rewrite the characteristic equation as follows

$$\rho^2 - D\rho + \det\Phi(T) = 0, \quad D = u_1(T) + v_2(T). \tag{4.12}$$

We notice that the eigenvalues ρ also are called the *multipliers* of the fractional Δ-difference Hamiltonian system (4.1). As we know, under considered system by means of the Floquet theory can not be stable unless $\det\Phi(T) = \rho_1\rho_2 = 1$. Thus we can concentrate on the following characteristic equation

$$\rho^2 - D\rho + 1 = 0.$$

It is clear that the roots ρ of quadratic polynomial (4.12) are defined by

$$\rho_{1,2} = \frac{D \pm \sqrt{D^2-4}}{2}. \tag{4.13}$$

Lemma 4.3 ([13]). *Fractional Δ-difference Hamiltonian system (4.1) is unstable provided that $|D| > 2$, and stable if $|D| < 2$. If $|D| = 2$, then the system (4.1) will be stable in the case $v_1(T) = u_2(T) = 0$, but conditionally stable and not stable otherwise.*

In this position, we can represent a stability criterion for the Hamiltonian system (4.1) as below.
Let us verify connection between the fractional Δ-difference Hamiltonian system (4.1) and fractional Δ-difference Sturm-Liouville problem (3.1) as follows. Taking

$$e_1 = e_4 \equiv 0, \quad e_2 = \frac{1}{p}, \quad e_3 = q - \lambda, \tag{4.14}$$

obviously the system (4.1) reduces to the fractional Δ-difference Sturm-Liouville problem (3.1). Using the Floquet theory that briefly discussed above, one can investigate the stability of the fractional Δ-difference Sturm-Liouville problem (3.1).

Theorem 4.4. *Assume that $p > 0$ and q are real-valued functions defined by (1.11). Also suppose that p and q are T-periodic functions that is*

$$p(t+T) = p(t), \quad q(T+t+\alpha-1) = q(t+\alpha-1), \; t \in [a,b]_{\mathbb{N}_0}. \tag{4.15}$$

Let

(*i*)

$$\begin{aligned} &\left| 1 - \Delta_{a^+}^{-\alpha} \left[\frac{\Delta_{b_-}^{-\alpha} \left[[q(t+\alpha-1)-\lambda]u(t+\alpha-1) \right]}{u(a+\alpha-1)p(t)} \right]_{t=T+\alpha+a-1} \right| \\ &= \left| \frac{(T+\alpha-1)^{\lfloor \alpha-1 \rfloor} - c\Delta_{a^+}^{-\alpha} \left[\frac{(b-t)^{\lfloor \alpha-1 \rfloor}}{p(t)} \right]_{t=T+\alpha+a-1}}{\Gamma(\alpha)} \right|, \quad c \in \mathbb{R}. \end{aligned} \tag{4.16}$$

(*ii*)

$$\sum_{s=a}^{b} \sum_{w=a}^{b} \left(\frac{|q(w+\alpha-1)-\lambda|}{p(s)} \right) < \frac{1}{2}. \tag{4.17}$$

Then the fractional Δ-difference Sturm-Liouville problem (3.1) is stable.

Proof. Suppose on the contrary that, the fractional Δ-difference Sturm-Liouville problem (3.1) is unstable. So according to the Floquet theory, there exists a multiplier ρ with $|\rho| \neq 1$ and nontrivial solution u of (3.1) such that

$$u(t+T) = \rho u(t), \quad t \in \mathbb{N}_{\alpha-1}. \tag{4.18}$$

Thus in order to apply Theorem 3.1, we must prove that $u(t)$ has at least one zero in $[\alpha-1, T+\alpha-1]_{\mathbb{N}_{\alpha-1}}$. If this is not true, then by using (4.18) we conclude that $u(t) \not\equiv 0$ for all $t \in [\alpha-1, T+\alpha-1]_{\mathbb{N}_{\alpha-1}}$.
Turning back to the main problem, we know that the fractional Δ-difference equation
(4.19)
$$\Delta_{b_-}^{\alpha}\left(p(t)\Delta_{a^+}^{\alpha}u(t)\right) + [q(t+\alpha-1)-\lambda]u(t+\alpha-1) = 0, \ t = a, a+1, ..., b-1, b,$$

is equal to the fractional Δ-sum equation

$$u(t) = c_2 (t-a)^{\lfloor \alpha-1 \rfloor} + c_1 \Delta_{a^+}^{-\alpha} \frac{(b-t)^{\lfloor \alpha-1 \rfloor}}{p(t)} - \Delta_{a^+}^{-\alpha} \frac{\Delta_{b_-}^{-\alpha}\left[[q(t+\alpha-1)-\lambda]u(t+\alpha-1)\right]}{p(t)}.$$

Indeed considering property (Q_2) of Lemma 2.2, we have

$$\Delta_{a^+}^{-\alpha}\Delta_{a^+}^{\alpha}u(t) = u(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\lfloor \alpha-n+k \rfloor}}{\Gamma(\alpha-n+k+1)} \Delta_{a^+}^{k}\left[\Delta_{a^+}^{\alpha-n}u(a)\right],$$

where $0 \leq n-1 < \alpha \leq n, \ n \in \mathbb{N}$. So we have the following fractional Δ-sum form

$$u(t) = \left[\frac{\Delta_{a^+}^{\alpha-1}u(a)}{\Gamma(\alpha)}\right](t-a)^{\lfloor \alpha-1 \rfloor} - \left[\frac{\Delta_{b_-}^{\alpha-1}u(b)}{\Gamma(\alpha)}\right]\Delta_{a^+}^{-\alpha}\left[\frac{(b-t)^{\lfloor \alpha-1 \rfloor}}{p(t)}\right] - \Delta_{a^+}^{-\alpha}\left[\frac{\Delta_{b_-}^{-\alpha}\left[[q(t+\alpha-1)-\lambda]u(t+\alpha-1)\right]}{p(t)}\right].$$

Note that $u(t)$ is defined on $\mathbb{N}_{\alpha-1}$. Hence by using property (C_3) of Lemma 2.2, equivalently we can derive the following

$$\begin{aligned} u(t) = \frac{u(a+\alpha-1)}{\Gamma(\alpha)}(t-a)^{\lfloor \alpha-1 \rfloor} - \frac{u(b+1-\alpha)}{\Gamma(\alpha)}\Delta_{a^+}^{-\alpha}\left[\frac{(b-t)^{\lfloor \alpha-1 \rfloor}}{p(t)}\right] \\ - \Delta_{a^+}^{-\alpha}\left[\frac{\Delta_{b_-}^{-\alpha}\left[[q(t+\alpha-1)-\lambda]u(t+\alpha-1)\right]}{p(t)}\right]. \end{aligned} \tag{4.20}$$

Taking $t = T+\alpha+a-1$ in both sides of the equality (4.20), applying (4.18) and using the fact that there exists $c \in \mathbb{R}$ such that $u(b+1-\alpha) = cu(a+\alpha-1)$, we conclude that

$$\begin{aligned}\rho u(\alpha+a-1)=&\frac{u(a+\alpha-1)}{\Gamma(\alpha)}(T+\alpha-1)^{\lfloor\alpha-1\rfloor}\\&-\frac{cu(a+\alpha-1)}{\Gamma(\alpha)}\Delta_{a^+}^{-\alpha}\left[\frac{(b-t)^{\lfloor\alpha-1\rfloor}}{p(t)}\right]_{t=T+\alpha+a-1}\\&-\Delta_{a^+}^{-\alpha}\left[\frac{\Delta_{b_-}^{-\alpha}\left[[q(t+\alpha-1)-\lambda]u(t+\alpha-1)\right]}{p(t)}\right]_{t=T+\alpha+a-1}.\end{aligned}$$

Equivalently, we find that

$$\begin{aligned}&\rho u(\alpha+a-1)\left[1-\Delta_{a^+}^{-\alpha}\left[\frac{\Delta_{b_-}^{-\alpha}\left[[q(t+\alpha-1)-\lambda]u(t+\alpha-1)\right]}{u(a+\alpha-1)p(t)}\right]_{t=T+\alpha+a-1}\right]\\&=u(\alpha+a-1)\left[\frac{(T+\alpha-1)^{\lfloor\alpha-1\rfloor}-c\Delta_{a^+}^{-\alpha}\left[\frac{(b-t)^{\lfloor\alpha-1\rfloor}}{p(t)}\right]_{t=T+\alpha+a-1}}{\Gamma(\alpha)}\right].\end{aligned}$$

Now using the assumption (i) of (4.16), we have

$$|\rho|=1,$$

which contradicts with the assumption $|\rho|\neq 1$. Thus there exists at least one solution as $a+\alpha-1$ in $[\alpha-1,T+\alpha-1]_{\mathbb{N}_{\alpha-1}}$ for (4.19) and using (4.18) there is another as $T+a+\alpha-1$. Now applying Theorem 3.1, we conclude that

$$\sum_{s=0}^{T}\sum_{w=0}^{T}\left(\frac{|q(w+\alpha-1)-\lambda|}{p(s)}\right)\geq\frac{1}{2}.$$

At the end, as a result of the following property for any N-periodic function $f(t)$ on $\mathbb{Z}$, that is

$$\sum_{t_0}^{t_0+N}f(t)=\sum_{0}^{N}f(t),$$

and taking $T=b-a$, we have

$$\sum_{s=a}^{b}\sum_{w=a}^{b}\left(\frac{|q(w+\alpha-1)-\lambda|}{p(s)}\right)\geq\frac{1}{2},$$

which contradicts with (4.17). Therefore the fractional Δ-difference Sturm-Liouville problem (3.1) is stable. □

In the second step, we are interested to study of some spectral properties related to the fractional Δ-difference Sturm-Liouville problem (3.17). To this aim, we define so called Δ-Mittag-Leffler type functions as below.

Definition 4.5. Δ-Mittag-Leffler type function of order i is given by

$$\overline{\mathcal{E}}_\alpha(t;\lambda,a,i)=\sum_{k=0}^{\infty}\lambda^k\frac{(t-a+k(\alpha-1))^{\lfloor(k+1)\alpha-1-i\rfloor}}{\Gamma((k+1)\alpha-i)}, \tag{4.21}$$

where $t\in\mathbb{N}_a,\ a,\lambda\in\mathbb{R},\ \alpha>0,\ i=0,1,...,[\alpha]$.

Let us consider the fractional Δ-difference initial value problem

$$\begin{cases}\Delta^\alpha_{a^+}u(t)=\lambda u(t+\alpha-1), \quad 1<\alpha\le 2,\ t=a,a+1,a+2,...,\\ \Delta^{\alpha-2}_{a^+}u(t)\big|_{t=a}=a_0, \quad \Delta^{\alpha-1}_{a^+}u(t)\big|_{t=a}=a_1.\end{cases} \tag{4.22}$$

As we know in the case $1<\alpha\le 2$, $\Delta^\alpha_{a^+}u(t)=\lambda u(t+\alpha-1)$ is equivalent to

$$u(t)=a_0\frac{(t-a)^{\lfloor\alpha-2\rfloor}}{\Gamma(\alpha-1)}+a_1\frac{(t-a)^{\lfloor\alpha-1\rfloor}}{\Gamma(\alpha)}+\lambda\Delta^{-\alpha}u(t+\alpha-1).$$

In order to find an explicit solutions of (4.22), we use the successive approximations method. So we set

$$u_0(t)=a_0\frac{(t-a)^{\lfloor\alpha-2\rfloor}}{\Gamma(\alpha-1)}+a_1\frac{(t-a)^{\lfloor\alpha-1\rfloor}}{\Gamma(\alpha)},$$
$$u_m(t)=u_0(t)+\lambda\Delta^{-\alpha}_{a^+}u_{m-1}(t+\alpha-1),\ m=1,2,\dots .$$

Now applying the power rule (Q_4) in Lemma 2.7, shows that

$$\begin{aligned}u_1(t)=a_0&\left[\frac{(t-a)^{\lfloor\alpha-2\rfloor}}{\Gamma(\alpha-1)}+\lambda\frac{(t-a+\alpha-1)^{\lfloor2\alpha-2\rfloor}}{\Gamma(2\alpha-1)}\right]\\ &+a_1\left[\frac{(t-a)^{\lfloor\alpha-1\rfloor}}{\Gamma(\alpha)}+\lambda\frac{(t-a+\alpha-1)^{\lfloor2\alpha-1\rfloor}}{\Gamma(2\alpha)}\right].\end{aligned}$$

Continuing this process by induction, implies that

$$\begin{aligned}u_m(t)=a_0&\sum_{k=0}^{m}\lambda^k\frac{(t-a+k(\alpha-1))^{\lfloor(k+1)\alpha-2\rfloor}}{\Gamma((k+1)\alpha-1)}\\ &+a_1\sum_{k=0}^{m}\lambda^k\frac{(t-a+k(\alpha-1))^{\lfloor(k+1)\alpha-1\rfloor}}{\Gamma((k+1)\alpha)},\ m=0,1,2,\dots .\end{aligned} \tag{4.23}$$

Finally taking the limit as $m\to\infty$, we obtain

$$\begin{aligned}u(t)&=a_0\sum_{k=0}^{\infty}\lambda^k\frac{(t-a+k(\alpha-1))^{\lfloor(k+1)\alpha-2\rfloor}}{\Gamma((k+1)\alpha-1)}\\ &\quad+a_1\sum_{k=0}^{\infty}\lambda^k\frac{(t-a+k(\alpha-1))^{\lfloor(k+1)\alpha-1\rfloor}}{\Gamma((k+1)\alpha)}\\ &=a_0\overline{\mathcal{E}}_\alpha(t;\lambda,a,1)+a_1\overline{\mathcal{E}}_\alpha(t;\lambda,a,0).\end{aligned} \tag{4.24}$$

Let us point out that because of property (C_3) in Lemma 2.2, we conclude that $a_0=u(\alpha+a-2)$. So if we concentrate on the first initial condition $a_0=$

$u(\alpha+a-2)$ together with (4.24) and connecting mentioned initial condition to the boundary condition $u(\alpha+a-2)=0$ in the fractional Δ-difference boundary value problem (3.17)(taking $q\equiv 0$), without loss of generality one can conclude that $\overline{\mathcal{E}}_\alpha(\alpha+a-2;\lambda,a,0)=0$. Therefore by means of the fractional discrete Lyapunov type inequality (3.17), we can deduce that $\lambda\in\mathbb{R}$ is a real zero of Δ-Mittag-Leffler type function $\overline{\mathcal{E}}_\alpha(\alpha+a-2;\lambda,a,0)$ provided that

$$\sum_{a}^{b+1}|\lambda|\geq\Gamma(\alpha)\frac{b-a+2}{b-a+2\alpha}\frac{\Gamma(\alpha+b-a+2)\Gamma^2\left(\dfrac{b-a}{2}+1\right)}{\Gamma(b-a+2)\Gamma^2\left(\dfrac{b-a}{2}+\alpha\right)},$$

if $a+b$ is even and

$$\sum_{a}^{b+1}|\lambda|\geq\Gamma(\alpha)\frac{b-a+3}{b-a+2\alpha+1}\frac{\Gamma(\alpha+b-a+2)\Gamma^2\left(\dfrac{b-a+1}{2}+1\right)}{\Gamma(b-a+2)\Gamma^2\left(\dfrac{b-a+1}{2}+\alpha\right)},$$

if $a+b$ is odd. Equivalently $\lambda\in\mathbb{R}$ is a zero of Δ-Mittag-Leffler type function $\overline{\mathcal{E}}_\alpha(\alpha+a-2;\lambda,a,0)$ provided that

$$|\lambda|\geq\Gamma(\alpha)\frac{1}{b-a+2\alpha}\frac{\Gamma(\alpha+b-a+2)\Gamma^2\left(\dfrac{b-a}{2}+1\right)}{\Gamma(b-a+2)\Gamma^2\left(\dfrac{b-a}{2}+\alpha\right)},$$

if $a+b$ is even and

$$|\lambda|\geq\Gamma(\alpha)\frac{b-a+3}{(b-a+2)(b-a+2\alpha+1)}\frac{\Gamma(\alpha+b-a+2)\Gamma^2\left(\dfrac{b-a+1}{2}+1\right)}{\Gamma(b-a+2)\Gamma^2\left(\dfrac{b-a+1}{2}+\alpha\right)},$$

if $a+b$ is odd. So we can represent the following non-existence criterion for real zeros of Δ-Mittag-Leffler type function $\overline{\mathcal{E}}_\alpha(\alpha+a-2;\lambda,a,0)$.

Theorem 4.6. *For real parameter $1<\alpha\leq 2$, the Δ-Mittag-Leffler type function $\overline{\mathcal{E}}_\alpha(\alpha+a-2;\lambda,a,0)$ defined by (4.24) has no real zeros for*

$$\lambda\in(-A_{even},A_{even}),\quad a,b\in\mathbb{Z},\ a\geq 1,\ b\geq 2, \tag{4.25}$$

if $a+b$ is even and

$$\lambda\in(-A_{odd},A_{odd}),\quad a,b\in\mathbb{Z},\ a\geq 1,\ b\geq 2, \tag{4.26}$$

if $a+b$ is odd in which

$$A_{even}=\Gamma(\alpha)\frac{1}{b-a+2\alpha}\frac{\Gamma(\alpha+b-a+2)\Gamma^2\left(\dfrac{b-a}{2}+1\right)}{\Gamma(b-a+2)\Gamma^2\left(\dfrac{b-a}{2}+\alpha\right)},$$

$$A_{odd}=\Gamma(\alpha)\frac{b-a+3}{(b-a+2)(b-a+2\alpha+1)}\frac{\Gamma(\alpha+b-a+2)\Gamma^2\left(\dfrac{b-a+1}{2}+1\right)}{\Gamma(b-a+2)\Gamma^2\left(\dfrac{b-a+1}{2}+\alpha\right)}.$$

In order to represent third application for discrete Lyapunov type inequality (3.2), first we introduce in brief the concept of disconjugacy of fractional Δ-difference equations as below.
Consider a linear n-th order differential equation

$$D^n y+p_1(t)D^{n-1}y+...+p_{n-1}(t)Dy+p_n(t)y=0,\quad D\equiv\frac{d}{dt} \tag{4.27}$$

where $p_i(t),\ i=1,2,...,n$ are real-valued continuous functions defined on an interval I. Differential equation (4.27) is said to be disconjugate if every nontrivial solution has less than n zeros on I. Multiple zeros being counted according to their multiplicity. See [7] for details. Thus we can define disconjugacy of fractional linear Δ-difference equations as follows.

Definition 4.7. A fractional linear Δ-difference equation

$$\Delta^{\alpha_n}y+p_1(t)\Delta^{\alpha_{n-1}}y+...+p_{n-1}(t)\Delta^{\alpha_1}y+p_n(t)y=0, \tag{4.28}$$

where $\alpha_n>\alpha_{n-1}>...>\alpha_1>0$ and $p_i(t),\ i=1,2,...,n$ are real-valued continuous functions defined on an interval I, is said to be disconjugate if and only if every nontrivial solution has less than $[\alpha_n]+1$ zeros on I. We will show that there is a closed relationship between disconjugacy and solvability of fractional Δ-difference equations.

Definition 4.8. The fractional Δ-difference equation (3.1) is called disconjugate on the interval $[t_1,t_2]_{\mathbb{N}_0},\ t_1<t_2,\ t_1,t_2\in\mathbb{Z}$ if and only if there is no nontrivial real solution $u(t)$ having at least two zeros on $[t_1+\alpha-1,t_2+\alpha]_{\mathbb{N}_{\alpha-1}},\ t_1,t_2\in\mathbb{Z}$.

Theorem 4.9. *Assume that $p(t)>0$ and $q(t)$ are two real valued functions on $[t_1,t_2]_{\mathbb{N}_0}$ and $[t_1+\alpha-1,t_2+\alpha-1]_{\mathbb{N}_{\alpha-1}}$, respectively, such that $t_1\geq 1, t_2\geq 3$. Suppose that*

$$\sum_{s=t_1}^{t_2}\sum_{w=t_1}^{t_2}\left(\frac{|q(w+\alpha-1)-\lambda|}{p(s)}\right)<\frac{1}{2}. \tag{4.29}$$

Then the fractional Δ-difference Sturm-Liouville problem (3.1) is disconjugate on $[t_1,t_2]_{\mathbb{N}_0}$.

Proof. Suppose on the contrary that, there is a nontrivial real solution $u(t)$ having two zeros $s_1, s_2 \in [t_1+\alpha-1, t_2+\alpha]_{\mathbb{N}_{\alpha-1}}$, $s_1 < s_2$. By applying Theorem 3.1, we conclude that

$$\sum_{s=s_1}^{s_2}\sum_{w=s_1}^{s_2}\left(\frac{|q(w+\alpha-1)-\lambda|}{p(s)}\right) \geq \frac{1}{2}.$$

So it is clear that

$$\sum_{s=t_1}^{t_2}\sum_{w=t_1}^{t_2}\left(\frac{|q(w+\alpha-1)-\lambda|}{p(s)}\right) \geq \frac{1}{2}.$$

Obviously, the last inequality contradicts with assumption (4.29). Thus the fractional Δ-difference equation (3.1) is disconjugate on $[t_1, t_2]_{\mathbb{N}_0}$. This completes the proof. □

In similar way, we can represent the disconjugacy criterion for the fractional Δ-difference equation (3.17) as follows.

Lemma 4.10. *Let $q(t)$ be a real valued function on $[t_1+\alpha-2, t_2+\alpha+1]_{\mathbb{N}_{\alpha-2}}$, $t_1 < t_2$ such that $t_1 \geq 1, t_2 \geq 2$. Suppose that*

$$\sum_{t_1}^{t_2+1}|q(s+\alpha-1)| < \Gamma(\alpha)\frac{t_2-t_1+2}{t_2-t_1+2\alpha}\,\frac{\Gamma(\alpha+t_2-t_1+2)\Gamma^2\left(\dfrac{t_2-t_1}{2}+1\right)}{\Gamma(t_2-t_1+2)\Gamma^2\left(\dfrac{t_2-t_1}{2}+\alpha\right)},$$

if t_1+t_2 is even and

$$\sum_{t_1}^{t_2+1}|q(s+\alpha-1)| < \Gamma(\alpha)\frac{t_2-t_1+3}{t_2-t_1+2\alpha+1}\,\frac{\Gamma(\alpha+t_2-t_1+2)\Gamma^2\left(\dfrac{t_2-t_1+1}{2}+1\right)}{\Gamma(t_2-t_1+2)\Gamma^2\left(\dfrac{t_2-t_1+1}{2}+\alpha\right)},$$

if t_1+t_2 is odd. Then the fractional Δ-difference Sturm-Liouville problem (3.17) is disconjugate on $[t_1, t_2]_{\mathbb{N}_0}$.

In fourth step, one can observe that the disconjugacy criterions represented above also can be applied as straightforward nonexistence criterions for nontrivial solutions of the fractional Δ-difference Sturm-Liouville problems (3.1) and (3.17).

Theorem 4.11. *Assume that all of conditions Theorem 4.9 are satisfied. Then the fractional Δ-difference Sturm-Liouville problem (3.1) has no nontrivial solution.*

Proof. Suppose on the contrary that, $u(t)$ is a nontrivial solution of (3.1). Then there exist consecutive zeros t_1, t_2 of $u(t)$ such that $a+\alpha-1 \leq t_1 \leq t_2 \leq b+\alpha+1$.

Now continuing the proof of Theorem 4.9, desired proof will be reached. So we will not repeat it again. □

Lemma 4.12. *Assume that all conditions of Lemma 4.10 hold. Then the fractional Δ-difference Sturm-Liouville problem (3.17) has no nontrivial solution.*

In the last step, we are going to study the zeros count and distance between consecutive zeros of nontrivial solutions of the fractional Δ-difference Sturm-Liouville problems (3.1) and (3.17). The first couple of results will give us the estimate of the number of zeros.

Theorem 4.13. *Assume that $u(t)$ is a nontrivial solution of the fractional Δ-difference Sturm-Liouville problem (3.1). Suppose that $\{t_k\}_{k=1}^{2N+1}$, $N \geq 1$, is an increasing sequence of zeros of $u(t)$ in a compact interval I with length l. Then*

$$\sum_{k=1}^{N} \sum_{s=t_{2k-1}}^{t_{2k+1}} \sum_{w=t_{2k-1}}^{t_{2k+1}} \frac{|q(w+\alpha-1)-\lambda|}{p(s)} \geq \frac{N}{2}. \tag{4.30}$$

Proof. Applying Theorem 3.1 for the interval $[t_{2k-1}, t_{2k+1}] \subset I$, with $k = 1, 2, ..., N$, implies the following inequality

$$\sum_{s=t_{2k-1}}^{t_{2k+1}} \sum_{w=t_{2k-1}}^{t_{2k+1}} \left(\frac{|q(w+\alpha-1)-\lambda|}{p(s)} \right) \geq \frac{1}{2}.$$

Taking the sum on both sides of this inequality, gives us

$$\sum_{k=1}^{N} \sum_{s=t_{2k-1}}^{t_{2k+1}} \sum_{w=t_{2k-1}}^{t_{2k+1}} \frac{|q(w+\alpha-1)-\lambda|}{p(s)} \geq \frac{N}{2},$$

that completes the proof. □

Lemma 4.14. *Suppose that $u(t)$ is a nontrivial solution of the fractional Δ-difference Sturm-Liouville problem (3.17). Assume $\{t_k\}_{k=1}^{2N+1}$, $N \geq 1$, is an increasing sequence of zeros of $u(t)$ in a compact interval I with length l. Then*

$$\sum_{k=1}^{N} \sum_{s=t_{2k-1}}^{t_{2k+1}+1} |q(s+\alpha-1)-\lambda| > \frac{N\Gamma(\alpha)}{(l+2\alpha)\Gamma(l+2)\Gamma^2\left(\frac{l}{2}+\alpha\right)}, \tag{4.31}$$

if $t_{2k} + t_{2k+1}$ is even and

$$\sum_{k=1}^{N} \sum_{s=t_{2k-1}}^{t_{2k+1}+1} |q(s+\alpha-1)-\lambda| > \frac{N\Gamma(\alpha)}{(l+2\alpha+1)\Gamma(l+2)\Gamma^2\left(\frac{l+1}{2}+\alpha\right)}, \tag{4.32}$$

if $t_{2k} + t_{2k+1}$ is odd.

The last application for the fractional discrete Lyapunov type inequalities (3.2) and (3.18),(3.19), deals with considering nontrivial oscillatory solutions of the fractional Δ-difference Sturm-Liouville problems (3.1) and (3.17) and representing a criterion for estimate distance between consecutive zeros as follows.

Theorem 4.15. *Assume that $u(t)$ is an oscillatory solution of the fractional Δ-difference Sturm-Liouville problem (3.1). Suppose that $\{t_k\}_{k=1}^{\infty} \subset \mathbb{N}_{a+\alpha-1}$ with $a \in \mathbb{Z}_{\geq 1}$, $0.5 < \alpha < 1$, is an increasing sequence of zeros of $u(t)$. Suppose that there exists a real constant $\sigma \geq 1$, such that for any positive integer M we have*

$$\sum_{t}^{t+M} |q(s+\alpha-1)-\lambda|^{\sigma} \to 0, \quad t \in \mathbb{N}_{a+\alpha-1},\ t \to \infty, \tag{4.33}$$

Then $t_{n+1} - t_n \to \infty$.

Proof. Applying discrete Holder's inequality for $p = \sigma$, $q = \frac{\sigma-1}{\sigma}$ on the $\sum_{t}^{t+M} |q(s)-\lambda|$, implies that

$$\sum_{t}^{t+M} |q(s+\alpha-1)-\lambda| \leq \left(\sum_{t}^{t+M} |q(s+\alpha-1)-\lambda|^{\sigma}\right)^{\frac{1}{\sigma}} M^{\frac{\sigma-1}{\sigma}} \to 0, \quad t \to \infty. \tag{4.34}$$

Now assume on the contrary that, there exist a positive constant M and a subsequence $\{t_{n_k}\}_{k=1}^{\infty}$ of $\{t_n\}_{n=1}^{\infty}$ such that $t_{n_k+1} - t_{n_k} \leq M$ for all large k. Therefore by the recent assumption and (4.34), we conclude that

$$\sum_{t_{n_k}}^{t_{n_k+1}} |q(s+\alpha-1)-\lambda| \leq \sum_{t_{n_k}}^{t_{n_k}+M} |q(s+\alpha-1)-\lambda| \to 0, \quad k \to \infty. \tag{4.35}$$

Applying Theorem 3.1 for interval $[t_{n_k}, t_{n_k+1}]$, we have

$$\sum_{s=t_{n_k}}^{t_{n_k+1}} \sum_{w=t_{n_k}}^{t_{n_k+1}} \left(\frac{|q(w+\alpha-1)-\lambda|}{p(s)}\right) \geq \frac{1}{2}.$$

At last using (4.35), we have

$$1 \leq 2 \sum_{s=t_{n_k}}^{t_{n_k+1}} \sum_{w=t_{n_k}}^{t_{n_k+1}} \left(\frac{|q(w+\alpha-1)-\lambda|}{p(s)}\right) \to 0, \quad k \to \infty.$$

This contradiction completes the proof. □

Lemma 4.16. *Assume that $u(t)$ is an oscillatory solution of the fractional Δ-difference Sturm-Liouville problem (3.17). Suppose that $\{t_k\}_{k=1}^{\infty} \subset \mathbb{N}_{a+\alpha-2}$*

with $a \in \mathbb{Z}_{\geq 1},\ 1 < \alpha \leq 2$, is an increasing sequence of zeros of $u(t)$. If there exists a real constant $\sigma \geq 1$, such that for any integer $M > 0$

$$\sum_{t}^{t+M+1} |q(s+\alpha-1)-\lambda|^{\sigma} \to 0, \quad t \in \mathbb{N}_{a+\alpha-2},\ t \to \infty, \tag{4.36}$$

then $t_{n+1} - t_n \to \infty$.

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