ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 43 (2017), No. 2, pp. 409–425

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 2, pp. 409–425
Online ISSN: 1735-8515

SOME EXTENDED SIMPSON-TYPE INEQUALITIES AND APPLICATIONS

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(Communicated by Abbas Salemi)

Abstract. In this paper, we shall establish some extended Simpson-type inequalities for differentiable convex functions and differentiable concave functions which are connected with Hermite-Hadamard inequality. Some error estimates for the midpoint, trapezoidal and Simpson formula are also given.
Keywords: Hermite-Hadamard inequality, Simpson inequality, midpoint inequality, trapezoid inequality, convex function, concave functions, special means, quadrature rules.
MSC(2010): Primary; 26D15; Secondary: 26A51.

1. Introduction

Throughout this paper, let $a < b$ in $\mathbb{R}$.

The inequality

$$f\left(\frac{a+b}{2}\right) \leq (\geq) \frac{1}{b-a}\int_a^b f(t)\,dt \leq (\geq) \frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for all convex (concave) functions $f : [a,b] \to \mathbb{R}$, is known in the literature as Hermite-Hadamard inequality [8].

For some results which generalize, improve, and extend the inequality (1.1), see [1–7] and [9–16].

In [12], Tseng *et al.* established the following Hermite-Hadamard-type inequality which refines the inequality (1.1).

Article electronically published on 30 April, 2017.
Received: 25 November 2014, Accepted: 29 November 2015.
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Theorem 1.1. *Suppose that $f : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$. Then we have the inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

The third inequality in (1.2) is known in the literature as Bullen's inequality.

Using the similar proof of Theorem 1.1, we also note that the inequalities in (1.2) are reversed when f is concave on $[a, b]$.

In [4], Dragomir and Agarwal established the following results connected with the second inequality in the inequality (1.1).

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $|f'|$ is convex on $[a, b]$, then we have*

$$\left|\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{8}\left(\left|f'(a)\right|+\left|f'(b)\right|\right) \tag{1.3}$$

which is the trapezoid inequality provided $|f'|$ is convex on $[a, b]$.

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) and let $p > 1$. If $|f'|^{p/(p-1)}$ is convex on $[a, b]$, then we have*

$$\left|\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}} \tag{1.4}$$

which is the trapezoid inequality provided $|f'|^{p/(p-1)}$ is convex on $[a, b]$.

In [11], Pearce and Pečarić established the following results that give an improved and simplified constant in Theorem 1.3 to obtain Theorem 1.4 as follows:

Theorem 1.4. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) and $q \geq 1$. If the mapping $|f'|^q$ is convex on $[a, b]$, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}, \tag{1.5}$$

which is the trapezoid inequality provided $|f'|^q$ is convex on $[a, b]$.

Theorem 1.5. *Under the assumptions of Theorem 1.4, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}, \tag{1.6}$$

which is the midpoint inequality provided $|f'|^q$ *is convex on* $[a,b]$.

The comparable results to Theorem 1.4 and Theorem 1.5 with a concavity property instead of convexity.

Theorem 1.6. *Under the assumptions of Theorem 1.4 and* $|f'|^q\,(q\geq 1)$ *is concave on* $[a,b]$*, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right| \tag{1.7}$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|, \tag{1.8}$$

which are the trapezoid inequality and the midpoint inequality provided $|f'|^q$ *is concave on* $[a,b]$*, respectively.*

From the above results, it is natural to consider the extended Simpson-type formula in the following lemma.

Remark 1.7. Let $0\leq\alpha\leq 1, x\in\left[a,\frac{a+b}{2}\right]$ and $y\in\left[\frac{a+b}{2},b\right]$. Then we have the extended Simpson-type formulas as follows:

(1) *The trapezoid-type formula*

$$\begin{aligned}&\left|\alpha\left[\frac{f(x)+f(y)}{2}\right]+(1-\alpha)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ =&\left|\frac{f((1-\gamma)a+\gamma b)+f(\gamma a+(1-\gamma)b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\end{aligned}$$

as $\alpha=1, x=(1-\gamma)a+\gamma b$ *and* $y=\gamma a+(1-\gamma)b$.

(2) *The trapezoid formula*

$$\begin{aligned}&\left|\alpha\left[\frac{f(x)+f(y)}{2}\right]+(1-\alpha)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ =&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\end{aligned}$$

as $\alpha=1, x=a$ *and* $y=b$.

(3) *The midpoint formula*

$$\begin{aligned}&\left|\alpha\left[\frac{f(x)+f(y)}{2}\right]+(1-\alpha)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ =&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\end{aligned}$$

as $\alpha=0$.

(4) *The Simpson-type formula*

$$\begin{aligned}
&\left|\alpha\left[\frac{f(x)+f(y)}{2}\right]+(1-\alpha)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
=\;&\left|\alpha\left[\frac{f((1-\gamma)a+\gamma b)+f(\gamma a+(1-\gamma)b)}{2}\right]\right. \\
&\left.\qquad+(1-\alpha)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|
\end{aligned}$$

as $0\le\gamma\le\frac{1}{2}, x=(1-\gamma)a+\gamma b$ *and* $y=\gamma a+(1-\gamma)b$.

(5) *The Simpson formula*

$$\begin{aligned}
&\left|\alpha\left[\frac{f(x)+f(y)}{2}\right]+(1-\alpha)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
=\;&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|
\end{aligned}$$

as $\alpha=\frac{1}{3}, x=a$ *and* $y=b$.

(6) *The Bullen formula*

$$\begin{aligned}
&\left|\alpha\left[\frac{f(x)+f(y)}{2}\right]+(1-\alpha)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
=\;&\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|
\end{aligned}$$

as $\alpha=\frac{1}{2}, x=a$ *and* $y=b$.

In this paper, we establish some extended Simpson-type inequalities which reduce the Simpson-type, trapezoid-type, midpoint-type, Bullen-type inequalities, and generalize Theorems 1.2 and 1.4-1.6. Some applications to special means of real numbers are given. Finally, the approximations for quadrature formula are also given.

2. **Extended Simpson-type inequality**

Throughout this section, let $0\le\alpha\le 1, x\in\left[a,\frac{a+b}{2}\right], y\in\left[\frac{a+b}{2},b\right]$, and let $J_1, J_2, h(t), h_1(t)$ $(t\in[a,b])$ be defined as follows.

$$\begin{aligned}
J_1=\;&\frac{\alpha}{3(b-a)^3}\left[(x-a)^2\left(\frac{3b-a}{2}-x\right)+\left(\frac{a+b}{2}-x\right)^2\left(\frac{5b-a}{4}-x\right)\right. \\
&\left.+(b-y)^3+\left(y-\frac{a+b}{2}\right)^2\left(\frac{5b-a}{4}-y\right)\right]+\frac{1-\alpha}{8}.
\end{aligned} \tag{2.1}$$

$$J_2 = \frac{\alpha}{3(b-a)^3}\left[(x-a)^3 + \left(\frac{a+b}{2}-x\right)^2\left(x-\frac{5a-b}{4}\right)\right. \tag{2.2}$$
$$\left. + (b-y)^2\left(y-\frac{3a-b}{2}\right) + \left(y-\frac{a+b}{2}\right)^2\left(y-\frac{5a-b}{4}\right)\right] + \frac{1-\alpha}{8}.$$

$$h(t) = \begin{cases} t-a, & a \le t < x \\ \alpha\left(t-\frac{a+b}{2}\right) + (1-\alpha)(t-a), & x \le t < \frac{a+b}{2} \\ \alpha\left(t-\frac{a+b}{2}\right) + (1-\alpha)(t-b), & \frac{a+b}{2} \le t < y \\ t-b, & y \le t \le b \end{cases}.$$

$$h_1(t) = \begin{cases} t-a, & a \le t < x \\ \alpha\left(\frac{a+b}{2}-t\right) + (1-\alpha)(t-a), & x \le t < \frac{a+b}{2} \\ \alpha\left(t-\frac{a+b}{2}\right) + (1-\alpha)(b-t), & \frac{a+b}{2} \le t < y \\ b-t, & y \le t \le b \end{cases}.$$

In order to prove our main results, we need the following lemma and remark whose proof can be obtained by simple computation.

Lemma 2.1. *Let $a, b, x, y, \alpha, J_1, J_2, h(t), h_1(t)$ $(t \in [a,b])$ be defined as above. Then we have*

$$J_1 = \frac{1}{(b-a)^3}\int_a^b h_1(t)(b-t)\,dt,\ \ J_2 = \frac{1}{(b-a)^3}\int_a^b h_1(t)(t-a)\,dt,$$

$$\begin{aligned} & J_1 + J_2 \\ = \ & \frac{1}{(b-a)^2}\int_a^b h_1(t)\,dt \\ = \ & \frac{\alpha}{2(b-a)^2}\left[(x-a)^2 + \left(\frac{a+b}{2}-x\right)^2 + (b-y)^2 + \left(y-\frac{a+b}{2}\right)^2\right] + \frac{1-\alpha}{4} \\ = \ & \frac{1}{4} - \alpha\left[\frac{(x-a)\left(\frac{a+b}{2}-x\right)}{(b-a)^2} + \frac{(b-y)\left(y-\frac{a+b}{2}\right)}{(b-a)^2}\right], \end{aligned}$$

$$\begin{aligned} & aJ_1 + bJ_2 \\ = \ & \frac{1}{(b-a)^2}\int_a^b h_1(t)\,t\,dt \\ = \ & \frac{\alpha}{6(b-a)^2}\left[(x-a)^2(2x+a) + \left(\frac{a+b}{2}-x\right)^2\left(2x+\frac{a+b}{2}\right)\right. \\ & \left. + (b-y)^2(2y+b) + \left(y-\frac{a+b}{2}\right)^2\left(2y+\frac{a+b}{2}\right)\right] + \frac{1-\alpha}{4}\left(\frac{a+b}{2}\right), \end{aligned}$$

$$0 < J_1 \le J_1 + J_2 \le \frac{1}{4} \ \textit{and}\ 0 < J_2 \le J_1 + J_2 \le \frac{1}{4}.$$

Remark 2.2. Let $0 \leq \gamma, \rho \leq \frac{1}{2}$, $x = (1-\gamma)a + \gamma b$ and $y = \rho a + (1-\rho) b$ in the identities (2.1)and (2.2). Then we have

$$J_1 = \frac{1}{8} - \alpha\gamma\left(\frac{1}{2} - \gamma\right), J_2 = \frac{1}{8} - \alpha\rho\left(\frac{1}{2} - \rho\right)$$

and

$$J_1 + J_2 = \frac{1}{4} - \alpha\left[\gamma\left(\frac{1}{2} - \gamma\right) + \rho\left(\frac{1}{2} - \rho\right)\right].$$

Further, if $\gamma = \rho$, then

$$J_1 = J_2 = \frac{1}{8} - \alpha\gamma\left(\frac{1}{2} - \gamma\right) \text{ and } J_1 + J_2 = \frac{1}{4} - \alpha\gamma(1 - 2\gamma).$$

Now, we are ready to state and prove the main results.

Theorem 2.3. *Let $a, b, x, y, \alpha, J_1, J_2, h(t), h_1(t)$ $(t \in [a, b])$ be defined as above and let q, f be defined as in Theorem 1.4. Then we have the extended Simpson-type inequality*

$$\left|\alpha\left[\frac{f(x) + f(y)}{2}\right] + (1-\alpha) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.3}$$
$$\leq (J_1 + J_2)(b-a)\left(\frac{J_1|f'(a)|^q + J_2|f'(b)|^q}{J_1 + J_2}\right)^{\frac{1}{q}}.$$

Proof. Using the integration by parts and simple computation, we have the following identity:

$$\frac{1}{b-a}\int_a^b h(t) f'(t)\,dt \tag{2.4}$$
$$= \alpha\left[\frac{f(x) + f(y)}{2}\right] + (1-\alpha) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Now, using Hölder's inequality, the convexity of $|f'|^q$ and Lemma 2.1, we have the inequality

$$\left|\frac{1}{b-a}\int_a^b h(t) f'(t)\,dt\right| \tag{2.5}$$
$$\leq \frac{1}{b-a}\int_a^b |h(t)|\left|f'(t)\right| dt$$
$$\leq \frac{1}{b-a}\int_a^b h_1(t)\left|f'(t)\right| dt$$
$$\leq \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left(\int_a^b h_1(t)\left|f'(t)\right|^q dt\right)^{\frac{1}{q}}$$
$$= \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left(\int_a^b h_1(t)\left|f'\left(\frac{b-t}{b-a}\cdot a + \frac{t-a}{b-a}\cdot b\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$
\begin{aligned}
&\leq \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left[\int_a^b h_1(t)\frac{b-t}{b-a}\left|f'(a)\right|^q + h_1(t)\frac{t-a}{b-a}\left|f'(b)\right|^q dt\right]^{\frac{1}{q}} \\
&= \left(\frac{1}{(b-a)^2}\int_a^b h_1(t)\,dt\right)^{1-\frac{1}{q}}\left(\frac{1}{(b-a)^3}\int_a^b h_1(t)(b-t)\,dt\cdot\left|f'(a)\right|^q\right. \\
&\qquad\qquad \left.+\frac{1}{(b-a)^3}\int_a^b h_1(t)(t-a)\,dt\cdot\left|f'(b)\right|^q\right)^{\frac{1}{q}}\cdot(b-a) \\
&= (J_1+J_2)^{1-\frac{1}{q}}\left(J_1\left|f'(a)\right|^q + J_2\left|f'(b)\right|^q\right)^{\frac{1}{q}}(b-a) \\
&= (J_1+J_2)(b-a)\left(\frac{J_1\left|f'(a)\right|^q + J_2\left|f'(b)\right|^q}{J_1+J_2}\right)^{\frac{1}{q}}.
\end{aligned}
$$

The inequality (2.3) follows from the identity (2.4) and the inequality (2.5). This completes the proof. □

Under the conditions of Theorem 2.3 and Remark 2.2, we have the following corollaries and remarks.

Corollary 2.4. *Using Theorem 2.3 and Remark 1.7, we have*

$$
\begin{aligned}
&\left|\alpha\left[\frac{f((1-\gamma)a+\gamma b)+f(\gamma a+(1-\gamma)b)}{2}\right]\right. \\
&\qquad\left. +(1-\alpha)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \left[\frac{1}{4}-\alpha\gamma(1-2\gamma)\right](b-a)\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}
\end{aligned}
$$

which is the Simpson-type inequality provided $|f'|^q$ *is convex on* $[a,b]$.

Corollary 2.5. *In Corollary 2.4, let* $\gamma = 0$. *Then, we have*

$$
\begin{aligned}
&\left|\alpha\left(\frac{f(a)+f(b)}{2}\right)+(1-\alpha)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \frac{b-a}{4}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.6}
$$

Remark 2.6. In Corollary 2.5, let $\alpha = \frac{1}{3}$. Then, we have

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \frac{b-a}{4}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}},
\end{aligned}
$$

which is the Simpson inequality [5] provided $|f'|^q$ is convex on $[a,b]$.

Remark 2.7. In Corollary 2.5, let $\alpha = \frac{1}{2}$. Then, we have

$$\left|\frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}},$$

which is the Bullen's inequality provided $|f'|^q$ is convex on $[a,b]$.

Remark 2.8. If we choose $\alpha = 1$ in Corollary 2.5, then the inequality (2.6) reduces the trapezoid inequality (1.5).

Remark 2.9. If we choose $\alpha = 0$ in Corollary 2.5, then the inequality (2.6) reduces the midpoint inequality (1.6).

Corollary 2.10. *In Corollary 2.4, let $\alpha = 1$. Then, we have*

$$\left|\frac{f((1-\gamma)a + \gamma b) + f(\gamma a + (1-\gamma)b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4} - \gamma(1-2\gamma)\right](b-a)\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}},$$

which is the trapezoid-type inequality provided $|f'|^q$ is convex on $[a,b]$.

Remark 2.11. In Corollary 2.10, let $\gamma = \frac{1}{4}$. Then, we have

$$\left|\frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{8}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}},$$

which is the second inequality in (1.2) provided $|f'|^q$ is convex on $[a,b]$.

Theorem 2.12. *Let $a, b, x, y, \alpha, J_1, J_2, h(t), h_1(t)$ $(t \in [a,b])$ be defined as above and let q, f be defined as in Theorem 1.6. Then we have the extended Simpson-type inequality*

$$\left|\alpha\left[\frac{f(x) + f(y)}{2}\right] + (1-\alpha)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq (J_1 + J_2)(b-a)\left|f'\left(\frac{J_1 a + J_2 b}{J_1 + J_2}\right)\right|. \tag{2.7}$$

Proof. Since $q>1$ and $|f'|^q$ is concave on $[a,b]$, $|f'|$ is also concave on $[a,b]$. Using the Jensen's integral inequality and Lemma 2.1, we have

$$
\begin{aligned}
(2.8)\qquad & \left|\frac{1}{b-a}\int_a^b h(t)\,f'(t)\,dt\right| \\
\leq\ & \frac{1}{b-a}\int_a^b |h(t)|\,|f'(t)|\,dt \\
\leq\ & \frac{1}{b-a}\int_a^b h_1(t)\,|f'(t)|\,dt \\
\leq\ & \frac{1}{b-a}\left(\int_a^b h_1(t)\,dt\right)\left|f'\left(\frac{\int_a^b h_1(t)\,tdt}{\int_a^b h_1(t)\,dt}\right)\right| \\
=\ & (b-a)\left(\frac{1}{(b-a)^2}\int_a^b h_1(t)\,dt\right)\left|f'\left(\frac{\frac{1}{(b-a)^2}\int_a^b h_1(t)\,tdt}{\frac{1}{(b-a)^2}\int_a^b h_1(t)\,dt}\right)\right| \\
=\ & (b-a)(J_1+J_2)\left|f'\left(\frac{J_1a+J_2b}{J_1+J_2}\right)\right|.
\end{aligned}
$$

The inequality (2.7) follows from the identity (2.4) and the inequality (2.8). This completes the proof. □

Under the conditions of Theorem 2.12 and Remark 1.7, we have the following corollaries and remarks.

Corollary 2.13. *In Theorem 2.12, let* $x=(1-\gamma)a+\gamma b$ *and* $y=\gamma a+(1-\gamma)b$ *where* $0\leq\gamma\leq\frac{1}{2}$. *Then, using Remark 2.2, we have*

$$J_1=J_2=\frac{1}{8}-\alpha\gamma\left(\frac{1}{2}-\gamma\right),\ J_1+J_2=\frac{1}{4}-\alpha\gamma(1-2\gamma)$$

and

$$
\begin{aligned}
&\left|\alpha\left[\frac{f((1-\gamma)a+\gamma b)+f(\gamma a+(1-\gamma)b)}{2}\right]\right. \\
&\qquad\qquad\left.+(1-\alpha)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq\ & \left[\frac{1}{4}-\alpha\gamma(1-2\gamma)\right](b-a)\left|f'\left(\frac{a+b}{2}\right)\right|,
\end{aligned}
$$

which is the Simpson-type inequality provided $|f'|^q$ *is concave on* $[a,b]$.

Corollary 2.14. *In Corollary 2.13, let $\gamma = 0$. Then, we have*

$$\left|\alpha\left[\frac{f(a)+f(b)}{2}\right]+(1-\alpha)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{2.9}$$

Remark 2.15. In Corollary 2.14, let $\alpha = \frac{1}{3}$. Then, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|,$$

which is the Simpson inequality [5] provided $|f'|^q$ is concave on $[a,b]$.

Remark 2.16. In Corollary 2.14, let $\alpha = \frac{1}{2}$. Then, we have

$$\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|,$$

which is the Bullen's inequality provided $|f'|^q$ is concave on $[a,b]$.

Remark 2.17. If we choose $\alpha = 1$ in Corollary 2.14, then the inequality (2.9) reduces the trapezoid inequality (1.7).

Remark 2.18. If we choose $\alpha = 0$ in Corollary 2.14, then the inequality (2.9) reduces the midpoint inequality (1.8).

Corollary 2.19. *In Corollary 2.13, let $\alpha = 1$. Then, we have*

$$\left|\frac{f((1-\gamma)a+\gamma b)+f(\gamma a+(1-\gamma)b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4}-\gamma(1-2\gamma)\right](b-a)\left|f'\left(\frac{a+b}{2}\right)\right|,$$

which is the trapezoid-type inequality provided $|f'|^q$ is concave on $[a,b]$.

Remark 2.20. In Corollary 2.19, let $\gamma = \frac{1}{4}$. Then, we have

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{b-a}{8}\left|f'\left(\frac{a+b}{2}\right)\right|,$$

which is the second inequality in (1.2) provided $|f'|^q$ is concave on $[a,b]$.

3. **Applications for special means**

In the literature, let us recall the following special means of the two real numbers u and v:

(1) The weighted arithmetic mean

$$A_\alpha(u,v) := \alpha u + (1-\alpha)v,\ \ u,v \in \mathbb{R}.$$

(2) The unweighted arithmetic mean

$$A(u,v) := \frac{u+v}{2},\ \ u,v \in \mathbb{R}.$$

(3) The harmonic mean

$$H(u,v) := \frac{2}{\frac{1}{u}+\frac{1}{v}},\ \ u,v > 0.$$

(4) The identric mean

$$I(u,v) := \begin{cases} \frac{1}{e}\left(\frac{v^v}{u^u}\right)^{\frac{1}{v-u}} & \text{if } u \neq v \\ u & \text{if } u = v \end{cases},\ u,v > 0.$$

(5) The logarithmic mean

$$L(u,v) := \begin{cases} \frac{v-u}{\ln v - \ln u} & \text{if } u \neq v \\ u & \text{if } u = v \end{cases},\ u,v > 0.$$

(6) The p-logarithmic mean

$$L_p(u,v) := \begin{cases} \left[\frac{v^{p+1}-u^{p+1}}{(p+1)(v-u)}\right]^{\frac{1}{p}} & \text{if } u \neq v \\ u & \text{if } u = v \end{cases},\ \ u,v > 0,\ \ p \in \mathbb{R}\backslash\{-1,0\}.$$

(7) The p-power mean

$$M_p(u,v) := \left(\frac{u^p+v^p}{2}\right)^{\frac{1}{p}},\ \ u,v > 0, p \in \mathbb{R}\backslash\{0\}.$$

Using the above results, we have the following results about the above special means:

Proposition 3.1. *In Corollary 2.4, let* $s \in (-\infty,1] \cup \left[1+\frac{1}{q},\infty\right) \backslash \{-1,0\}$, $q \geq 1, a > 0, b > 0$ *and let* $f(t) = t^s$ *on* $[a,b]$. *Then we have*

$$\begin{aligned} &\left|A_\alpha\left(A\left(A_\gamma^s(b,a), A_\gamma^s(a,b)\right), A^s(a,b)\right) - L_s^s(a,b)\right| \\ \leq\ &\left[\frac{1}{4} - \alpha\gamma(1-2\gamma)\right] |s|(b-a) M_q\left(a^{s-1}, b^{s-1}\right). \end{aligned}$$

Corollary 3.2. *In Proposition 3.1, let* $\gamma = 0$. *Then, we have*

$$|A_\alpha(A(a^s,b^s), A^s(a,b)) - L_s^s(a,b)| \leq \frac{|s|(b-a)}{4} M_q\left(a^{s-1}, b^{s-1}\right). \tag{3.1}$$

Corollary 3.3. *In Proposition 3.1, let* $\alpha = 1$. *Then, we have*

$$\begin{aligned} &\left|A\left(A_\gamma^s(b,a), A_\gamma^s(a,b)\right) - L_s^s(a,b)\right| \\ \leq\ &\left[\frac{1}{4} - \gamma(1-2\gamma)\right]|s|(b-a)M_q\left(a^{s-1}, b^{s-1}\right). \end{aligned} \tag{3.2}$$

Proposition 3.4. *In Corollary 2.13, let* $s \in \left[1, 1+\frac{1}{q}\right]$, $q \geq 1$, $a \geq 0, b \geq 0$ *and let* $f(t) = t^s$ *on* $[a,b]$. *Then we have*

$$\begin{aligned} &\left|A_\alpha\left(A\left(A_\gamma^s(b,a), A_\gamma^s(a,b)\right), A^s(a,b)\right) - L_s^s(a,b)\right| \\ \leq\ &\left[\frac{1}{4} - \alpha\gamma(1-2\gamma)\right]s(b-a)A^{s-1}(a,b). \end{aligned}$$

Corollary 3.5. *In Proposition 3.4, let* $\gamma = 0$. *Then, we have*

$$|A_\alpha(A(a^s, b^s), A^s(a,b)) - L_s^s(a,b)| \leq \frac{s(b-a)}{4}A^{s-1}(a,b). \tag{3.3}$$

Corollary 3.6. *In Proposition 3.4, let* $\alpha = 1$. *Then, we have*

$$\begin{aligned} &\left|A\left(A_\gamma^s(b,a), A_\gamma^s(a,b)\right) - L_s^s(a,b)\right| \\ \leq\ &\left[\frac{1}{4} - \gamma(1-2\gamma)\right]s(b-a)A^{s-1}(a,b). \end{aligned} \tag{3.4}$$

Proposition 3.7. *In Corollary 2.4, let* $q \geq 1, a > 0, b > 0$ *and let* $f(t) = \frac{1}{t}$ *on* $[a,b]$. *Then we have*

$$\begin{aligned} &\left|A_\alpha\left(H^{-1}(A_\gamma(b,a), A_\gamma(a,b)), A^{-1}(a,b)\right) - L^{-1}(a,b)\right| \\ \leq\ &\left[\frac{1}{4} - \alpha\gamma(1-2\gamma)\right](b-a)M_q\left(a^{-2}, b^{-2}\right). \end{aligned}$$

Corollary 3.8. *In Proposition 3.7, let* $\gamma = 0$. *Then, we have*

$$\left|A_\alpha\left(H^{-1}(a,b), A^{-1}(a,b)\right) - L^{-1}(a,b)\right| \leq \frac{b-a}{4}M_q\left(a^{-2}, b^{-2}\right). \tag{3.5}$$

Corollary 3.9. *In Proposition 3.7, let* $\alpha = 1$. *Then, we have*

$$\begin{aligned} &\left|H^{-1}(A_\gamma(b,a), A_\gamma(a,b)) - L^{-1}(a,b)\right| \\ \leq\ &\left[\frac{1}{4} - \gamma(1-2\gamma)\right](b-a)M_q\left(a^{-2}, b^{-2}\right). \end{aligned} \tag{3.6}$$

Proposition 3.10. *In Corollary 2.13, let* $a > 0, b > 0$ *and let* $f(t) = \ln t$ *on* $[a,b]$. *Then we have*

$$\begin{aligned} &|A_\alpha(A(\ln A_\gamma(b,a), \ln A_\gamma(a,b)), \ln A(a,b)) - \ln I(a,b)| \\ \leq\ &\left[\frac{1}{4} - \alpha\gamma(1-2\gamma)\right](b-a)A^{-1}(a,b). \end{aligned}$$

Corollary 3.11. *In Proposition 3.10, let* $\gamma = 0$. *Then, we have*

$$|A_{\alpha}(A(\ln a, \ln b), \ln A(a,b)) - \ln I(a,b)| \leq \frac{b-a}{4} A^{-1}(a,b). \tag{3.7}$$

Corollary 3.12. *In Proposition 3.10, let* $\alpha = 1$. *Then, we have*

$$\begin{aligned} &|A(\ln A_{\gamma}(b,a), \ln A_{\gamma}(a,b)) - \ln I(a,b)| \\ \leq \ & \left[\frac{1}{4} - \gamma(1-2\gamma)\right](b-a) A^{-1}(a,b). \end{aligned} \tag{3.8}$$

4. Applications for the extended Simpson quadrature formula

Throughout this section, let $0 \leq \alpha \leq 1, I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a partition of the interval $[a,b], l_i = x_{i+1} - x_i, \xi_i \in \left[x_i, \frac{x_i + x_{i+1}}{2}\right]$ and $\zeta_i \in \left[\frac{x_i + x_{i+1}}{2}, x_{i+1}\right]$ $(i = 0, 1, \cdots, n-1)$. Define the extended Simpson quadrature formula

$$\int_a^b f(t)dt = S_{\alpha}(f, I_n, \xi, \zeta) + R_{\alpha}(f, I_n, \xi, \zeta), \tag{4.1}$$

where

$$S_{\alpha}(f, I_n, \xi, \zeta) := \alpha \sum_{i=0}^{n-1} \frac{f(\xi_i) + f(\zeta_i)}{2} l_i + (1-\alpha) \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) l_i,$$

and $\xi_i \in [x_i, (x_i + x_{i+1})/2]$, $\zeta_i \in [(x_i + x_{i+1})/2, x_{i+1}]$, and the remainder term $R_{\alpha}(f, I_n, \xi_i, \zeta_i)$ denotes the associated approximation error of $\int_a^b f(t)dt$ by $S_{\alpha}(f, I_n, \xi, \zeta)$.

Let $\alpha \in \left\{0, \frac{1}{3}, 1\right\}$ in the identity (4.1). Then we have the following special formulae.

(1) The midpoint formula

$$S_0(f, I_n, \xi, \zeta) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) l_i.$$

(2) The trapezoid formula

$$S_1(f, I_n, \xi, \zeta) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} l_i,$$

where $\xi_i = x_i$ and $\zeta_i = x_{i+1}$ $(i = 0, 1, \cdots, n-1)$.

(3) The Simpson formula

$$S_{\frac{1}{3}}(f, I_n, \xi, \zeta) = \sum_{i=0}^{n-1} \frac{1}{6}\left[f(x_i) + 4f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_{i+1})\right] l_i,$$

where $\xi_i = x_i$ and $\zeta_i = x_{i+1}$ $(i = 0, 1, \cdots, n-1)$.

Theorem 4.1. *Let f be defined as in Theorem 2.3 and let $\int_a^b f(t)dt, S_\alpha(f, I_n, \xi, \zeta)$ and $R_\alpha(f, I_n, \xi, \zeta)$ be defined as in the identity (4.1). Then, the remainder term $R_\alpha(f, I_n, \xi, \zeta)$ satisfies the estimate*

$$
\begin{aligned}
&|R_\alpha(f, I_n, \xi, \zeta)| \\
\leq\ & \sum_{i=0}^{n-1} (T_1(i) + T_2(i)) l_i^2 \left(\frac{T_1(i) |f'(x_i)|^q + T_2(i) |f'(x_{i+1})|^q}{T_1(i) + T_2(i)} \right)^{\frac{1}{q}} \\
\leq\ & \max\{|f'(a)|, |f'(b)|\} \sum_{i=0}^{n-1} (T_1(i) + T_2(i)) l_i^2,
\end{aligned}
\tag{4.2}
$$

where

$$
\begin{aligned}
T_1(i) = \frac{\alpha}{3l_i^3} \Bigg[& (\xi_i - x_i)^2 \left(\frac{3x_{i+1} - x_i}{2} - \xi_i \right) \\
& + \left(\frac{x_i + x_{i+1}}{2} - \xi_i \right)^2 \left(\frac{5x_{i+1} - x_i}{4} - \xi_i \right) + (x_{i+1} - \zeta_i)^3 \\
& + \left(\zeta_i - \frac{x_i + x_{i+1}}{2} \right)^2 \left(\frac{5x_{i+1} - x_i}{4} - \zeta_i \right) \Bigg] + \frac{1-\alpha}{8}
\end{aligned}
$$

and

$$
\begin{aligned}
T_2(i) = \frac{\alpha}{3l_i^3} \Bigg[& \left(\frac{x_i + x_{i+1}}{2} - \xi_i \right)^2 \left(\xi_i - \frac{5x_i - x_{i+1}}{4} \right) \\
& + (\xi_i - x_i)^3 + (x_{i+1} - \zeta_i)^2 \left(\zeta_i - \frac{3x_i - x_{i+1}}{2} \right) \\
& + \left(\zeta_i - \frac{x_i + x_{i+1}}{2} \right)^2 \left(\zeta_i - \frac{5x_i - x_{i+1}}{4} \right) \Bigg] + \frac{1-\alpha}{8}
\end{aligned}
$$

for all $i = 0, 1, \cdots, n-1$.

Proof. Apply Theorem 2.3 on the intervals $[x_i, x_{i+1}]$ $(i = 0, 1, \cdots, n-1)$ to get

$$
\begin{aligned}
& \left| \left[\alpha \frac{f(\xi_i) + f(\zeta_i)}{2} + (1-\alpha) f\left(\frac{x_i + x_{i+1}}{2} \right) \right] l_i - \int_{x_i}^{x_{i+1}} f(t)dt \right| \\
\leq\ & (T_1(i) + T_2(i)) l_i^2 \left(\frac{T_1(i) |f'(x_i)|^q + T_2(i) |f'(x_{i+1})|^q}{T_1(i) + T_2(i)} \right)^{\frac{1}{q}}
\end{aligned}
\tag{4.3}
$$

for all $i = 0, 1, \cdots, n-1$.

Using the convexity of $|f'|^q$, we have

$$\begin{aligned}
(4.4)\qquad & \left(\frac{T_1(i)\,|f'(x_i)|^q + T_2(i)\,|f'(x_{i+1})|^q}{T_1(i)+T_2(i)}\right)^{\frac{1}{q}} \\
\leq\ & \left[\frac{T_1(i)}{T_1(i)+T_2(i)}\left(\frac{b-x_i}{b-a}\,|f'(a)|^q + \frac{x_i-a}{b-a}\,|f'(b)|^q\right)\right. \\
& \left. + \frac{T_2(i)}{T_1(i)+T_2(i)}\left(\frac{b-x_{i+1}}{b-a}\,|f'(a)|^q + \frac{x_{i+1}-a}{b-a}\,|f'(b)|^q\right)\right]^{\frac{1}{q}} \\
\leq\ & \left(\max\left\{|f'(a)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}} = \max\left\{|f'(a)|, |f'(b)|\right\}
\end{aligned}$$

for all $i = 0, 1, \cdots, n-1$.

The inequality (4.2) follows from the inequalities (4.3), (4.4) and the generalized triangle inequality. This completes the proof. □

Corollary 4.2. *In Theorem 4.1, let* $\alpha = \frac{1}{3}$ *and* $\xi_i = x_i, \zeta_i = x_{i+1}(i = 0, 1, \cdots, n-1)$. *Then* $T_1(i) = T_2(i) = \frac{1}{8}$ $(i = 0, 1, \cdots, n-1)$ *and the Simpson-type error satisfies*

$$\begin{aligned}
\left|R_{\frac{1}{3}}(f, I_n, \xi, \zeta)\right| &\leq \sum_{i=0}^{n-1} \frac{l_i^2}{4}\left[\frac{|f'(x_i)|^q + |f'(x_{i+1})|^q}{2}\right]^{\frac{1}{q}} \\
&\leq \max\left\{|f'(a)|, |f'(b)|\right\} \sum_{i=0}^{n-1} \frac{l_i^2}{4}.
\end{aligned}$$

Corollary 4.3. *In Theorem 4.1, let* $\alpha = 1$ *and* $\xi_i = x_i, \zeta_i = x_{i+1}(i = 0, 1, \cdots, n-1)$. *Then* $T_1(i) = T_2(i) = \frac{1}{8}$ $(i = 0, 1, \cdots, n-1)$ *and the trapzoid-type error satisfies*

$$\begin{aligned}
\left|R_1(f, I_n, \xi, \zeta)\right| &\leq \sum_{i=0}^{n-1} \frac{l_i^2}{4}\left[\frac{|f'(x_i)|^q + |f'(x_{i+1})|^q}{2}\right]^{\frac{1}{q}} \\
&\leq \max\left\{|f'(a)|, |f'(b)|\right\} \sum_{i=0}^{n-1} \frac{l_i^2}{4}
\end{aligned}$$

which is Proposition 3 *in* [11].

Corollary 4.4. *In Theorem 4.1, let* $\alpha = 0$ *and* $\xi_i = x_i, \zeta_i = x_{i+1}$ $(i = 0, 1, \cdots, n-1)$. *Then* $T_1(i) = T_2(i) = \frac{1}{8}$ $(i = 0, 1, \cdots, n-1)$ *and the midpoint-type error satisfies*

$$\begin{aligned}
\left|R_0(f, I_n, \xi, \zeta)\right| &\leq \sum_{i=0}^{n-1} \frac{l_i^2}{4}\left[\frac{|f'(x_i)|^q + |f'(x_{i+1})|^q}{2}\right]^{\frac{1}{q}} \\
&\leq \max\left\{|f'(a)|, |f'(b)|\right\} \sum_{i=0}^{n-1} \frac{l_i^2}{4}.
\end{aligned}$$

Similarly, using Theorem 2.12 we can prove the following theorem.

Theorem 4.5. *Let f be defined as in Theorem 2.12, $T_1(i), T_2(i)$ $t(i = 0, 1, \cdots, n-1)$ be defined as in Theorem 4.1 and let $\int_a^b f(t)dt$, $S_\alpha(f, I_n, \xi, \zeta)$ and $R_\alpha(f, I_n, \xi, \zeta)$ be defined as in the identity (4.1). Then, the remainder term $R_\alpha(f, I_n, \xi, \zeta)$ satisfies the estimate*

$$|R_\alpha(f, I_n, \xi, \zeta)| \leq \sum_{i=0}^{n-1} (T_1(i) + T_2(i))\, l_i^2 \left| f'\left(\frac{T_1(i)\, x_i + T_2(i)\, x_{i+1}}{T_1(i) + T_2(i)}\right)\right|$$

for all $i = 0, 1, ..., n-1$.

Corollary 4.6. *In Theorem 4.5, let $\alpha = \frac{1}{3}$ and $\xi_i = x_i, \zeta_i = x_{i+1}(i = 0, 1, \cdots, n-1)$. Then $T_1(i) = T_2(i) = \frac{1}{8}$ $(i = 0, 1, \cdots, n-1)$ and the Simpson-type error satisfies*

$$\left|R_{\frac{1}{3}}(f, I_n, \xi, \zeta)\right| \leq \sum_{i=0}^{n-1} \frac{l_i^2}{4} \left| f'\left(\frac{x_i + x_{i+1}}{2}\right)\right|.$$

Corollary 4.7. *In Theorem 4.5, let $\alpha = 1$ and $\xi_i = x_i, \zeta_i = x_{i+1}(i = 0, 1, \cdots, n-1)$. Then $T_1(i) = T_2(i) = \frac{1}{8}$ $(i = 0, 1, \cdots, n-1)$ and the trapzoid-type error satisfies*

$$|R_1(f, I_n, \xi, \zeta)| \leq \sum_{i=0}^{n-1} \frac{l_i^2}{4} \left| f'\left(\frac{x_i + x_{i+1}}{2}\right)\right|.$$

Corollary 4.8. *In Theorem 4.5, let $\alpha = 0$ and $\xi_i = x_i, \zeta_i = x_{i+1}$ $(i = 0, 1, \cdots, n-1)$. Then $T_1(i) = T_2(i) = \frac{1}{8}$ $(i = 0, 1, \cdots, n-1)$ and the midpoint-type error satisfies*

$$|R_0(f, I_n, \xi, \zeta)| \leq \sum_{i=0}^{n-1} \frac{l_i^2}{4} \left| f'\left(\frac{x_i + x_{i+1}}{2}\right)\right|$$

which is Proposition 4 in [11].

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