Title:

Left derivable or Jordan left derivable mappings on Banach algebras

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LEFT DERIVABLE OR JORDAN LEFT DERIVABLE MAPPINGS ON BANACH ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$ be a unital Banach algebra, $\mathcal{M}$ be a left $\mathcal{A}$-module, and W in $\mathcal{Z}(\mathcal{A})$ be a left separating point of $\mathcal{M}$. We show that if $\mathcal{M}$ is a unital left $\mathcal{A}$-module and δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$, then the following four conditions are equivalent: (i) δ is a Jordan left derivation; (ii)δ is left derivable at W; (iii) δ is Jordan left derivable at W; (iv)$A\delta(B) + B\delta(A) = \delta(W)$ for each A, B in $\mathcal{A}$ with $AB = BA = W$. Let $\mathcal{A}$ have property ($\mathbb{B}$) (see Definition 3.1), $\mathcal{M}$ be a Banach left $\mathcal{A}$-module, and δ be a continuous linear operator from $\mathcal{A}$ into $\mathcal{M}$. Then δ is a generalized Jordan left derivation if and only if δ is Jordan left derivable at zero. In addition, if there exists an element $C \in \mathcal{Z}(\mathcal{A})$ which is a left separating point of $\mathcal{M}$, and $Rann_{\mathcal{M}}(\mathcal{A}) = \{0\}$, then δ is a generalized left derivation if and only if δ is left derivable at zero.

Keywords: (Jordan) left derivation, generalized (Jordan) left derivation, (Jordan) left derivable mapping.
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1. Introduction

Let $\mathcal{A}$ be a Banach algebra over the complex field $\mathbb{C}$. As usual, for each A, B in $\mathcal{A}$, we write $A \circ B$ for the *Jordan product* $AB + BA$, while $A \cdot B$ or AB denotes the ordinary *product* of A and B. The *center* of $\mathcal{A}$ is $\mathcal{Z}(\mathcal{A}) = \{A \in \mathcal{A} : AB = BA \text{ for each } B \text{ in } \mathcal{A}\}$. Let $\mathcal{M}$ be a left $\mathcal{A}$-module. The *right annihilator* of $\mathcal{A}$ on $\mathcal{M}$ is $Rann_{\mathcal{M}}(\mathcal{A}) = \{M \in \mathcal{M} : AM = 0 \text{ for each } A \text{ in } \mathcal{A}\}$. An element W in $\mathcal{A}$ is said to be *a left separating point of* $\mathcal{M}$, if for each M in $\mathcal{M}$ satisfying $WM = 0$, we have $M = 0$.

Let δ be a linear mapping from $\mathcal{A}$ into a left $\mathcal{A}$-module $\mathcal{M}$. δ is called a *left derivation* if $\delta(AB) = A\delta(B) + B\delta(A)$ for each A, B in $\mathcal{A}$, and is called a *Jordan left derivation* if $\delta(A \circ B) = 2A\delta(B) + 2B\delta(A)$ for each A, B in $\mathcal{A}$. Let $C \in \mathcal{A}$. δ is said to be *left derivable at* C if $\delta(AB) = A\delta(B) + B\delta(A)$

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for each A, B in $\mathcal{A}$ satisfying $AB = C$, and to be *Jordan left derivable at C* if $\delta(A \circ B) = 2A\delta(B) + 2B\delta(A)$ for each A, B in $\mathcal{A}$ satisfying $A \circ B = C$.

The concepts of left derivations and Jordan left derivations were introduced by Bresar and Vukman in [5]. For results concerning the relationship between left derivations and Jordan left derivations on prime rings, we refer the reader to [4, 5, 14, 15]. It's natural that every (Jordan) left derivation is (Jordan) left derivable at each point. There have been a number of papers concerning the study of conditions under which (Jordan) derivations can be completely determined by the action on some sets of points [1–3, 6, 7, 10, 12, 16, 17]. Using the techniques of researching (Jordan) derivations, several authors were devoted to study linear (or additive) mappings on some algebras behaving like (Jordan) left derivations when acting on special products. In [13], Li and Zhou study left derivable mappings at zero on some algebras. In [8, 9], the authors considered a continuous mapping δ satisfying $A\delta(A^{-1}) + A^{-1}\delta(A) = 0$ for each invertible element A in von Neumann algebras or Banach algebras. In [9, 11, 13], the authors characterized continuous (Jordan) left derivable mappings at the identity or non-trivial idempotents on some algebras. In this paper, we study (Jordan) left derivable mappings at zero or non-zero central elements.

The paper is organized as follows. In Section 2, we characterize linear mappings (Jordan) left derivable at non-zero central elements without continuity assumption. Let δ be a linear mapping from a unital Banach algebra $\mathcal{A}$ into a unital left $\mathcal{A}$-module $\mathcal{M}$, and C be a non-zero element in $\mathcal{A}$. If δ is left derivable at C, then $C \cdot \delta$ is a Jordan left derivation. When $C \in \mathcal{Z}(\mathcal{A})$, we prove that if δ is Jordan left derivable at C, or if $A\delta(B) + B\delta(A) = \delta(C)$ for each A, B in $\mathcal{A}$ with $AB = BA = C$, then $C \cdot \delta$ is also a Jordan left derivation. Let $W \in \mathcal{Z}(\mathcal{A})$ be a left separating point of $\mathcal{M}$. As applications of the preceding results, we conclude that δ is a Jordan left derivation if δ is (Jordan) left derivable at W, or if $A\delta(B) + B\delta(A) = \delta(W)$ for each A, B in $\mathcal{A}$ with $AB = BA = W$, which generalizes the corresponding results in [8, 9, 11].

In Section 3, we consider the relations between generalized (Jordan) left derivations (see Definition 3.2) and (Jordan) left derivable mappings at zero. Let $\mathcal{A}$ be a unital Banach algebra with property ($\mathbb{B}$) (see Definition 3.1), and δ be a continuous linear operator from $\mathcal{A}$ into a Banach left $\mathcal{A}$-module $\mathcal{M}$. Then δ is a generalized Jordan left derivation if and only if δ is Jordan left derivable at zero. In addition, if there exists an element $C \in \mathcal{Z}(\mathcal{A})$ which is a left separating point of $\mathcal{M}$, and $Rann_{\mathcal{M}}(\mathcal{A}) = \{0\}$, then δ is a generalized left derivation if and only if δ is left derivable at zero.

2. (Jordan) left derivable mappings at non-zero central elements

In this section, we suppose that $\mathcal{A}$ is a Banach algebra with identity I and $\mathcal{M}$ is a unital left $\mathcal{A}$-module, unless stated otherwise. We consider linear mappings (Jordan) left derivable at non-zero central elements without continuity assumption.

At first, we considered a linear mapping δ satisfying $A\delta(A^{-1}) + A^{-1}\delta(A)$ $= 0$ for each invertible element A in $\mathcal{A}$. Fadaee and Ghahramani [9] also consider it when δ is continuous. In this paper, we use a different technique to consider it while δ is not necessarily continuous. Replacing [9, Lemma 2.1] with the following Lemma 2.1, we easily know that assumption of continuity of δ is not necessary in [9, Proposition 2.2 and Theorem 2.5].

Lemma 2.1. *Let $\delta : \mathcal{A} \to \mathcal{M}$ be a linear mapping. If for each invertible element A in $\mathcal{A}$, we have*

$$A\delta(A^{-1}) + A^{-1}\delta(A) = \delta(I),$$

then δ is a Jordan left derivation.

Proof. By assumption, we have $\delta(I) + \delta(I) = \delta(I)$. Thus, $\delta(I) = 0$. Let A be invertible in $\mathcal{A}$. By $A\delta(A^{-1}) + A^{-1}\delta(A) = \delta(I) = 0$, it follows that

$$(2.1) \qquad\qquad\qquad \delta(A) = -A^2\delta(A^{-1}).$$

Let $T \in \mathcal{A}$, $n \in \mathbb{N}^+$ with $n \geq \|T\| + 1$, and $B = nI + T$. Then B and $(I - B)$ are both invertible in $\mathcal{A}$. By (2.1),

$$\begin{aligned}
\delta(B) = & -B^2\delta(B^{-1}) = -B^2\delta(B^{-1}(I - B)^2 - B) \\
= & -B^2\delta(B^{-1}(I - B)^2) + B^2\delta(B) \\
= & B^2(B^{-1}(I - B)^2)^2\delta((I - B)^{-2}B) + B^2\delta(B) \\
= & (I - B)^4\delta((I - B)^{-2}) - (I - B)^2(I - B)^2\delta((I - B)^{-1}) + B^2\delta(B) \\
= & -\delta((I - B)^2) + (I - B)^2\delta(I - B) + B^2\delta(B) \\
= & \delta(B) - \delta(B^2) + 2B\delta(B).
\end{aligned}$$

Thus, $\delta(B^2) = 2B\delta(B)$. Since $B = nI + T$, we have $\delta((nI + T)^2) = 2(nI + T)\delta(nI + T)$, and so $\delta(T^2) = 2T\delta(T)$ for each T in $\mathcal{A}$. That is, δ is a Jordan left derivation. $\qquad\square$

Lemma 2.2. *Let $\delta : \mathcal{A} \to \mathcal{M}$ be a linear mapping, and let C be a non-zero element in $\mathcal{A}$. If for each invertible element A in $\mathcal{A}$, we have*

$$A\delta(A^{-1}C) + A^{-1}C\delta(A) = \delta(C),$$

then $C \cdot \delta$ is a Jordan left derivation.

Proof. By assumption, we have $\delta(C) + C\delta(I) = \delta(C)$. That is, $C\delta(I) = 0$. Let A be invertible in $\mathcal{A}$. By $A\delta(A^{-1}C) + A^{-1}C\delta(A) = \delta(C)$, it follows that

(2.2) $$A^{-1}C\delta(A) = \delta(C) - A\delta(A^{-1}C),$$

(2.3) $$\delta(A^{-1}C) = A^{-1}\delta(C) - A^{-2}C\delta(A).$$

Let $T \in \mathcal{A}$, $n \in \mathbb{N}^+$ with $n \geq \|T\| + 1$, and $B = nI + T$. Then B and $(I - B)$ are both invertible in $\mathcal{A}$. By (2.2) and (2.3),

$$\begin{aligned}
B^{-1}C\delta(B) =&\delta(C) - B\delta(B^{-1}C) = \delta(C) - B\delta(B^{-1}(I - B)C + C)\\
=&(I - B)\delta(C) - B\delta(B^{-1}(I - B)C)\\
=&(I - B)\delta(C) - B[B^{-1}(I - B)\delta(C) - (B^{-1}(I - B))^2 C\delta((I - B)^{-1}B)]\\
=&(I - B)B^{-1}(I - B)C\delta((I - B)^{-1})\\
=&(I - B)B^{-1}[\delta(C) - (I - B)^{-1}\delta((I - B)C)]\\
=&-\delta(C) + B^{-1}\delta(BC).
\end{aligned}$$

Thus, $B^{-1}\delta(BC) = \delta(C) + B^{-1}C\delta(B)$. Hence,

$$\delta(BC) = B \cdot B^{-1}\delta(BC) = B\delta(C) + C\delta(B).$$

Since $B = nI + T$, $\delta(nC + TC) = (nI + T)\delta(C) + C\delta(nI + T)$, that is,

(2.4) $$\delta(TC) = T\delta(C) + C\delta(T)$$

for each T in $\mathcal{A}$. Hence, for each invertible element A in $\mathcal{A}$, by (2.4),

$$\begin{aligned}
\delta(C) =&A\delta(A^{-1}C) + A^{-1}C\delta(A)\\
=&AA^{-1}\delta(C) + AC\delta(A^{-1}) + A^{-1}C\delta(A)\\
=&\delta(C) + AC\delta(A^{-1}) + A^{-1}C\delta(A),
\end{aligned}$$

that is, $AC\delta(A^{-1}) + A^{-1}C\delta(A) = 0 = C\delta(I)$ for each invertible element A in $\mathcal{A}$. Let $\tilde{\delta} = C \cdot \delta$. By Lemma 2.1, $\tilde{\delta}$ is a Jordan left derivation. $\square$

Theorem 2.3. *Let $\delta : \mathcal{A} \to \mathcal{M}$ be a linear mapping, and let C be a non-zero element in $\mathcal{A}$. If δ is left derivable at C, then $C \cdot \delta$ is a Jordan left derivation.*

Proof. For each invertible element A in $\mathcal{A}$, we have $\delta(C) = \delta(A \cdot (A^{-1}C)) = A\delta(A^{-1}C) + A^{-1}C\delta(A)$. Thus, by Lemma 2.2, $C \cdot \delta$ is a Jordan left derivation. $\square$

Theorem 2.4. *Let $\delta : \mathcal{A} \to \mathcal{M}$ be a linear mapping, and let C be a non-zero element in $\mathcal{Z}(\mathcal{A})$. If δ satisfies one of the following conditions, then $C \cdot \delta$ is a Jordan left derivation:*

    (i) *δ is Jordan left derivable at C;*
    (ii) *$A\delta(B) + B\delta(A) = \delta(C)$ for each A, B in $\mathcal{A}$ with $AB = BA = C$.*

Proof. Suppose that (i) holds, i.e., $2A\delta(B) + 2B\delta(A) = \delta(C)$ for each A, B in $\mathcal{A}$ with $A \circ B = C$. For each invertible element A in $\mathcal{A}$, since $C \in \mathcal{Z}(\mathcal{A})$ and $2 \cdot \delta(C) = \delta(A \cdot (A^{-1}C) + (A^{-1}C) \cdot A)$, we have $\delta(C) = A\delta(A^{-1}C) + A^{-1}C\delta(A)$. Thus, by Lemma 2.2, $C \cdot \delta$ is a Jordan left derivation.

Now we suppose that (ii) holds. For each invertible element A in $\mathcal{A}$, since $C \in \mathcal{Z}(\mathcal{A})$ and $A \cdot (A^{-1}C) = (A^{-1}C) \cdot A = C$, we have $\delta(C) = A\delta(A^{-1}C) + A^{-1}C\delta(A)$. By Lemma 2.2, $C \cdot \delta$ is a Jordan left derivation. $\qquad\square$

As applications of Theorems 2.3 and 2.4, we conclude the following results.

Theorem 2.5. *Let $\delta : \mathcal{A} \to \mathcal{M}$ be a linear mapping, and let $W \in \mathcal{Z}(\mathcal{A})$ be a left separating point of $\mathcal{M}$. Then the following conditions are equivalent:*

    (i) *δ is a Jordan left derivation,*
    (ii) *δ is left derivable at W,*
    (iii) *δ is Jordan left derivable at W,*
    (iv) *$A\delta(B) + B\delta(A) = \delta(W)$ for each A, B in $\mathcal{A}$ with $AB = BA = W$.*

Proof. (i)$\Rightarrow$(iii) and (ii)$\Rightarrow$(iv) are obvious. We only need to prove (i)$\Rightarrow$(ii), (ii)$\Rightarrow$(i), (iii)$\Rightarrow$(i), and (iv)$\Rightarrow$(i).

Suppose that δ satisfies (ii), (iii), or (iv). By Theorems 2.3 and 2.4, $W \cdot \delta$ is a Jordan left derivation. For each A in $\mathcal{A}$, $W\delta(A^2) = 2AW\delta(A)$. Since $W \in \mathcal{Z}(\mathcal{A})$, we have $W\delta(A^2) = W \cdot 2A\delta(A)$. Since W is a left separating point of $\mathcal{M}$, we obtain $\delta(A^2) = 2A\delta(A)$ for each A in $\mathcal{A}$. That is, (i) holds.

Now we suppose that (i) holds. For each B in $\mathcal{A}$, since $W \in \mathcal{Z}(\mathcal{A})$, we have $BW = WB$ and

$$\delta(BW) = \delta(WB) = B\delta(W) + W\delta(B).$$

On the other hand, for each A, B in $\mathcal{A}$ with $AB = W$, we have

$$\delta(BW) = \delta(BAB) = B^2\delta(A) + 3BA\delta(B) - AB\delta(B).$$

Thus,

$$B\delta(AB) + 2AB\delta(B) = B^2\delta(A) + 3BA\delta(B)$$

for each A, B in $\mathcal{A}$ with $AB = W$. Multiply the equation by A at the left side,

$$AB\delta(AB) + 2AAB\delta(B) = AB^2\delta(A) + 3ABA\delta(B).$$

Since $W = AB \in \mathcal{Z}(\mathcal{A})$, we have $W\delta(AB) = WB\delta(A) + WA\delta(B)$. And since W is a left separating point of $\mathcal{M}$, we obtain $\delta(AB) = B\delta(A) + A\delta(B)$ for each A, B in $\mathcal{A}$ with $AB = W$, i.e., (ii) holds. $\qquad\square$

Remark 2.6. In Theorem 2.5, if δ is a Jordan left derivation, then $\delta(BW + WB) = 2B\delta(W) + 2W\delta(B)$ for each B in $\mathcal{A}$. Since $W \in \mathcal{Z}(\mathcal{A})$, we have $BW = WB$ and $\delta(BW) = \delta(WB) = B\delta(W) + W\delta(B)$. Thus, Theorem 2.5 improves the result of [8, Theorem 1.1].

Corollary 2.7. *Let $\delta : \mathcal{A} \to \mathcal{M}$ be a linear mapping. Then the following conditions are equivalent:*

(i) δ *is a Jordan left derivation,*

(ii) δ *is left derivable at* I,

(iii) δ *is Jordan left derivable at* I,

(iv) $A\delta(A^{-1}) + A^{-1}\delta(A) = \delta(I)$ *for each invertible element* A *in* $\mathcal{A}$.

Corollary 2.8. *Let* $\delta : \mathcal{A} \to \mathcal{M}$ *be a linear mapping, and* X, Y *be elements in* $\mathcal{A}$ *with* $X + Y = I$. *If* δ *is left derivable at* X *and* Y, *then* δ *is a Jordan left derivation.*

In addition, when $X, Y \in \mathcal{Z}(\mathcal{A})$, *then* δ *is a Jordan left derivation if and only if* δ *is Jordan left derivable at* X *and* Y.

By [13, 15], if $\mathcal{A}$ is a CSL algebra or a unital semisimple Banach algebra, then every continuous Jordan left derivation from $\mathcal{A}$ into itself is zero. So from Theorems 2.3 and 2.4 we have the following corollaries.

Corollary 2.9. *Let* $\mathcal{A}$ *be a CSL algebra or a unital semisimple Banach algebra, and* δ *be a continuous linear mapping from* $\mathcal{A}$ *into itself. If* δ *is left derivable at* I, *or if* δ *is Jordan left derivable at* I, *then* $\delta = 0$.

Corollary 2.10. *Let* $\mathcal{A}$ *be as in Corollary* 2.9, δ *be a continuous linear mapping from* $\mathcal{A}$ *into itself, and* $W \in \mathcal{A}$ *be a left separating point of* $\mathcal{A}$. *If* δ *is left derivable at* W, *then* $\delta = 0$.

In addition, if $W \in \mathcal{Z}(\mathcal{A})$ *and if* δ *is Jordan left derivable at* W, *then* $\delta = 0$.

Remark 2.11. Let $\mathcal{A}$ be a unital Banach algebra, $\mathcal{M}$ be a unital Banach left $\mathcal{A}$-module, and $W \in \mathcal{Z}(\mathcal{A})$ be a left separating point of $\mathcal{M}$. Fadaee and Ghahramani [9] proved that a continuous linear mapping $\delta : \mathcal{A} \to \mathcal{M}$ is a Jordan left derivation if $A\delta(A^{-1}) + A^{-1}\delta(A) = 0$ for each invertible element A in $\mathcal{A}$, or if δ is Jordan left derivable at I. Ghahramani [11] proved that every continuous linear mapping $\delta : \mathcal{A} \to \mathcal{M}$ is a Jordan left derivation if δ is left derivable at I. Ebadiana and Gordji [8] proved that a bounded linear mapping $\delta : \mathcal{A} \to \mathcal{M}$ is a Jordan left derivation if $\delta(AB) = A\delta(B) + B\delta(A)$ for each A, B in $\mathcal{A}$ with $AB = BA = W$. In this section, we improve the results in [8, 9, 11] without assumption that δ is bounded or continuous. In fact, we only assume that $\mathcal{M}$ is a unital left $\mathcal{A}$-module. Correspondingly, we conclude that every linear mapping $\delta : \mathcal{A} \to \mathcal{M}$ is a Jordan left derivation if δ is (Jordan) left derivable at W, or if $A\delta(B) + B\delta(A) = \delta(W)$ for each A, B in $\mathcal{A}$ with $AB = BA = W$.

Let $\mathcal{R}$ be a 2-torsion free ring with identity I which satisfies that for each T in $\mathcal{R}$, there is some integer n such that $nI - T$ and $(n-1)I - T$ are invertible or $nI + T$ and $(n+1)I + T$ are invertible. If we replace $\mathcal{A}$ with $\mathcal{R}$ and replace linear mappings with additive mappings, then all of the above results in this section are still true.

3. (Jordan) left derivable mappings at zero

In this section, we consider continuous linear operators (Jordan) left derivable at zero. At first, we introduce a class of Banach algebras with property

($\mathbb{B}$). Let $\mathcal{A}$ be a Banach algebra and ϕ be a continuous bilinear mapping from $\mathcal{A} \times \mathcal{A}$ into a Banach space $\mathcal{X}$. We say that ϕ *preserves zero products* if for each A, B in $\mathcal{A}$, $AB = 0$ implies $\phi(A, B) = 0$. Then property ($\mathbb{B}$) is defined as follows.

Definition 3.1. [3] A Banach algebra $\mathcal{A}$ is said to have *property ($\mathbb{B}$)*, if for every Banach space $\mathcal{X}$ and every continuous bilinear mapping $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{X}$, ϕ preserving zero products implies that for each A, B, C in $\mathcal{A}$,

$$\phi(AB, C) = \phi(A, BC).$$

The class of Banach algebras with property ($\mathbb{B}$) includes C^*-algebras, group-algebras, unitary algebras, and Banach algebras generated by idempotents.

Definition 3.2. Let δ be a linear operator from a unital Banach algebra $\mathcal{A}$ into a Banach left $\mathcal{A}$-module $\mathcal{M}$. Then δ is called

(i) a *generalized left derivation*, if there exists an element ξ in $\mathcal{M}$, such that for each A, B in $\mathcal{A}$,

$$\delta(AB) = A\delta(B) + B\delta(A) - AB\xi.$$

(ii) a *generalized Jordan left derivation*, if there exists an element ξ in $\mathcal{M}$, such that for each A, B in $\mathcal{A}$,

$$\delta(A \circ B) = 2A\delta(B) + 2B\delta(A) - (A \circ B)\xi.$$

Remark 3.3. If δ is a generalized left derivation, then there exists an operator $d : \mathcal{A} \to \mathcal{M}$ satisfying $d(A) = \delta(A) - A\xi$ for each A in $\mathcal{A}$. We denote that $R_\xi(A) = A\xi$ for each A in $\mathcal{A}$, then $\delta = d + R_\xi$. We can not confirm that d is a left derivation since $\mathcal{A}$ is not necessarily commutative. But if δ is a generalized Jordan left derivation, and $d : \mathcal{A} \to \mathcal{M}$ is an operator satisfying $\delta = d + R_\xi$, then d is a Jordan left derivation.

We consider the following conditions on a continuous linear operator δ from a unital Banach algebra $\mathcal{A}$ into a Banach left $\mathcal{A}$-module $\mathcal{M}$:

(D1) for each A, B, C in $\mathcal{A}$, $AB = BC = 0$ implies $AC\delta(B) = 0$;
(D2) for each A, B in $\mathcal{A}$, $AB = 0$ implies $A\delta(B) + B\delta(A) = 0$;
(D3) for each A, B in $\mathcal{A}$, $AB = BA = 0$ implies $A\delta(B) + B\delta(A) = 0$;
(D4) for each A, B in $\mathcal{A}$, $A \circ B = 0$ implies $A\delta(B) + B\delta(A) = 0$.

Theorem 3.4. *Let $\mathcal{A}$ be a Banach algebra with identity I and have property ($\mathbb{B}$), $\mathcal{M}$ be a Banach left $\mathcal{A}$-module with $Rann_\mathcal{M}(\mathcal{A}) = \{0\}$, and $\delta : \mathcal{A} \to \mathcal{M}$ be a continuous linear operator. If δ satisfies (D1) or (D2), then for each C in $\mathcal{A}$, $\tilde{\delta} = C \cdot \delta$ is a generalized left derivation.*

Furthermore, $\tilde{\delta}$ is a left derivation if and only if $I \cdot \tilde{\delta}(I) = 0$.

In addition, if there exists an element $C \in \mathcal{Z}(\mathcal{A})$ which is a left separating point of $\mathcal{M}$, then the following conditions are equivalent:

(i) *δ is a generalized left derivation,*

(ii) δ *satisfies (D1),*
(iii) δ *satisfies (D2).*

Theorem 3.5. *Let $\mathcal{A}$ be a Banach algebra with identity I and have property ($\mathbb{B}$), $\mathcal{M}$ be a Banach left $\mathcal{A}$-module, and $\delta : \mathcal{A} \to \mathcal{M}$ be a continuous linear operator. Then the following conditions are equivalent:*

 (i) δ *is a generalized Jordan left derivation,*
(ii) δ *satisfies (D3),*
(iii) δ *satisfies (D4).*

Furthermore, δ is a Jordan left derivation if and only if $I \cdot \delta(I) = 0$.

We will complete the proofs of Theorems 3.4 and 3.5 by showing the following lemmas.

Lemma 3.6. *Let $\mathcal{A}$ and $\mathcal{M}$ be as in Theorem 3.4, and $\delta : \mathcal{A} \to \mathcal{M}$ be a continuous linear operator. Then δ satisfies (D1) if and only if for each C in $\mathcal{A}$, $C \cdot \delta$ is a generalized left derivation.*

Proof. If δ satisfies (D1), then for each A', B' in $\mathcal{A}$ with $A'B' = 0$, we define the bilinear mapping $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ by

$$\phi(A, B) = AB'\delta(BA'), A, B \in \mathcal{A}.$$

For each A, B in $\mathcal{A}$ with $AB = 0$, since $ABA' = BA'B' = 0$, we have $\phi(A, B) = 0$. Since $\mathcal{A}$ has property ($\mathbb{B}$), for each A, B, C in $\mathcal{A}$, $\phi(AB, C) = \phi(A, BC)$, i.e.,

$$(3.1) \qquad\qquad ABB'\delta(CA') = AB'\delta(BCA').$$

Now we fix A, B, C in $\mathcal{A}$, and consider the bilinear mapping $\psi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ such that for each A', B' in $\mathcal{A}$,

$$\psi(A', B') = ABB'\delta(CA') - AB'\delta(BCA').$$

On account of (3.1), $\psi(A', B') = 0$ for each A', B' in $\mathcal{A}$ with $A'B' = 0$. Since $\mathcal{A}$ has property ($\mathbb{B}$), it follows that $\psi(A'B', C') = \psi(A', B'C')$ for each A', B', C' in $\mathcal{A}$. Thus, for each A', B', C', A, B, C in $\mathcal{A}$,

$$ABC'\delta(CA'B') - AC'\delta(BCA'B') = ABB'C'\delta(CA') - AB'C'\delta(BCA').$$

Put $\tilde{\delta} = C'\delta$, where C' as in $\mathcal{A}$. Take $C = A' = I$ and $\xi = \tilde{\delta}(I)$, then for each B', A, B in $\mathcal{A}$,

$$A(\tilde{\delta}(BB') - B\tilde{\delta}(B') - B'\tilde{\delta}(B) + BB'\xi) = 0.$$

Since $Rann_{\mathcal{M}}(\mathcal{A}) = \{0\}$, for each B', B in $\mathcal{A}$, we have

$$\tilde{\delta}(BB') = B\tilde{\delta}(B') + B'\tilde{\delta}(B) - BB'\xi.$$

Thus, $\tilde{\delta}$ is a generalized left derivation.

If for each C in $\mathcal{A}$, $C \cdot \delta$ is a generalized left derivation, then for each A, B, C in $\mathcal{A}$ with $AB = BC = 0$, we have

$$0 = C\delta(AB) = AC\delta(B) + BC\delta(A) - AB\xi = AC\delta(B).$$

Hence, $AC\delta(B) = 0$ for each A, B, C in $\mathcal{A}$ with $AB = BC = 0$. □

Lemma 3.7. *Let $\mathcal{A}$ and $\mathcal{M}$ be as in Theorem 3.4, and $\delta : \mathcal{A} \to \mathcal{M}$ be a continuous linear operator. If δ satisfies (D2), then for each C in $\mathcal{A}$, $C \cdot \delta$ is a generalized left derivation.*

Proof. Take A, B, C in $\mathcal{A}$ satisfying $AB = BC = 0$, then

$$0 = A \cdot (B\delta(C) + C\delta(B)) = AB\delta(C) + AC\delta(B) = AC\delta(B).$$

For each C in $\mathcal{A}$, let $\tilde{\delta} = C \cdot \delta$. By Lemma 3.6, $\tilde{\delta}$ is a generalized left derivation.
□

Remark 3.8. In Lemma 3.7, if there exists an element $C \in \mathcal{Z}(\mathcal{A})$ and C is a left separating point of $\mathcal{M}$, then δ satisfies (D2) if and only if δ is a generalized left derivation.

Lemma 3.9. *Let $\mathcal{A}$ and $\mathcal{M}$ be as in Theorem 3.5, and $\delta : \mathcal{A} \to \mathcal{M}$ be a continuous linear operator. Then δ satisfies (D3) if and only if δ is a generalized Jordan left derivation.*

Proof. If δ satisfies (D3), then for each A', B' in $\mathcal{A}$ with $A'B' = 0$, we define the bilinear mapping $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ by

$$\phi(A, B) = B'A\delta(BA') + BA'\delta(B'A)$$

for each A, B in $\mathcal{A}$. For each A, B in $\mathcal{A}$ with $AB = 0$, since $B'ABA' = BA'B'A = 0$, we have $\phi(A, B) = 0$. Since $\mathcal{A}$ has property ($\mathbb{B}$), for each A, B, C in $\mathcal{A}$, $\phi(AB, C) = \phi(A, BC)$, i.e.,

$$(3.2) \qquad B'AB\delta(CA') + CA'\delta(B'AB) = B'A\delta(BCA') + BCA'\delta(B'A).$$

Now we fix A, B, C in $\mathcal{A}$, and consider the bilinear mapping $\psi : \mathcal{A} \times \mathcal{A} \to \mathcal{M}$ such that for each A', B' in $\mathcal{A}$,

$$\psi(A', B') = B'AB\delta(CA') + CA'\delta(B'AB) - B'A\delta(BCA') - BCA'\delta(B'A).$$

For each A', B' in $\mathcal{A}$ with $A'B' = 0$, on account of (3.2), $\psi(A', B') = 0$. Since $\mathcal{A}$ has property ($\mathbb{B}$), it follows that for each A', B', C' in $\mathcal{A}$, $\psi(A'B', C') = \psi(A', B'C')$. Thus, for each A', B', C', A, B, C in $\mathcal{A}$,

$$C'AB\delta(CA'B') + CA'B'\delta(C'AB) - C'A\delta(BCA'B') - BCA'B'\delta(C'A)$$
$$= B'C'AB\delta(CA') + CA'\delta(B'C'AB) - B'C'A\delta(BCA') - BCA'\delta(B'C'A).$$

Take $A = C = A' = C' = I$ and $\xi = \delta(I)$, then for each B, B' in $\mathcal{A}$,

$$\delta(B \circ B') = 2B\delta(B') + 2B'\delta(B) - (B \circ B')\xi.$$

Thus, δ is a generalized Jordan left derivation.

If δ is a generalized Jordan left derivation, it is obvious that for each A, B in $\mathcal{A}$ with $AB = BA = 0$, we have

$$0 = \delta(AB) + \delta(BA) = \delta(A \circ B) = 2A\delta(B) + 2B\delta(A) - (A \circ B)\xi = 2(A\delta(B) + B\delta(A)).$$

Thus, $A\delta(B) + B\delta(A) = 0$ for each A, B in $\mathcal{A}$ with $AB = BA = 0$. $\qquad\square$

Lemma 3.10. *Let $\mathcal{A}$ and $\mathcal{M}$ be as in Theorem 3.5, and $\delta : \mathcal{A} \to \mathcal{M}$ be a continuous linear operator. Then δ satisfies (D4) if and only if δ is a generalized Jordan left derivation.*

Proof. If δ satisfies (D4), we consider A, B in $\mathcal{A}$ with $AB = BA = 0$, then $A \circ B = 0$. With property (D4), we have $A\delta(B) + B\delta(A) = 0$, and δ satisfies (D3). By Lemma 3.9, δ is a generalized Jordan left derivation.

And if δ is a generalized Jordan left derivation, it is obvious that for each A, B in $\mathcal{A}$ with $A \circ B = 0$, we have

$$0 = \delta(A \circ B) = 2A\delta(B) + 2B\delta(A) - (A \circ B)\xi = 2(A\delta(B) + B\delta(A)).$$

Thus, $A\delta(B) + B\delta(A) = 0$ for each A, B in $\mathcal{A}$ with $A \circ B = 0$. $\qquad\square$

Remark 3.11. If $\mathcal{A}$ is an arbitrary Banach algebra with property ($\mathbb{B}$) (without unital assumption), we can obtain similar results as Theorems 3.4 and 3.5, and use similar techniques to prove them. But the proofs are complicated. For brief, in this paper, we only consider the case that $\mathcal{A}$ is unital.

Acknowledgements

References

[1] J. Alaminos, J. Extremera, A.R. Villena and M. Bresar, Characterizing homomorphisms and derivations on C*-algebras, *Proc. Roy. Soc. Edinburgh Sect. A* **137** (2007) 1–7.

[2] J. Alaminos, M. Bresar, J. Extremera and A.R. Villena, Characterizing Jordan maps on C*-algebras through zero products, *Proc. Edinb. Math. Soc. (2)* **53** (2010) 543–555.

[3] J. Alaminos, M. Bresar, J. Extremera and A.R. Villena, Maps preserving zero products, *Studia Math.* **193** (2009) 131–159.

[4] M. Ashraf, N. Rehman and S. Ali, On Jordan left derivations of Lie ideals in prime rings, *Southeast Asian Bull. Math.* **25** (2002) 379–382.

[5] M. Brešar and J. Vukman, On left derivations and related mappings, *Proc. Amer. Math. Soc.* **110** (1990) 7–16.

[6] M.A. Chebotar, W. Ke and P. Lee, Maps characterrized by action on zero products, *Pacific J. Math.* **216** (2004) 217–228.

[7] M.A. Chebotar, W. Ke, P. Lee and R. Zhang, On maps preserving zero Jordan products, *Monatsh. Math.* **149** (2006) 91–101.

[8] A. Ebadiana and M. Eshaghi Gordji, Left Jordan derivations on Banach algebras, *Iran. J. Math. Sci. Inform.* **6** (2011) 1–6.

[9] B. Fadaee and H. Ghahramani, Continuous linear maps on reflexive algebras behaving like Jordan left derivations at idempotent-product elements, ArXiv:1312.6953 [math.OA] (2014).

[10] H. Ghahramani, Additive mappings derivable at nontrivial idempotents on Banach algebras, *Linear Multilinear Algebra* **60** (2012) 725–742.

[11] H. Ghahramani, Characterizations of left derivable maps at non-trivial idempotents on nest algebras, ArXiv:1312.6959 [math. OA] (2014).

[12] J. Hou and L. Zhao, Zero-product preserving additive maps on symmetric operator spaces and self-adjoint operator spaces, *Linear Algebra Appl.* **399** (2005) 235–244.

[13] J. Li and J. Zhou, Jordan left derivations and some left derivable maps, *Oper. Matrices* **4** (2010), no. 1, 127–138.

[14] J. Vukman, Jordan left derivations on semiprime rings, *Math. J. Okayama Univ.* **39** (1997) 1–6.

[15] J. Vukman, On left Jordan derivations of rings and Banach algebras, *Aequationes Math.* **75** (2008), no. 3, 260–266.

[16] J. Zhu and C. Xiong, Generalized derivable mappings at the point zero on nest algebras, *Acta Math. Sinica (Chin. Ser.)* **45** (2002), no. 4, 783–788.

[17] J. Zhu and C. Xiong, Generalized derivable mappings at zero point on some reflexive operator algebras, *Linear Algebra Appl.* **397** (2005) 367–379.

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