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THE STEINER DIAMETER OF A GRAPH

Y. MAO

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Abstract. The Steiner distance of a graph, introduced by Chartrand, Oellermann, Tian and Zou in 1989, is a natural generalization of the concept of classical graph distance. For a connected graph G of order at least 2 and $S \subseteq V(G)$, the *Steiner distance* $d(S)$ among the vertices of S is the minimum size among all connected subgraphs whose vertex sets contain S. Let n, k be two integers with $2 \leq k \leq n$. Then the *Steiner k-eccentricity* $e_k(v)$ of a vertex v of G is defined by $e_k(v) = \max\{d(S) \,|\, S \subseteq V(G),\ |S| = k,\ and\ v \in S\}$. Furthermore, the *Steiner k-diameter* of G is $sdiam_k(G) = \max\{e_k(v) \,|\, v \in V(G)\}$. In 2011, Chartrand, Okamoto and Zhang showed that $k-1 \leq sdiam_k(G) \leq n-1$. In this paper, graphs with $sdiam_3(G) = 2, 3, n-1$ are characterized, respectively. We also consider the Nordhaus-Gaddum-type results for the parameter $sdiam_k(G)$. We determine sharp upper and lower bounds of $sdiam_k(G) + sdiam_k(\overline{G})$ and $sdiam_k(G) \cdot sdiam_k(\overline{G})$ for a graph G of order n. Some graph classes attaining these bounds are also given.

Keywords: Diameter, Steiner tree, Steiner k-diameter, complementary graph.
MSC(2010): Primary: 05C05; Secondary: 05C12, 05C76.

1. **Introduction**

All graphs in this paper are undirected, finite and simple. We refer to [5] for graph theoretical notation and terminology not described here. Distance is one of the most basic concepts of graph-theoretic subjects. If G is a connected graph and $u, v \in V(G)$, then the *distance* $d(u, v)$ between u and v is the length of a shortest path connecting u and v. If v is a vertex of a connected graph G, then the *eccentricity* $e(v)$ of v is defined by $e(v) = \max\{d(u, v) \,|\, u \in V(G)\}$. Furthermore, the *radius* $rad(G)$ and *diameter* $diam(G)$ of G are defined by $rad(G) = \min\{e(v) \,|\, v \in V(G)\}$ and $diam(G) = \max\{e(v) \,|\, v \in V(G)\}$. These last two concepts are related by the inequalities $rad(G) \leq diam(G) \leq 2rad(G)$. The *center* $C(G)$ of a connected graph G is the subgraph induced by the vertices

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u of G with $e(u) = rad(G)$. Goddard and Oellermann gave a survey on this subject, see [19].

The distance between two vertices u and v in a connected graph G also equals the minimum size of a connected subgraph of G containing both u and v. This observation suggests a generalization of the classical graph distance. The Steiner distance of a graph, introduced by Chartrand, Oellermann, Tian and Zou [8] in 1989, is a natural and nice generalization of the concept of classical graph distance. For a graph $G(V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (or simply, *an S-tree*) is a subgraph $T(V', E')$ of G that is a tree with $S \subseteq V'$. Let G be a connected graph of order at least 2 and let S be a nonempty set of vertices of G. Then the *Steiner distance* $d_G(S)$ among the vertices of S (or simply the distance of S) is the minimum size among all connected subgraphs whose vertex sets contain S. When there is no S-Steiner tree, we set $d_G(S) = \infty$ by convention. Note that if H is a connected subgraph of G such that $S \subseteq V(H)$ and $|E(H)| = d(S)$, then H is a tree. Clearly, $d(S) = \min\{e(T) \mid S \subseteq V(T)\}$, where T is subtree of G. Furthermore, if $S = \{u, v\}$, then $d(S) = d(u, v)$ is nothing new but the classical distance between u and v. Clearly, if $|S| = k$, then $d(S) \geq k - 1$. If G is the graph of Figure 1 (a) and $S = \{u, v, x\}$, then $d(S) = 4$. There are several trees of size 4 containing S. One such tree T is also shown in Figure 1 (b), see [8].

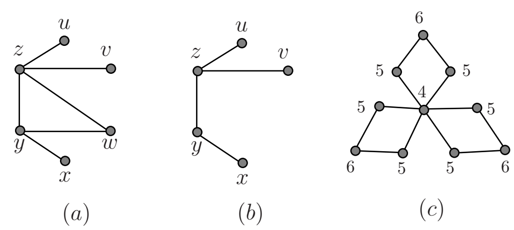


FIGURE 1. Graphs for the basic definition

Let n and k be two integers with $2 \leq k \leq n$. The *Steiner k-eccentricity* $e_k(v)$ of a vertex v of G is defined by $e_k(v) = \max\{d(S) \mid S \subseteq V(G), |S| = k, \textit{ and } v \in S\}$. The *Steiner k-radius* of G is $srad_k(G) = \min\{e_k(v) \mid v \in V(G)\}$, while the *Steiner k-diameter* of G is $sdiam_k(G) = \max\{e_k(v) \mid v \in V(G)\}$. Note for every connected graph G, $e_2(v) = e(v)$ for all vertices v of G, $srad_2(G) = rad(G)$ and $sdiam_2(G) = diam(G)$. Each vertex of the graph G of Figure 1 (c) is labeled with its Steiner 3-eccentricity, so that $srad_3(G) = 4$ and $sdiam_3(G) = 6$.

In [12], Dankelmann, Swart and Oellermann obtained an upper bound on $sdiam_k(G)$ for a graph G in terms of the order of G and the minimum degree of G, that is, $sdiam_k(G) \leq \frac{3n}{\delta+1} + 3k$. Recently, Ali, Dankelmann, Mukwembi [2] improved the bound of $sdiam_n(G)$ and showed that $sdiam_k(G) \leq \frac{3n}{\delta+1} + 2k - 5$, for all connected graphs G. Moreover, they constructed graphs to show that the bounds are asymptotically best possible.

The Steiner tree problem in networks, and particularly in graphs, was formulated in 1971 by Hakimi (see [20]) and Levi (see [24]). In the case of an unweighted, undirected graph, this problem consists of finding, for a subset of vertices S, a minimal-size connected subgraph that contains the vertices in S. The computational side of this problem has been widely studied, and it is known that it is an NP-hard problem for general graphs (see [21]). The determination of a Steiner tree in a graph is a discrete analogue of the well-known geometric Steiner problem: In a Euclidean space (usually a Euclidean plane) find the shortest possible network of line segments interconnecting a set of given points. Steiner trees have application to multiprocessor computer networks. For example, it may be desired to connect a certain set of processors with a subnetwork that uses the least number of communication links. A Steiner tree for the vertices, corresponding to the processors that need to be connected, corresponds to such a desired subnetwork. The problem of determining the Steiner distance is known to be NP-hard [17].

Let G be a k-connected graph and u, v be any pair of vertices of G. Let $P_k(u,v)$ be a family of k vertex-disjoint paths between u and v, i.e., $P_k(u,v) = \{p_1, p_2, \cdots, p_k\}$, where $p_1 \leq p_2 \leq \cdots \leq p_k$ and p_i denotes the number of edges of path p_i. The *k-distance* $d_k(u,v)$ between vertices u and v is the minimum $|p_k|$ among all $P_k(u,v)$ and the *k-diameter* $d_k(G)$ of G is defined as the maximum k-distance $d_k(u,v)$ over all pairs u, v of vertices of G. The concept of k-diameter emerges rather naturally when one looks at the performance of routing algorithms. Its applications to network routing in distributed and parallel processing are studied and discussed by various authors including Chung [10], Du et al. [14], Hsu [22,23], Meyer and Pradhan [16].

In the sequel, let $K_{s,t}$, K_n, P_n and C_n denote the complete bipartite graph of order $s+t$ with part sizes s and t, complete graph of order n, path of order n, and cycle of order n, respectively. The degree of a vertex v in G is denoted by $d_G(v)$. For $S \subseteq V(G)$, we denote $G - S$ the subgraph by deleting the vertices of S together with the edges incident with them from G. If $S = \{v\}$, we simply write $G - v$ for $G - \{v\}$. Let $N_G(v)$ denote the neighbors of the vertex v in G.

From the above definitions, the following observation is easily seen.

Observation 1.1. Let k, n be two integers with $2 \leq k \leq n$.

(1) For a complete graph K_n, $sdiam_k(K_n) = k - 1$;

(2) For a path P_n, $sdiam_k(P_n) = n - 1$;

(3) For a cycle C_n, $sdiam_k(C_n) = \lfloor \frac{n(k-1)}{k} \rfloor$.

In [9], Chartrand et al. derived the upper and lower bounds for $sdiam_k(G)$.

Proposition 1.2 ([9]). *Let k, n be two integers with $2 \leq k \leq n$, and let G be a connected graph of order n. Then*

$$k-1 \leq sdiam_k(G) \leq n-1.$$

Moreover, the bounds are sharp.

The following observation is immediate.

Observation 1.3. Let G be a connected graph of order n. Then
(1) $sdiam_2(G)=1$ if and only if G is a complete graph;
(2) $sdiam_2(G)=n-1$ if and only if G is a path of order n.

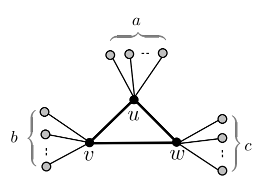
Let uv be an edge in G. A *double-star* on uv is a maximal tree in G which is the union of stars centered at u or v such that each star contains the edge uv. Bloom [4] characterized the graphs with $sdiam_2(G)=2$.

Theorem 1.4 ([4]). *Let G be a connected graph of order n. Then $sdiam_2(G)=2$ if and only if $\overline{G}$ is non-empty and $\overline{G}$ does not contain a double star of order n as its subgraph.*

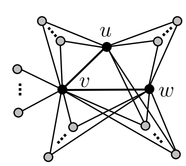
In this paper, we focus on the case $k=3$ and characterize the graphs with $sdiam_3(G)=2$ in Section 2, which can be seen as an extension of (1) of Observation 1.3.

Theorem 1.5. *Let G be a connected graph of order n. Then $sdiam_3(G)=2$ if and only if $0 \leq \Delta(\overline{G}) \leq 1$ if and only if $n-2 \leq \delta(G) \leq n-1$.*

We now define two graph classes. A *triple-star* H_1 is defined as a connected graph of order n obtained from a triangle and three stars $K_{1,a}, K_{1,b}, K_{1,c}$ by identifying the center of a star and one vertex of the triangle, where $0 \leq a \leq b \leq c$, $c \geq 1$ and $a+b+c=n-3$; see Figure 2 (a). Let H_2 be a connected graph of order n obtained from a path $P=uvw$ and $n-3$ vertices such that for each $x \in V(H_2)-\{u,v,w\}$, $xu, xv, xw \in E(H_2)$, or $xu, xv \in E(H_2)$ but $xw \notin E(H_2)$, or $xv, xw \in E(H_2)$ but $xu \notin E(H_2)$, or $xu, xw \in E(H_2)$ but $xv \notin E(H_2)$, or $xv \in E(H_2)$ but $xu, xw \notin E(H_2)$; see Figure 2 (b).



(a) H_1



(b) H_2

FIGURE 2. Graphs for Theorem 1.6

Graphs with $sdiam_3(G)=3$ are also characterized in Section 2, which can be seen as an extension of Theorem 1.4.

Theorem 1.6. *Let G be a connected graph of order n. Then $sdiam_3(G) = 3$ if and only if G satisfies the following conditions.*
- $\Delta(\overline{G}) \geq 2$;
- $\overline{G}$ *does not contain a triple-star H_1 as its subgraph;*
- $\overline{G}$ *does not contain H_2 as its subgraph.*

Denote by $T_{a,b,c}$ a tree with a vertex v of degree 3 such that $T_{a,b,c} - v = P_a \cup P_b \cup P_c$, where $0 \leq a \leq b \leq c$ and $1 \leq b \leq c$ and $a + b + c = n - 1$; see Figure 3 (a). Observe that $T_{0,b,c}$, where $b + c = n - 1$, is a path of order n. Denote by $\triangle_{p,q,r}$ a unicyclic graph containing a triangle K_3 and satisfying $\triangle_{p,q,r} - V(K_3) = P_p \cup P_q \cup P_r$, where $0 \leq p \leq q \leq r$ and $p+q+r = n-3$; see Figure 3 (b).

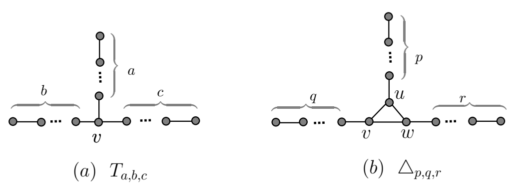


FIGURE 3. Graphs for Theorem 1.7

In Section 2, graphs with $sdiam_3(G) = n - 1$ are also characterized, which can be seen as an extension of (2) of Observation 1.3.

Theorem 1.7. *Let G be a connected graph of order n $(n \geq 3)$. Then $sdiam_3(G) = n-1$ if and only if $G = T_{a,b,c}$, where $a \geq 0$ and $1 \leq b \leq c$ and $a+b+c = n-1$, or $G = \triangle_{p,q,r}$, where $0 \leq p \leq q \leq r$ and $p + q + r = n - 3$.*

Let $\mathcal{G}(n)$ denote the class of simple graphs of order n. Give a graph theoretic parameter $f(G)$ and a positive integer n, the *Nordhaus-Gaddum Problem* is to determine sharp bounds for: (1) $f(G) + f(\overline{G})$ and (2) $f(G) \cdot f(\overline{G})$, as G ranges over the class $\mathcal{G}(n)$, and characterize the extremal graphs. The Nordhaus-Gaddum type relations have received wide investigations. Recently, Aouchiche and Hansen published a survey paper on this subject, see [3].

Xu [25] obtained the Nordhaus-Gaddum results for the Steiner 2-diameter of graphs. In Section 3, we obtain the Nordhaus-Gaddum results for the Steiner k-diameter of graphs.

Theorem 1.8. *Let $G \in \mathcal{G}(n)$ be a connected graph, and $\overline{G}$ be its connected complement. Let k be an integer with $3 \leq k \leq n$. Then*
(i) $2k - 1 - x \leq sdiam_k(G) + sdiam_k(\bar{G}) \leq \max\{n + k - 1, 4k - 2\}$;
(ii) $sdiam_n(G) \cdot sdiam_n(\overline{G}) = (n - 1)^2$,
where $x = 0$ if $n \geq 2k - 2$ and $x = 1$ if $n < 2k - 2$.

For $k = n, n-1, n-2, 3$, we improve the above Nordhaus-Gaddum results of Steiner k-diameter and obtain the following results.

Observation 1.9. Let $G \in \mathcal{G}(n)$ be a connected graph, and $\overline{G}$ be its connected complement. Let k be an integer with $3 \leq k \leq n$. Then
(i) $sdiam_n(G) + sdiam_n(\overline{G}) = 2n-2$;
(ii) $sdiam_n(G) \cdot sdiam_n(\overline{G}) = (n-1)^2$.

Akiyama and Harary [1] characterized the graphs for which G and $\overline{G}$ both have connectivity one.

Lemma 1.10 ([1])**.** *Let G be a graph with n vertices. Then $\kappa(G) = \kappa(\bar{G}) = 1$ if and only if G satisfies the following conditions.*
(i) *$\kappa(G) = 1$ and $\Delta(G) = n-2$;*
(ii) *$\kappa(G) = 1$, $\Delta(G) \leq n-3$ and G has a cut vertex v with pendant edge e and pendant vertex u such that $G - u$ contains a spanning complete bipartite subgraph.*

By Lemma 1.10, we obtain the following result.

Proposition 1.11. *Let $G \in \mathcal{G}(n)$ $(n \geq 5)$ be a connected graph with a connected complement $\overline{G}$. Then*
(i) $2n-4 \leq sdiam_{n-1}(G) + sdiam_{n-1}(\overline{G}) \leq 2n-2$;
(ii) $(n-2)^2 \leq sdiam_{n-1}(G) \cdot sdiam_{n-1}(\overline{G}) \leq (n-1)^2$.
Moreover,
(a) *$sdiam_{n-1}(G) + sdiam_{n-1}(\overline{G}) = 2n-4$ or $sdiam_{n-1}(G) \cdot sdiam_{n-1}(\overline{G}) = (n-2)^2$ if and only if both G and $\overline{G}$ are 2-connected;*
(b) *$sdiam_{n-1}(G) + sdiam_{n-1}(\overline{G}) = 2n-3$ or $sdiam_{n-1}(G) \cdot sdiam_{n-1}(\overline{G}) = (n-1)(n-2)$ if and only if $\lambda(G) = 1$ and $\overline{G}$ are 2-connected, or $\lambda(\overline{G}) = 1$ and G are 2-connected.*
(c) *$sdiam_{n-1}(G) + sdiam_{n-1}(\overline{G}) = 2n-2$ or $sdiam_{n-1}(G) \cdot sdiam_{n-1}(\overline{G}) = (n-1)^2$ if and only if G satisfies one of the following conditions.*
- *$\kappa(G) = 1$, $\Delta(G) = n-2$;*
- *$\kappa(G) = 1$, $\Delta(G) \leq n-3$ and G has a cut vertex v with pendant edge e and pendant vertex u such that $G - u$ contains a spanning complete bipartite subgraph.*

Proposition 1.12. *Let $G \in \mathcal{G}(n)$ $(n \geq 5)$ be a connected graph with a connected complement $\overline{G}$. If both G and $\overline{G}$ contains at least two cut vertices, then*
(i) $2n-6 \leq sdiam_{n-2}(G) + sdiam_{n-2}(\overline{G}) \leq 2n-2$;
(ii) $(n-3)^2 \leq sdiam_{n-2}(G) \cdot sdiam_{n-2}(\overline{G}) \leq (n-1)^2$.
Otherwise,
(iii) $2n-6 \leq sdiam_{n-2}(G) + sdiam_{n-2}(\overline{G}) \leq 2n-3$;
(iv) $(n-3)^2 \leq sdiam_{n-2}(G) \cdot sdiam_{n-2}(\overline{G}) \leq (n-1)(n-2)$.
Moreover, the upper and lower bounds are sharp.

For Steiner 3-diameter, we improve the result in Theorem 1.8 and prove the following result in Section 3.

Proposition 1.13. *Let $G \in \mathcal{G}(n)$ $(n \geq 10)$ be a connected graph with a connected complement $\overline{G}$. Then*
(i) $6 \leq sdiam_3(G) + sdiam_3(\overline{G}) \leq n+2$;
(ii) $9 \leq sdiam_3(G) \cdot sdiam_3(\overline{G}) \leq 3(n-1)$.
Moreover, the bounds are sharp.

2. Graphs with given Steiner 3-diameter

In this section, we characterize graphs with $sdiam_3(G) = 2, 3, n-1$ and give the proofs of Theorems 1.5, 1.6 and 1.7.

The following observation is easily seen.

Observation 2.1. If H is a spanning subgraph of G, then $sdiam_k(G) \leq sdiam_k(H)$.

When $G = T$ is a tree of order n, graphs attaining the upper bound of Proposition 1.2 can be characterized in the following, which will be used later.

Proposition 2.2. *Let k, n be two integers with $2 \leq k \leq n$, and let T be a tree of order n. Then $sdiam_k(T) = n-1$ if and only if $r \leq k$, where r is the number of leaves in T.*

Proof. Suppose $r \leq k$. Let $v_1, v_2, \cdots, v_r$ be all the leaves of T. Choose $S \subseteq V(T)$ and $|S| = k$ such that $v_1, v_2, \cdots, v_r \in S$. Then any S-Steiner tree must use all edges of T. Since $|E(T)| = n-1$, it follows that $d_T(S) \geq |E(T)| = n-1$ and hence $sdiam_k(T) \geq n-1$. Combining this with Proposition 1.2, we have $sdiam_k(T) = n-1$.

Conversely, suppose $sdiam_k(T) = n-1$. If $s \geq k+1$, then for any $S \subseteq V(G)$ with $|S| = k$, there exists a leaf v in T such that $v \notin S$. Let $T' = T - v$. Then T' is a S-Steiner tree and hence $d_T(S) \leq n-2$. From the arbitrariness of S, we have $sdiam_k(T) \leq n-2 < n-1$, a contradiction. So $s \leq k$. □

From Proposition 1.2, we have $k-1 \leq sdiam_k(G) \leq n-1$. We now show a property of the graphs attaining the lower bound.

Lemma 2.3. *Let n, k be two integers with $2 \leq k \leq n$, and let G be a connected graph of order n. If $sdiam_k(G) = k-1$, then $0 \leq \Delta(\overline{G}) \leq k-2$, namely, $n-k+1 \leq \delta(G) \leq n-1$.*

Proof. Suppose $\Delta(\overline{G}) \geq k-1$. Then there exists a vertex $u \in V(\overline{G})$ such that $d_{\overline{G}}(u) \geq k-1$. Pick up $v_1, v_2, \cdots, v_{k-1} \in N_{\overline{G}}(u)$. Let $S = \{u, v_1, v_2, \cdots, v_{k-1}\}$. Since $uv_i \in E(\overline{G})$ $(1 \leq i \leq k-1)$, it follows that $uv_i \notin E(G)$ and hence u is an isolated vertex in $G[S]$. Thus, any S-Steiner tree must use k edges of $E(G)$, which implies that $d_G(S) \geq k$. Therefore, $sdiam_k(G) \geq k$, a contradiction. So $0 \leq \Delta(\overline{G}) \leq k-2$, namely, $n-k+1 \leq \delta(G) \leq n-1$. □

Proof of Theorem 1.5. For Lemma 2.3, if $sdiam_3(G) = 2$, then $0 \leq \Delta(\overline{G}) \leq 1$. Conversely, if $0 \leq \Delta(\overline{G}) \leq 1$, then $n-2 \leq \delta(G) \leq n-1$. Thus, G is a graph obtained from the complete graph of order n by deleting some independent edges. For any $S = \{u, v, w\} \subseteq V(G)$, at least two elements in $\{uv, vw, uw\}$ belong to $E(G)$. Without loss of generality, let $uv, vw \in E(G)$. It is clear that the tree T induced by the edges in $\{uv, vw\}$ is an S-Steiner tree and hence $d_G(S) \leq 2$. From the arbitrariness of S, we have $sdiam_3(G) \leq 2$ and hence $sdiam_3(G) = 2$ by Proposition 1.2. □

Proof of Theorem 1.6. Suppose that G is a graph with $sdiam_3(G) = 3$. From Theorem 1.5, we have $\Delta(\overline{G}) \geq 2$. It suffices to prove the following two claims.

Claim 1. $\overline{G}$ does not contain a triple-star as its subgraph.

Assume, to the contrary, that $\overline{G}$ contains a triple-star H_1 as its subgraph. Choose $S = \{u, v, w\}$. Then $uv, uw, vw \in E(\overline{G})$ and hence $uv, uw, vw \notin E(G)$. For any $x \in V(G)-S$, one can see that $xu \notin E(G)$ or $xv \notin E(G)$ or $xw \notin E(G)$. Observe that any S-Steiner tree T must occupy at least one vertex of $V(G)-S$, say y. Then $yu \notin E(G)$ or $yv \notin E(G)$ or $yw \notin E(G)$. Without loss of generality, let $yu \notin E(G)$. Therefore, the tree T must occupy at least one vertex of $V(G) - \{u, v, w, y\}$. Thus the tree T contains at least 5 vertices in G, which implies that $d_G(S) \geq 4$ and hence $sdiam_3(G) \geq 4$, a contradiction. So $\overline{G}$ does not contain H_1 as its subgraph.

Claim 2. $\overline{G}$ does not contain H_2 as its subgraph.

Assume, to the contrary, that G contains H_2 as its subgraph. Choose $S = \{u, v, w\} \subseteq V(G)$. Since $uv, vw \in E(\overline{G})$, it follows that $uv, vw \notin E(G)$. It is clear that any S-Steiner tree T uses at least one vertex in $V(G) - S$. For each $x \in V(G) - S$, we have $xu, xv, xw \in E(\overline{G})$ or $xu, xv \in E(\overline{G})$ or $xv, xw \in E(\overline{G})$ or $xu, xw \in E(\overline{G})$ or $xv \in E(\overline{G})$, that is, $xu, xv, xw \notin E(G)$ or $xu, xv \notin E(G)$ or $xv, xw \notin E(G)$ or $xu, xw \notin E(G)$ or $xv \notin E(G)$. One can see that the tree T connecting S uses at least two vertices in $V(G)-S$. Therefore, $e(T) \geq 4$ and $d_G(S) \geq 4$, and hence $sdiam_3(G) \geq 4$, a contradiction. So $\overline{G}$ does not contain H_2 as its subgraph.

From the above arguments, we know that the result holds.

Conversely, suppose that G is a connected graph such that $\Delta(\overline{G}) \geq 2$ and $\overline{G}$ does not contain both H_1 and H_2 as its subgraph. From the definition of $sdiam_3(G)$, it suffices to show that $d_G(S) = 3$ for any $S \subseteq V(G)$. Set $S = \{u, v, w\}$. Then $0 \leq |E(G[S])| \leq 3$.

If $2 \leq |E(G[S])| \leq 3$, then there are two edges in $G[S]$ belonging to $E(G)$, say uv, vw. Therefore, the tree T induced by the edges in $\{uv, vw\}$ is an S-Steiner tree in G, and hence $d_G(S) = 2 < 3$, as desired.

Suppose $|E(G[S])| = 0$. Then $uv, vw, uw \notin E(G)$ and hence $uv, vw, uw \in E(\overline{G})$. Because $\overline{G}$ does not contain the subgraph H_1 as its subgraph, there exists a vertex $y \in V(G) - S$ such that $yu, yv, yw \notin E(\overline{G})$, which implies

$yu, yv, yw \in E(G)$. It is clear that the tree T induced by the edges in $\{yu, yv, yw\}$ is an S-Steiner tree in G and hence $d_G(S) \leq 3$, as desired.

Suppose $|E(G[S])| = 1$. Without loss of generality, let $uw \in E(G)$. Then $uv, vw \in E(\overline{G})$. Since $\overline{G}$ does not contain H_2 as its subgraph, there exists a vertex $x \in V(G) - S$ such that $xu \in E(\overline{G})$ but $xv, xw \notin E(\overline{G})$, or $xw \in E(\overline{G})$ but $xu, xv \notin E(\overline{G})$. By symmetry, we only need to consider the former case. Then $xv, xw \in E(G)$. Combining this with $uw \in E(G)$, the tree T induced by the edges in $\{xv, xw, uw\}$ is an S-Steiner tree in G, and hence $d_G(S) \leq 3$, as desired.

From the arbitrariness of S, we know that $sdiam_3(G) \leq 3$. Since $\Delta(\overline{G}) \geq 2$, it follows from Theorem 1.5 that $sdiam_3(G) = 3$. □

We are now in a position to give the proof of Theorem 1.7.

Lemma 2.4. *Let G be a connected graph of order n $(n \geq 5)$. If $4 \leq c(G) \leq n$, then $sdiam_3(G) \leq n - 2$, where $c(G)$ is the circumference of the graph G.*

Proof. If $c(G) = n$, then there is a Hamilton cycle C_n in G. From Observations 1.1 and 2.1, we have $sdiam_3(G) \leq sdiam_3(C_n) = \lfloor \frac{2}{3}n \rfloor \leq n - 2$. Let $c(G) = t$ $(4 \leq t \leq n-1)$. Then there exists a cycle of order t in G, say $C_t = v_1v_2 \cdots v_tv_1$. Let $G_1, G_2, \cdots, G_r$ be the connected components of $G - V(C_t)$.

Suppose $r \geq 4$. Clearly, each connected component G_i $(1 \leq i \leq r)$ contains a spanning tree T_i (note that if G_i is trivial, then T_i is trivial). Since G is connected, there is an edge e_i such that one endpoint of e_i belongs to $V(T_i)$ and the other endpoint belongs to $V(C_t)$. Furthermore, we choose one edge from the cycle C_t, say e, and delete it. Then the tree T induced by the edges in $(\bigcup_{i=1}^{r} E(T_i)) \cup (\bigcup_{i=1}^{r} e_j) \cup (E(C_t) - e)$ is a spanning tree of G with at least four leaves. From Proposition 2.2 and Observation 2.1, $sdiam_3(G) \leq sdiam_3(T) \leq n - 2$, as desired.

We now assume $r \leq 3$. It suffices to show that $d_G(S) \leq n - 2$ for any $S \subseteq V(G)$ with $|S| = 3$. We have the following four cases to consider. If $|S \cap V(C_t)| = 3$, then it follows from Observation 1.1 that $d_G(S) \leq sdiam_3(C_t) = \lfloor \frac{2}{3}t \rfloor \leq \frac{2}{3}t \leq \frac{2}{3}(n-1) \leq n - 2$, as desired. If $|S \cap V(C_t)| = 2$, then there exists a vertex $x \in S$ such that $x \in V(G - V(C_t))$. Then x must belong to some connected component in $G - V(C_t)$. Without loss of generality, let $x \in V(G_1)$, and let $S = \{x, v_i, v_j\}$ where $v_i, v_j \in V(C_t)$ $(1 \leq i \neq j \leq t)$. Because G_1 is connected, G_1 contains a spanning tree, say T_1. Since G is connected, we can find an edge e_1 with one endpoint belonging to $V(T_1)$ and the other, say v_k, belonging to $V(C_t)$ (note that v_k, v_i or v_k, v_j are not necessarily different). Since $d(\{v_i, v_j, v_k\}) \leq sdiam_3(C_t) = \lfloor \frac{2}{3}t \rfloor$, we have $d_G(S) \leq d(\{v_i, v_j, v_k\}) + |E(T_1)| + 1 = d(\{v_i, v_j, v_k\}) + |V(T_1)| \leq \lfloor \frac{2}{3}t \rfloor + n - t \leq n - \frac{1}{3}t < n - 1$ and hence $d_G(S) \leq n - 2$, as desired.

Suppose $|S \cap V(C_t)| = 1$. Then there exist two vertices $x, y \in S$ such that $x, y \in V(G - V(C_t))$. Set $S = \{x, y, v_i\}$ where $v_i \in V(C_t)$ $(1 \leq i \leq t)$. Thus,

x, y must belong to the same connected component of $G - V(C_t)$, or x, y belong to two different connected components. Consider the former case. Without loss of generality, let $x, y \in V(G_1)$. Since G_1 is connected, it follows that G_1 contains a spanning tree, say T_1. Because G is connected, we can find an edge e_1 with one endpoint belonging to $V(T_1)$ and the other, say v_j, belonging to $V(C_t)$ (note that v_i and v_j are not necessarily different). Since $d(\{v_i, v_j\}) \leq \lfloor \frac{1}{2}t \rfloor$, it follows that $d_G(S) \leq d(\{v_i, v_j\}) + |E(T_1)| + 1 = d(\{v_i, v_j\}) + |V(T_1)| \leq n - t + \lfloor \frac{1}{2}t \rfloor \leq n - \lceil \frac{1}{2}t \rceil$. Since $t \geq 4$, we have $d_G(S) \leq n - 2$, as desired. Consider the latter case. Without loss of generality, let $x \in V(G_1)$ and $y \in V(G_2)$. Clearly, G_i $(i = 1, 2)$ contains a spanning tree T_i. We can find the edges e_1, e_2 with one endpoint belonging to $V(T_1), V(T_2)$ and the other, say v_j, v_k, belonging to $V(C_t)$, respectively (note that v_i, v_j, v_k are not necessarily different). Since $d(\{v_i, v_j, v_k\}) \leq sdiam_3(C_t) = \lfloor \frac{2}{3}t \rfloor$, we have $d_G(S) \leq d(\{v_i, v_j, v_k\}) + |E(T_1)| + |E(T_2)| + 2 = d(\{v_i, v_j, v_k\}) + |V(T_1)| + |V(T_2)| \leq \lfloor \frac{2}{3}t \rfloor + n - t \leq n - \lceil \frac{1}{3}t \rceil$ and hence $d_G(S) \leq n - 2$, as desired.

Suppose $|S \cap V(C_t)| = 0$. Then $S \subseteq V(G - V(C_t))$. Let $S = \{x, y, z\}$. Thus, x, y, z belong to three different connected components, or x, y, z belong to two different connected components, or x, y, z must belong to one connected component. We only prove the first case, the other two cases can be proved similarly. Without loss of generality, let $x \in V(G_1)$, $y \in V(G_2)$ and $z \in V(G_3)$. For $i = 1, 2, 3$, G_i contains a spanning tree T_i. Since G is connected, we can find the edges e_1, e_2, e_3 with one endpoint belonging to $V(T_1), V(T_2), V(T_3)$ and the other, say v_i, v_j, v_k, belonging to $V(C_t)$, respectively (note that v_j, v_k, v_j are not necessarily different). Since $d_G(\{v_i, v_j, v_k\}) \leq sdiam_3(C_t) = \lfloor \frac{2}{3}t \rfloor$, we have $d_G(S) \leq d(\{v_i, v_j, v_k\}) + \sum_{i=1}^{3} |E(T_i)| + 3 = d(\{v_i, v_j, v_k\}) + \sum_{i=1}^{3} |V(T_i)| \leq \lfloor \frac{2}{3}t \rfloor + n - t \leq n - \lceil \frac{1}{3}t \rceil$ and hence $d_G(S) \leq n - 2$, as desired.

From the above arguments, we conclude that $sdiam_3(G) \leq n-2$. The proof is now complete. □

If T is a nontrivial tree and $S \subseteq V(T)$, where $|S| \geq 2$, then there is a unique subtree T_s of size $d(S)$ containing the vertices of S. We refer to such a tree as the *tree generated by* S.

Chartrand, Oellermann, Tian and Zou [8] obtained the following result.

Lemma 2.5 ([8]). *If H is a subgraph of a graph G and v is a vertex of G, then $d(v, H)$ denotes the minimum distance from v to a vertex of H. Therefore,*

$$d(S \cup \{v\}) = d(S) + d(v, T_s).$$

Proof of Theorem 1.7. For $n = 3$, $sdiam_3(G) = n - 1 = 2$ if and only if $G = P_3 = T_{0,1,1}$ or $G = K_3 = \triangle_{0,0,0}$. For $n = 4$, $sdiam_3(G) = n - 1 = 3$ if and only if $G = P_4 = T_{0,1,2}$ or $G = \triangle_{0,0,1}$. We now assume $n \geq 5$.

Suppose $G = T_{a,b,c}$, where $0 \leq a \leq b \leq c$ and $1 \leq b \leq c$ and $a+b+c = n-1$. Since there are at most three leaves in G, it follows from Proposition 2.2 that

$sdiam_3(G) = n-1$. Suppose $G = \triangle_{p,q,r}$, where $0 \leq p \leq q \leq r$ and $p+q+r = n-3$. From Proposition 1.2, we have $sdiam_3(G) \leq n-1$. It suffices to show that $sdiam_3(G) \geq n-1$. Choose the three leaves in $T_{a,b,c}$, say x, y, z, such that $x \in V(P_a)$, $y \in V(P_b)$ and $z \in V(P_c)$. Let $S' = \{x, z\}$ and $S = \{x, y, z\}$. From Lemma 2.5, $d_G(S) = d_G(S' \cup \{y\}) = d_G(S') + d(y, T_s) = (b+c) + a = n-1$, and hence $sdiam_3(G) = sdiam_3(T_{a,b,c}) = n-1$, as desired. Similarly, we can get $sdiam_3(\triangle_{p,q,r}) = n-1$, as desired.

Conversely, suppose $sdiam_3(G) = n-1$. If G is a tree, then it follows from Proposition 2.2 that G contains at most three leaves. Thus, $G = T_{a,b,c}$, where $0 \leq a \leq b \leq c$ and $1 \leq b \leq c$ and $a+b+c = n-1$. Now, we consider the graph G containing cycles. Recall that $c(G)$ is the circumference of the graph G. Obviously, $3 \leq c(G) \leq n$. If $4 \leq c(G) \leq n$, then it follows from Lemma 2.4 that $sdiam_3(G) \leq n-2$, a contradiction. Therefore, $c(G) = 3$. Suppose that G contains at least two triangles. If there exist two triangles having at most one common vertex, then G contains a spanning tree with at least four leaves, say T. From Observation 2.1 and Proposition 1.2, we have $sdiam_3(G) \leq sdiam_3(T) \leq n-2$, a contradiction. So we assume that there exist two triangles having two common vertices in G. Therefore, G contains K_4^- as its subgraph, where K_4^- is a graph obtained from a clique K_4 by deleting one edge. Now, we consider the two vertices of degree 3 in K_4^-. If the degree of each such vertex in K_4^- is larger than 4 in G, then G contains a spanning tree with four leaves. Again from Observation 2.1 and Proposition 1.2, $sdiam_3(G) \leq sdiam_3(T) \leq n-2$, a contradiction. Then G contains the graph H as its subgraph, where H is a graph obtained from K_4^- and two paths by identifying one endvertex of each path and each vertex of degree 2 in K_4^-. One can also that $sdiam_3(H) \leq n-2$ and hence $sdiam_3(G) \leq sdiam_3(H) \leq n-2$, a contradiction. From the above arguments, we conclude that G only contains one triangle and hence $G = \triangle_{a,b,c}$. The proof is complete. □

3. Nordhaus-Gaddum results

The following proposition is a preparation of the proof of Theorem 1.8.

Proposition 3.1. *Let G be a connected graph. If $sdiam_k(G) \geq 2k$, then $sdiam_k(\overline{G}) \leq k$.*

Proof. For any $S \subseteq V(G)$ and $|S| = k$, if $G[S]$ is not connected, then $\overline{G}[S]$ is connected, and hence $d_{\overline{G}}(S) = k-1 < k$. Suppose that $G[S]$ is connected. Then we have the following claim.

Claim 1. There exists a vertex $u \in V(G) - S$ such that $|E_G[u, S]| = 0$.

Assume, to the contrary, that $|E_G[x, S]| \geq 1$ for any $x \in V(G) - S$. For any $S' \subseteq V(G)$ and $|S'| = k$, since $G[S]$ is connected and $|E_G[x, S]| \geq 1$ for any $x \in S' - S$, it follows that $G[S \cup S']$ is connected, and hence $d_G(S') \leq 2k-1$. From the arbitrariness of S', we have $sdiam_k(G) \leq 2k-1$, a contradiction.

From Claim 1, there exists a vertex $u \in V(G) - S$ such that $|E_{\overline{G}}[u, S]| = k$, and the tree induced by these k edges is an S-Steiner tree in $\overline{G}$. So $d_{\overline{G}}(S) = k$. From the arbitrariness of S, we have $sdiam_k(\overline{G}) \leq k$, as desired. $\square$

Proof of Theorem 1.8. We first give the proof of the upper bounds. If $sdiam_k(G) \geq 2k$, then it follows from Proposition 3.1 that $sdiam_k(\overline{G}) \leq k$. Furthermore, since $sdiam_k(G) \leq n-1$, we have $sdiam_k(G) + sdiam_k(\overline{G}) \leq n+k-1$ and $sdiam_k(G) \cdot sdiam_k(\overline{G}) \leq k(n-1)$. By the same reason, if $sdiam_k(\overline{G}) \geq 2k$, then $sdiam_k(G) \leq k$, and hence $sdiam_k(G) + sdiam_k(\overline{G}) \leq n+k-1$ and $sdiam_k(G) \cdot sdiam_k(\overline{G}) \leq k(n-1)$. We now assume that $sdiam_k(G) \leq 2k-1$ and $sdiam_k(\overline{G}) \leq 2k-1$. Then $sdiam_k(G) + sdiam_k(\overline{G}) \leq 4k-2$, and hence $sdiam_k(G) + sdiam_k(\overline{G}) \leq \max\{n+k-1, 4k-2\}$ and $sdiam_k(G) \cdot sdiam_k(\overline{G}) \leq \max\{k(n-1), (2k-1)^2\}$.

Next, we find the lower bounds. From Proposition 1.2, since $sdiam_k(G) \geq k-1$ and $sdiam_k(\overline{G}) \geq k-1$, we have $sdiam_k(G) + sdiam_k(\overline{G}) \geq 2k-2$ and $sdiam_k(G) \cdot sdiam_k(\overline{G}) \geq (k-1)^2$. Since $n \geq 2k-2$, we claim that $sdiam_k(G) + sdiam_k(\overline{G}) \geq 2k-1$ and $sdiam_k(G) \cdot sdiam_k(\overline{G}) \geq (k-1)k$. Assume, to the contrary, that $sdiam_k(G) + sdiam_k(\overline{G}) = 2k-2$ and $sdiam_k(G) \cdot sdiam_k(\overline{G}) \geq (k-1)^2$. Then $sdiam_k(G) = sdiam_k(\overline{G}) = k-1$. From Lemma 2.3, we have $n-k+1 \leq \delta(G) \leq n-1$ and $0 \leq \Delta(G) \leq k-2$, and hence $n \leq 2k-3$, a contradiction. So $sdiam_k(G) + sdiam_k(\overline{G}) \geq 2k-1$ and $sdiam_k(G) \cdot sdiam_k(\overline{G}) \geq (k-1)k$. $\square$

Lemma 3.2. *Let G be a graph. Then $sdiam_{n-1}(G) = n-2$ if and only if G is 2-connected.*

Proof. Suppose that G is 2-connected. For any $S \subseteq V(G)$ and $|S| = n-1$, there exists a unique vertex $V(G) - S$, say v, such that $G - v$ is connected, and hence $G - v$ contains a spanning tree, which implies $d_G(S) \leq n-2$. From the arbitrariness of S, we have $sdiam_{n-1}(G) \leq n-2$. From Proposition 1.2, $sdiam_{n-1}(G) = n-2$.

Conversely, we suppose $sdiam_{n-1}(G) = n-2$. If G is not 2-connected, then there exists a cut vertex in G, say v. Choose $S = V(G) - v$. Then $|S| = n-1$. Observe that any S-Steiner tree must use all the vertices of G. Thus $d_G(S) \geq n-1$, which contradicts $sdiam_{n-1}(G) = n-2$. $\square$ $\square$

By using Proposition 3.1 and Lemma 3.2, we can proof Proposition 1.11.

Proof of Proposition 1.11. From Proposition 1.2, we have $2n-4 \leq sdiam_{n-1}(G) + sdiam_{n-1}(\overline{G}) \leq 2n-2$ and $(n-2)^2 \leq sdiam_{n-1}(G) \cdot sdiam_{n-1}(\overline{G}) \leq (n-1)^2$. Clearly, $sdiam_{n-1}(G) + sdiam_{n-1}(\overline{G}) = 2n-4$ or $sdiam_{n-1}(G) \cdot sdiam_{n-1}(\overline{G}) = (n-2)^2$ if and only if $sdiam_{n-1}(G) = sdiam_{n-1}(\overline{G}) = n-2$. From Lemma 3.2, $sdiam_{n-1}(G) + sdiam_{n-1}(\overline{G}) = 2n-4$ or $sdiam_{n-1}(G) \cdot sdiam_{n-1}(\overline{G}) = (n-2)^2$ if and only if both G and $\overline{G}$ are 2-connected.

It is clear that $sdiam_{n-1}(G) + sdiam_{n-1}(\overline{G}) = 2n-3$ or $sdiam_{n-1}(G) \cdot sdiam_{n-1}(\overline{G}) = (n-1)(n-2)$ if and only if $sdiam_{n-1}(G) = n-2$ and $sdiam_{n-1}(\overline{G}) = n-1$, or $sdiam_{n-1}(G) = n-1$ and $sdiam_{n-1}(\overline{G}) = n-2$. Furthermore, $sdiam_{n-1}(G) + sdiam_{n-1}(\overline{G}) = 2n-3$ or $sdiam_{n-1}(G) \cdot sdiam_{n-1}(\overline{G}) = (n-1)(n-2)$ if and only if $\lambda(G) = 1$ and $\overline{G}$ is 2-connected, or $\lambda(\overline{G}) = 1$ and G is 2-connected.

For the remaining case, we have $sdiam_{n-1}(G) + sdiam_{n-1}(\overline{G}) = 2n-2$ or $sdiam_{n-1}(G) \cdot sdiam_{n-1}(\overline{G}) = (n-1)^2$ if and only if $sdiam_{n-1}(G) = sdiam_{n-1}(\overline{G}) = n-1$. From Lemma 1.10, $sdiam_{n-1}(G) + sdiam_{n-1}(\overline{G}) = 2n-2$ or $sdiam_{n-1}(G) \cdot sdiam_{n-1}(\overline{G}) = (n-1)^2$ if and only if G satisfies the following conditions.

- $\kappa(G) = 1$, $\Delta(G) = n-2$;
- $\kappa(G) = 1$, $\Delta(G) \leq n-3$ and G has a cut vertex v with pendant edge e and pendant vertex u such that $G - u$ contains a spanning complete bipartite subgraph. □

Proof of Proposition 1.12. From Proposition 1.2, $2n-6 \leq sdiam_{n-2}(G) + sdiam_{n-2}(\overline{G}) \leq 2n-2$ and $(n-3)^2 \leq sdiam_{n-2}(G) \cdot sdiam_{n-2}(\overline{G}) \leq (n-1)^2$. So the results follow for the case that both G and $\overline{G}$ contain at least two cut vertices. From now on, we assume that G or $\overline{G}$ contains only one cut vertex, or G or $\overline{G}$ is 2-connected. Without loss of generality, we assume that G contains only one cut vertex or G is 2-connected. For any $S \subseteq V(G)$ and $|S| = n-2$, there exists a vertex $v \in V(G) - S$ such that $G - v$ is connected, and hence $G - v$ contains a spanning tree, which implies $d_G(S) \leq n-2$. From the arbitrariness of S, we have $sdiam_{n-2}(G) \leq n-2$. From Proposition 1.2, we have $sdiam_{n-2}(\overline{G}) \leq n-1$. So $sdiam_{n-2}(G) + sdiam_{n-2}(\overline{G}) \leq 2n-3$ and $sdiam_{n-2}(G) \cdot sdiam_{n-2}(\overline{G}) \leq (n-1)(n-2)$. □

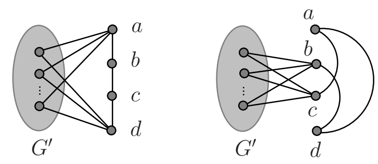
To show the sharpness of the bounds in Proposition 1.12, we consider the following example.

Example 3.3. Let $G = P_4$. Then $\overline{G} = P_4$, $sdiam_2(P_4) = sdiam_2(\overline{P_4}) = 3$. Therefore, we have $sdiam_2(P_4) + sdiam_2(\overline{P_4}) = 6 = 2n-4$ and $sdiam_2(P_4) \cdot sdiam_2(\overline{P_4}) = 9 = (n-1)^2$, which implies that the upper bounds are sharp for the case both G and $\overline{G}$ contain at least two cut vertices. Let S^* be a tree obtained from a star of order $n-2$ and a path of length 2 by identifying the center of the star and a vertex of degree one in the path. Then $\overline{S^*}$ is a graph obtained from a clique of order $n-1$ by deleting an edge uv and then adding an pendent edge vw at v. Choose $S = V(G) - \{u, w\}$. Then any S-Steiner tree uses all the vertices of $V(G)$, and hence $d_G(S) \geq n-1$. From the arbitrariness of S, we have $sdiam_{n-2}(G) \geq n-1$, and hence $sdiam_{n-2}(G) = n-1$, by Proposition 1.2. Choose $S \subseteq V(\overline{G}) - w$ and $|S| = n-2$. Then any S-Steiner tree uses $n-1$ vertices of $V(\overline{G})$, and hence $d_{\overline{G}}(S) \geq n-2$. From the arbitrariness of S, we have $sdiam_{n-2}(\overline{G}) \geq n-2$. One can easily check that $sdiam_{n-2}(\overline{G}) \leq n-2$. So $sdiam_{n-2}(\overline{G}) = n-2$, and hence $sdiam_{n-2}(G) + sdiam_{n-2}(\overline{G}) = 2n-3$ and $sdiam_{n-2}(G) \cdot sdiam_{n-2}(\overline{G}) = (n-1)(n-2)$. This implies that the upper bounds in Proposition 1.12 are sharp. Let G be a graph such that both G and $\overline{G}$ are 3-connected. For any $S \subseteq V(G)$ and $|S| = n-2$, there exist two vertices u, v in $V(G)-S$ such that $G-\{u,v\}$ is connected, and hence $G-\{u,v\}$ contains a spanning tree, which implies $d_G(S) \leq n-3$. From the arbitrariness of S, we have $sdiam_{n-2}(G) \leq n-3$, and hence $sdiam_{n-2}(G) = n-3$, by Proposition 1.2. Similarly, we have $sdiam_{n-2}(\overline{G}) = n-3$. Then $sdiam_{n-2}(G) + sdiam_{n-2}(\overline{G}) = 2n-6$ and $sdiam_{n-2}(G) \cdot sdiam_{n-2}(\overline{G}) = (n-3)^2$, which implies that the lower bounds in Proposition 1.12 are sharp.

Proof of Proposition 1.13. The upper bounds follow from Theorem 1.8. We now find the lower bounds of $sdiam_3(G) + sdiam_3(\overline{G})$ and $sdiam_3(G) \cdot sdiam_3(\overline{G})$. If $sdiam_3(G) + sdiam_3(\overline{G}) < 6$ or $sdiam_3(G) \cdot sdiam_3(\overline{G}) < 9$, then we have $sdiam_3(G) = 2$ or $sdiam_3(\overline{G}) = 2$. Without loss of generality, let $sdiam_3(G) = 2$. From Theorem 1.5, we have $0 \leq \Delta(\overline{G}) \leq 1$ and hence $\overline{G}$ is disconnected. Thus $sdiam_3(\overline{G}) = \infty$, which results in $sdiam_3(G) + sdiam_3(\overline{G}) = \infty$ and $sdiam_3(G) \cdot sdiam_3(\overline{G}) = \infty$, a contradiction. So $sdiam_3(G) + sdiam_3(\overline{G}) \geq 6$ and $sdiam_3(G) \cdot sdiam_3(\overline{G}) \geq 9$. $\square$

To show the sharpness of the bounds in Proposition 1.13, we consider the following example.

Example 3.4. One can check that $G = P_n$ is a sharp example for the upper bounds of this proposition. To show the sharpness of the lower bounds, we consider the following example. If $sdiam_3(G) + sdiam_3(\overline{G}) = 6$, then $sdiam_3(G) = sdiam_3(\overline{G}) = 3$. Let G' be a graph of order $n-4$, and let a, b, c, d be a path. Let G be the graph obtained from G' and the path by adding edges between the vertex a and all vertices of G' and adding edges between the vertex d and all vertices of G'; see FIHURE 4 (a). We now show that $sdiam_3(G) = sdiam_3(\overline{G}) = 3$. Choose $S = \{a, b, d\}$. Then it is easy to see that $d_G(S) \geq 3$ and hence $sdiam_3(G) \geq 3$. It suffices to prove that $d_G(S) \leq 3$ for any $S \subseteq V(G)$ with $|S| = 3$. Suppose $|S \cap V(G')| = 3$. Without loss of generality, let $S = \{x, y, z\}$. Then the tree T induced by the edges in $\{xa, ya, za\}$ is an S-Steiner tree and hence $d_G(S) \leq 3$. Suppose $|S \cap V(G')| = 2$. Without loss of generality, let $x, y \in S \cap V(G')$. If $a \in S$, then the tree T induced by the edges in $\{xa, ya\}$ is an S-Steiner tree, which implies $d_G(S) \leq 2$. If $b \in S$, then the tree T induced by the edges in $\{xa, ya, ab\}$ is an S-Steiner tree and hence $d_G(S) \leq 3$. Suppose $|S \cap V(G')| = 1$. Without loss of generality, let $x \in S \cap V(G')$. If $a, b \in S$, then the tree T induced by the edges in $\{xa, ab\}$ is an S-Steiner tree and hence $d_G(S) \leq 2$. If $b, c \in S$, then the tree T induced by the edges in $\{xd, cd, bc\}$ is an S-Steiner tree and hence $d_G(S) \leq 3$. If $a, c \in S$, then the tree T induced by the edges in $\{xa, ab, bc\}$ is an S-Steiner tree, which



(a) (b)

FIGURE 4. Graphs for Theorem 1.8

implies $d_G(S) \leq 3$. Suppose $|S \cap V(G')| = 0$. If $a, b, c \in S$, then the tree T induced by the edges in $\{ab, bc\}$ is an S-Steiner tree and hence $d_G(S) \leq 2$. If $a, b, d \in S$, then the tree T induced by the edges in $\{ab, bc, cd\}$ is an S-Steiner tree, which implies $d_G(S) \leq 3$. From the arbitrariness of S, we conclude that $sdiam_3(G) \leq 3$ and hence $sdiam_3(G) = 3$. Similarly, one can also check that $sdiam_3(\overline{G}) = 3$.

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