Title:

Module homomorphisms from Fréchet algebras

Author(s):

H. Shayanpour
MODULE HOMOMORPHISMS FROM FRÉCHET ALGEBRAS

H. SHAYANPOUR

(Communicated by Omid Ali S. Karamzadeh)

Dedicated to Prof. Taher Ghasemi Honary

Abstract. We first study some properties of $A$-module homomorphisms $\theta : X \to Y$, where $X$ and $Y$ are Fréchet $A$-modules and $A$ is a unital Fréchet algebra. Then we show that if there exists a continued bisection of the identity for $A$, then $\theta$ is automatically continuous under certain condition on $X$. In particular, every homomorphism from $A$ into certain Fréchet algebras (including Banach algebra) is automatically continuous. Finally, we show that every unital Fréchet algebra with a continued bisection of the identity, is functionally continuous.

Keywords: Automatic continuity, Fréchet algebras, module homomorphism, continued bisection of the identity, Fréchet $A$-module.

MSC(2010): Primary: 46H40; Secondary: 16D10, 46H05.

1. Introduction

In the following, we present the notations, definitions and known results, which are related to our work. For further details one can refer, for example, to [3].

A Fréchet algebra is a complete metrizable topological algebra whose topology is defined by a separating family $P = \{p_n\}$ of submultiplicative seminorms. Note that the topology of a Fréchet algebra $A$ can be generated by a sequence $\{p_n\}_{n \in \mathbb{N}}$ of separating submultiplicative seminorms, i.e., $p_n(xy) \leq p_n(x)p_n(y)$ for all $n \in \mathbb{N}$ and every $x, y \in A$, such that $p_n(x) \leq p_{n+1}(x)$ for all $x \in A$ and $n \in \mathbb{N}$. The Fréchet algebra $A$ with the above generating sequence of seminorms is denoted by $(A, \{p_n\})$. Note that a sequence $\{x_k\}$ in the Fréchet algebra $(A, \{p_n\})$ converges to $x \in A$ if and only if $p_n(x_k - x) \to 0$, for each $n \in \mathbb{N}$, as $k \to \infty$.

Let $A$ be a complex algebra. A character on $A$ is a non-zero homomorphism from $A$ into $\mathbb{C}$. The set of all characters of $A$ is denoted by $S_A$. If $A$ is a complex topological algebra, then the set of continuous characters of $A$ is denoted by $M_A$. If $A$ is a complex topological algebra with the dual space $A'$,
then, $M_A \cup \{0\} \subseteq A'$. The relative $w^*$-topology on $M_A \cup \{0\}$ is the Gelfand topology. A topological algebra $A$ is called functionally continuous if every character on $A$ is continuous, i.e., $M_A = S_A$. It is known that every Banach algebra is functionally continuous. Let $A$ be a complex algebra. A left [right] $A$-module is a complex linear space $E$ together with a bilinear map $(a, x) \mapsto a \cdot x \ [(a, x) \mapsto x \cdot a], \ A \times E \to E$, such that

$$a \cdot (b \cdot x) = ab \cdot x \ [(x \cdot a) \cdot b = x \cdot ab] \quad (a, b \in A, \ x \in E).$$

An $A$-bimodule is a linear space $E$ which is a left $A$-module and a right $A$-module such that

$$a \cdot (x \cdot b) = (a \cdot x) \cdot b \quad (a, b \in A, \ x \in E).$$

Let $A$ be an algebra, and let $E$ and $F$ be left [right] $A$-modules. A linear map $T : E \to F$ is a left [right] $A$-module homomorphism if

$$T(a \cdot x) = a \cdot T(x) \ [T(x \cdot a) = T(x) \cdot a] \quad (a \in A, \ x \in E).$$

Let $E$ and $F$ be $A$-bimodules. A linear map $T : E \to F$ is an $A$-bimodule homomorphism if it is both a left and a right $A$-module homomorphism. For a Fréchet algebra $A$, a Fréchet (Banach) left $A$-module is a Fréchet (Banach) space $X$ which is endowed with a structure of left $A$-module such that the binary action $\cdot : A \times X \to X, \ (a, x) \mapsto a \cdot x$ is continuous. When $A$ is unital with the unit element $e_A$, then a left $A$-module $X$ is called unital if $e_A \cdot x = x$ for every $x \in X$. Similarly, a Fréchet (Banach) right $A$-module is a Fréchet (Banach) space $X$ which is endowed with a structure of right $A$-module such that the binary action $\cdot : A \times X \to X, \ (a, x) \mapsto x \cdot a$ is continuous. A linear space $X$ is a Fréchet (Banach) $A$-bimodule if it is both left and right Fréchet (Banach) $A$-module and if

$$a \cdot (x \cdot b) = (a \cdot x) \cdot b \quad (a, b \in A, x \in X).$$

Let $A$ and $B$ be topological linear spaces, and let $\theta : A \to B$ be a linear mapping. The separating space of $\theta$ is defined by

$$\mathcal{S}(\theta) = \{b \in B : \text{there exists a net } (a_\delta) \text{ in } A \text{ such that } a_\delta \to 0 \text{ and } \theta(a_\delta) \to b\}.$$

The separating space $\mathcal{S}(\theta)$ is a closed linear subspace of $B$ and moreover, if $A$ and $B$ are $F$-spaces, by the Closed Graph Theorem, $\theta$ is continuous if and only if $\mathcal{S}(\theta) = \{0\}$.

Let $A$ be an algebra. Then the elements $a, b \in A$ are (mutually) orthogonal in $A$, written $a \perp b$, if $ab = 0 = ba$. Let $A$ be a unital algebra. A continued bisection of the identity for $A$ is a pair $(\{r_n\}, \{s_n\})$ of sequences of idempotents of $A$ such that $e_A = r_1 + s_1$ and for each $n \in \mathbb{N}$ we have $r_n = r_{n+1} + s_{n+1}$ and $Ar_nA = As_nA$. By the definition of the continued bisection $(\{r_n\}, \{s_n\})$ of the identity, $r_n, s_n$ and $r_n + s_n$ are idempotents and hence $(r_n + s_n)^2 = r_n + s_n$. Therefore,

$$r_n s_n + s_n r_n = 0.$$  

(1.1)
Now we have
\[
    r_n s_n + r_n s_n r_n = r_n (r_n s_n + s_n r_n) = 0 = (r_n s_n + s_n r_n) r_n
\]
and so \( r_n s_n = s_n r_n \). By (1.1) and (1.2), we have \( r_n s_n = 0 = s_n r_n \) and hence \( s_n \perp r_n \).

Continued bisections of the identity originate from Johnsons study of automatic continuity of homomorphisms from \( B(X) \) for a Banach space \( X \). In [5] Johnson proved \( B(X) \) has a continued bisections of the identity whenever \( X \) is isomorphic to \( X \oplus X \). However, there are operator algebras on some spaces without any continued bisection of the identity, for further details one can refer, for example, to [9] and [10]. Figiel in [4] gives an example of a reflexive space \( X \) with a continued bisections of the identity, which is not isomorphic to \( X \oplus X \). Laustsen in [8] proved that every unital, properly infinite ring \( R \) has a continued bisection of the identity. Here a properly infinite ring \( R \) is a ring with idempotents elements \( P_1 \) and \( P_2 \) in \( R \) that are orthogonal and satisfy \( P_n = ST \) and \( e_R = TS \) (\( e_R \) is identity of \( R \)), for some elements \( S \) and \( T \) in \( R \) and for \( n = 1, 2 \).

In the first section, we prove some inequalities for \( A \)-module homomorphisms between Fréchet \( A \)-modules as well as homomorphisms between Fréchet algebras.

In the second section, we show that if \((A, \{p_n\})\) is a unital Fréchet algebra with a continued bisection \( \{\{r_n\}, \{s_n\}\} \) of the identity, \((X, \{q_n\})\) is a Fréchet left \( A \)-module such that \( s_n X \not\subseteq \ker(q_{k_n}) \) for an increasing subsequence \( \{k_n\} \) of \( \mathbb{N} \), and \( Y \) is a Banach left \( A \)-module which is unital, then every left \( A \)-module homomorphism \( \theta : X \rightarrow Y \) is automatically continuous. Moreover, for every bounded subset \( E \) of \( X \), there exists an \( M > 0 \) such that \( \|s_n \theta(x)\| \leq M p_{k_n} (s_n)^2 \), for all \( x \in X \) and \( n \in \mathbb{N} \). We also show that if \((A, \{p_n\})\) is a unital Fréchet algebra with a continued bisection of the identity, then every homomorphism \( \theta : A \rightarrow B \), where \( B \) is a Banach algebra, is automatically continuous.

In particular, if \((A, \{p_n\})\) is a unital Fréchet algebra with a continued bisection of the identity, then \( A \) is functionally continuous.

2. Some inequalities for \( A \)-module homomorphisms

In 1974, Sinclair studied module homomorphisms between Banach \( A \)-modules, when \( A \) is a regular semisimple commutative unital Banach algebra [12]. Bade and Curtis in [1] showed that if \( \theta \) is a homomorphism defined on a Banach algebra \( A \), \( \{a_n\} \) is a sequence of mutually orthogonal elements of \( A \) and if \( \{b_n\} \) is an arbitrary sequence in \( A \) such that \( b_n a_n = b_n \), for all \( n \in \mathbb{N} \), then there exists an \( M > 0 \) such that \( \|\theta(b_n)\| \leq M \|a_n\| \|b_n\| \), for all \( n \in \mathbb{N} \). Also, Johnson in [6] studied module homomorphisms between Banach \( G \)-modules, when \( G \)
is a locally compact abelian group. We now extend some of these results for Fréchet algebras and obtain some results on the automatic continuity of module homomorphisms on Fréchet $A$-modules.

First we present the following general results on Fréchet $A$-modules.

**Remark 2.1.** (i) Let $(A, \{p_n\})$ be a Fréchet algebra and $\{a_n\}$ be a sequence in $A$. Since $\{p_n\}$ is separating, there exists $k_1 \in \mathbb{N}$ such that $p_{k_1}(a_1) \neq 0$. Since $\{p_n\}$ is also an increasing sequence, we can choose $p_{k_2} \geq p_{k_1}$ such that $p_{k_2}(a_2) \neq 0$. By a successive argument, we have a subsequence $\{p_{k_n}\}$ of the sequence $\{p_n\}$ such that $p_{k_n}(a_n) \neq 0$.

(ii) Let $(A, \{p_n\})$ be a Fréchet algebra, $(X, \{q_n\})$ be a Fréchet left $A$-module. Since $X$ is a Fréchet space, the sequence $\{q_n\}$ is separating. Therefore, $\bigcap_{n=1}^\infty \ker(q_n) = 0$.

The proof of the following proposition is inspired by the proof of the theorem 2.1 in [1].

**Proposition 2.2.** Let $(A, \{p_n\})$ be a Fréchet algebra, $(X, \{q_n\})$ be a Fréchet left $A$-module, $Y$ be a Banach left $A$-module, and $\theta : X \rightarrow Y$ be a left $A$-module homomorphism. Let $\{a_n\}$ be a sequence in $A$ such that $a_na_m = 0$ for $n \neq m$, and let $\{k_n\}$ be a sequence in $\mathbb{N}$ such that $p_{k_n}(a_n) \neq 0$ for all $n \in \mathbb{N}$.

(i) If $\{x_n\}$ is a sequence in $X$ such that $a_n \cdot x_m = 0$ for $n \neq m$, and if $\{r_n\}$ is a sequence in $\mathbb{N}$ such that $q_{r_n}(x_n) \neq 0$ for all $n \in \mathbb{N}$, then there is a constant $C > 0$ such that

\[
\|\theta(a_n \cdot x_n)\| \leq C p_{k_n}(a_n) q_{r_n}(x_n),
\]

for all $n \in \mathbb{N}$.

(ii) If $\{b_n\}$ is a sequence of elements of $A$ such that $a_nb_m = 0$ for all $n \neq m$ and $b_n \cdot X \subset \ker(q_{r_n})$ for some subsequence $\{q_{r_n}\}$, then the linear operator $a_nb_n \cdot \theta(\cdot)$ is continuous for large enough $n \in \mathbb{N}$. Moreover, if $p_{r_n}(b_n) \neq 0$ for all $n \in \mathbb{N}$, then for every bounded subset $E \subseteq X$ there exists an $M > 0$ such that for every $x \in E$, we have

\[
\|a_nb_n \cdot \theta(x)\| \leq M p_{k_n}(a_n)p_{r_n}(b_n),
\]

for large enough $n \in \mathbb{N}$.

**Proof.** (i) By the hypothesis we may take a sequence $\{a'_n\}$ in $A$ and a sequence $\{x'_n\}$ in $X$ such that $p_{k_n}(a'_n) = q_{r_n}(x'_n) = 1$. Renaming the sequence $\{a_n\}$ by $\{a'_n\}$ and the sequence $\{x_n\}$ by $\{x'_n\}$, we may assume that $p_{k_n}(a_n) = q_{r_n}(x_n) = 1$, for all $n \in \mathbb{N}$. Assume towards a contradiction that there is no $C$ for which (2.1) holds. Then there is a map

\[T : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}, \quad T((i, j)) = n(i, j),\]
such that it is increasing on both components, and moreover,

&(2.3) 
\|\theta(u_{i,j}, v_{i,j})\| \geq 4^{i+j},

where \(u_{i,j} = a_{n(i,j)}\) and \(v_{i,j} = x_{n(i,j)}\). We set

\[ S_n^i = \sum_{k=1}^{n} 2^{-k} u_{i,k}, \quad (i, n \in \mathbb{N}). \]

Fixing \(i \in \mathbb{N}\), for each \(m \in \mathbb{N}\) there exists \(k_m\) such that for every \(j > k_m\), we have \(m < k_{n(i,j)}\). On the other hand, we may assume that the sequence \(\{p_n\}\) is increasing and hence for every \(m \in \mathbb{N}\) and \(j > k_m\) we have \(p_m(\cdot) \leq p_{k_m(i,j)}(\cdot)\).

By taking \(p(i,j) = p_{k_m(i,j)}\), for each \(m \in \mathbb{N}\) and \(n > r > k_m\) we have

\[ p_m(S_n^i - S_r^i) \leq \sum_{k=r+1}^{n} \frac{p_m(u(k,i))}{2^k} \leq \sum_{k=r+1}^{n} \frac{p(k,i)u(k,i)}{2^k} = \sum_{k=r+1}^{n} \frac{1}{2^k}, \]

which shows that \(p_m(S_n^i)\) is a Cauchy sequence for each \(m \in \mathbb{N}\). Therefore, for each \(i\), the series

\[ f_i = \sum_{k=1}^{\infty} 2^{-k} u_{i,k}, \]

converges in \(A\). Let \(L_i\) be the operation of left multiplication by \(f_i\) in \(Y\). We have \(f_i \cdot v_{i,j} = 2^{-j} u_{i,i} \cdot v_{i,j}\) and

\[ L_i(\theta(v_{i,j})) = f_i \cdot \theta(v_{i,j}) = \theta(f_i \cdot v_{i,j}) = \theta(2^{-j} u_{i,j} \cdot v_{i,j}) = 2^{-j} \theta(u_{i,j} \cdot v_{i,j}). \]

Since \(Y\) is a Banach left \(A\)-module, the operation of left multiplication in \(Y\) is continuous and so \(L_i\) is a non-zero continuous linear operator on \(Y\). We now choose an integer \(j(i) > i\) so that \(\|L_i\| \leq 2^{j(i)}\) for each \(i\), and set

\[ S = \sum_{k=1}^{\infty} 2^{-k} v_{i,j(k)}. \]

For each \(m\), there exists \(r_m\) such that for \(i > r_m\) we have \(m < r_{n(i,1)} \leq r_{n(i,j(i))}\).

On the other hand, since the sequence \(\{q_n\}\) is increasing, for every \(m \in \mathbb{N}\) and \(i > r_m\) we have \(q_m(\cdot) \leq q_{n(i,j(i))}(\cdot)\). By a similar method as in the proof of the convergence of \(f_i\), \(S\) is convergent in \(X\). Hence \(f_i \cdot S = 2^{-i-j(i)} u_{i,j(i)} \cdot v_{i,j(i)}\) for each \(i\). By the hypothesis and (2.3), we have

\[ \|L_i(\theta(S))\| = \|f_i \theta(S)\| = \|2^{-i-j(i)} \theta(u_{i,j(i)} \cdot v_{i,j(i)})\| \geq 2^{i+j(i)}. \]

On the other hand,

\[ \|L_i(\theta(S))\| \leq \|\theta(S)\| \|L_i\| \leq 2^{j(i)} \|\theta(S)\|. \]

Then we have \(2^i \leq \|\theta(S)\|\) for every \(i \in \mathbb{N}\). The resulting contradiction completes the proof of (i).
(ii) By the hypothesis, there exists a sequence \( \{s_n\} \) in \( X \) such that \( q_{r_n}(b_n \cdot s_n) = 1 \). If \( a_n b_n \cdot \theta \) is discontinuous for infinitely many \( n \in \mathbb{N} \), without loss of generality, we may assume that for every \( n \in \mathbb{N} \) there exists a sequence \( \{x_m^n\} \in X \) such that \( \lim_{m \to \infty} x_m^n = 0 \) and \( \lim_{m \to \infty} a_n b_n \cdot \theta(x_m^n) = y_n \), but \( y_n \neq 0 \). Since for every \( n \), \( \lim_{m \to \infty} q_{r_n}(b_n \cdot x_m^n) = 0 \), it follows that for large enough \( m \) we have

\[
\|a_n b_n \cdot \theta(x_m^n)\| > n p_{k_n}(a_n) q_{r_n}(b_n \cdot x_m^n).
\]

Therefore, there exists a sequence \( \{x_n\} \subseteq X \) such that

\[
\|a_n b_n \cdot \theta(x_n)\| > n p_{k_n}(a_n) q_{r_n}(b_n \cdot x_n).
\]

So for each \( n \in \mathbb{N} \), there exists \( \varepsilon_n > 0 \) such that

\[
\|a_n b_n \cdot \theta(x_n)\| > \varepsilon_n,
\]

also we can choose \( \lambda_n \in \mathbb{N} \) such that

\[
\delta_n = \varepsilon_n - \frac{1}{\lambda_n} \left( 1 + \frac{n p_{k_n}(a_n)}{1 + \|a_n b_n \cdot \theta(s_n)\|} \right) > 0.
\]

If \( q_{r_n}(b_n \cdot x_n) = 0 \) for some \( n \), by taking \( z_n = x_n + s_n \frac{1}{\lambda_n(1 + \|a_n b_n \cdot \theta(s_n)\|)} \), we have

\[
\frac{1}{\lambda_n(1 + \|a_n b_n \cdot \theta(s_n)\|)} - q_{r_n}(b_n \cdot x_n) \leq q_{r_n}(b_n \cdot z_n)
\]

\[
\leq \frac{1}{\lambda_n(1 + \|a_n b_n \cdot \theta(s_n)\|)} + q_{r_n}(b_n \cdot x_n).
\]

Hence \( q_{r_n}(b_n \cdot z_n) = \frac{1}{\lambda_n(1 + \|a_n b_n \cdot \theta(s_n)\|)} \neq 0 \) and moreover,

\[
np_{k_n}(a_n) q_{r_n}(b_n \cdot z_n) = \frac{np_{k_n}(a_n)}{\lambda_n(1 + \|a_n b_n \cdot \theta(s_n)\|)} = -\delta_n + \varepsilon_n - \frac{1}{\lambda_n}.
\]

Thus

\[
\|a_n b_n \cdot \theta(z_n)\| \geq \|a_n b_n \cdot \theta(x_n)\| - \frac{\|a_n b_n \cdot \theta(s_n)\|}{\lambda_n(1 + \|a_n b_n \cdot \theta(s_n)\|)}
\]

\[
\geq \varepsilon_n - \frac{1}{\lambda_n} = np_{k_n}(a_n) q_{r_n}(b_n \cdot z_n) + \delta_n
\]

\[
> np_{k_n}(a_n) q_{r_n}(b_n \cdot x_n).
\]

Now by replacing \( x_n \) by \( z_n \), we may always assume that there exists \( \{x_n\} \subseteq X \) such that \( q_{r_n}(b_n \cdot x_n) \neq 0 \) and \( \|a_n b_n \cdot \theta(x_n)\| > np_{k_n}(a_n) q_{r_n}(b_n \cdot x_n) \) for all \( n \in \mathbb{N} \).
Now by applying (i) for the sequence \(\{b_n \cdot x_n\}\) instead of \(\{x_n\}\), there exists \(C > 0\) such that
\[
np_{k_n}(a_n)q_{r_n}(b_n \cdot x_n) < \|a_n b_n \cdot \theta(x_n)\| \leq Cp_{k_n}(a_n)q_{r_n}(b_n \cdot x_n),
\]
which implies that \(n < C\) for all \(n \in \mathbb{N}\). By this contradiction we conclude that the linear operator \(a_n b_n \cdot \theta(\cdot)\) is continuous for large enough \(n \in \mathbb{N}\).

For the proof of the second part of (ii), note that the operator
\[
\frac{a_n b_n \cdot \theta}{p_{k_n}(a_n)p_{r_n}(b_n)}
\]
is continuous for large enough \(n\). Hence it maps bounded sets into bounded sets, and the inverse images of open sets are open sets. If \(B(0,1)\) is the open unit ball in \(Y\), then there exists \(k\) such that
\[
\frac{a_n b_n \cdot \theta(V_k)}{p_{k_n}(a_n)p_{r_n}(b_n)} \subseteq B(0,1),
\]
where \(V_k = \{x \in X : q_k(x) < \frac{1}{k}\}\). On the other hand, if \(E\) is a bounded set in \(X\), there exists an \(M > 0\) such that \(E \subseteq MV_k\) and so \(\|a_n b_n \cdot \theta(x)\| \leqMp_{k_n}(a_n)p_{r_n}(b_n)\) for all \(x \in E\) and so (2.2) holds.

**Proposition 2.3.** Let \((A, \{p_n\})\) be a Fréchet algebra, \(B\) be a Banach algebra and \(\theta : A \to B\) be a homomorphism. Let \(\{a_n\}\) be a sequence in \(A\) such that \(a_n a_m = 0\) for \(n \neq m\), and let \(\{k_n\}\) be a sequence in \(\mathbb{N}\) such that \(p_{k_n}(a_n) \neq 0\) for all \(n \in \mathbb{N}\).

(i) If \(\{b_n\}\) is a sequence in \(A\) such that \(a_n b_m = 0\) for \(n \neq m\), and if \(\{x_n\}\) is a sequence in \(\mathbb{N}\) such that \(p_{r_n}(b_n) \neq 0\) for all \(n \in \mathbb{N}\), then there is a constant \(C > 0\) such that
\[
\|\theta(a_n b_n)\| \leq C p_{k_n}(a_n)p_{r_n}(b_n),
\]
for all \(n \in \mathbb{N}\).

(ii) If \(\{b_n\}\) is a sequence in \(A\) such that \(a_n b_m = 0\) for all \(n \neq m\) and \(b_n A \not\subseteq \ker(p_{r_n})\) for some subsequence \(\{p_{r_n}\}\), then the linear operator \(T_n : A \to B, T_n(x) = \theta(a_n b_n x)\) is continuous for large enough \(n \in \mathbb{N}\). Moreover, for every bounded subset \(E \subseteq A\) there exists an \(M > 0\) such that for every \(x \in E\), we have
\[
\|T_n(x)\| \leq Mp_{k_n}(a_n)p_{r_n}(b_n),
\]
for large enough \(n\).

**Proof.** (i) By the hypothesis we may assume that \(p_{k_n}(a_n) = p_{r_n}(b_n) = 1\). Assume towards a contradiction that there is no \(C\) for which (2.4) holds. Then, by following a similar method as in the previous proposition, there is a map
\[
T : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad T((i,j)) = n(i,j)
\]
such that it is increasing on both components, and moreover,
\[(2.6) \quad \|\theta(u_{i,j}v_{i,j})\| \geq 4^{i+j},\]
where \(u_{i,j} = a_{n_{i,j}}\) and \(v_{i,j} = b_{n_{i,j}}\). By the same argument as before, for each \(i \in \mathbb{N}\), the series \(f_i = \sum_{k=1}^{\infty} 2^{-k} u_{i,k}\) is convergent and so \(f_i \in A\). For each \(i \in \mathbb{N}\) choose an integer \(j(i) > i\) so that \(\|\theta(f_i)\| \leq 2^{i(i)}\), and set
\[S = \sum_{k=1}^{\infty} 2^{-k} v_{i,j(k)}(k),\]
which is a convergent series in \(A\), by a similar argument as in the previous proposition. Hence \(f_iS = 2^{-i-j(i)} u_{i,j(i)}v_{i,j(i)}\) for each \(i\) and so by the hypothesis and \((2.6)\) we have
\[\|\theta(f_iS)\| = \|2^{-i-j(i)}\theta(u_{i,j(i)}v_{i,j(i)})\| \geq 2^{i+j(i)}\]
On the other hand,
\[\|\theta(f_iS)\| \leq \|\theta(S)\|\|\theta(f_i)\| \leq 2^{i(i)}\|\theta(S)\|,\]
and hence \(2^i \leq \|\theta(S)\|\) for each \(i \in \mathbb{N}\), which is a contradiction. This completes the proof of (i).

(ii) By the hypothesis, there exists a sequence \(\{s_n\}\) in \(A\) such that \(p_{r_n}(s_n, s_n) = 1\). If \(T_n\) is discontinuous for infinitely many \(n \in \mathbb{N}\), by a similar argument as in the previous proposition, there exists a sequence \(\{x_n\}\) in \(A\) such that \(p_{r_n}(b_n x_n) \neq 0\) for all \(n \in \mathbb{N}\) and \(\|\theta(a_n b_n x_n)\| = \|T_n(x_n)\| > np_{r_n}(a_n) p_{r_n}(b_n x_n)\) for all \(n \in \mathbb{N}\). Now by applying (i) with the sequence \(\{b_n x_n\}\) instead of \(\{x_n\}\), there exists \(C > 0\) such that
\[\|\theta(a_n b_n x_n)\| \leq Cp_{r_n}(a_n) p_{r_n}(b_n x_n),\]
which implies that \(n < C\) for all \(n \in \mathbb{N}\). By this contradiction we conclude that the linear operator \(T_n\) is continuous for large enough \(n \in \mathbb{N}\).

Since the operator
\[\frac{T_n}{p_{r_n}(a_n) p_{r_n}(b_n)}\]
is continuous, it maps bounded sets into bounded sets and the inverse images of open sets are open sets. By a similar argument as before, there exists an \(M > 0\) such that \(\|T_n(x)\| \leq M p_{r_n}(a_n) p_{r_n}(b_n)\) for all \(x \in E\) and for large enough \(n\).

3. Automatic continuity of module homomorphisms from Fréchet algebras

In [7] the authors showed that if \(M = \{(x_n)_{n=0}^{\infty} \in L^1(w) : x_0 = 0\}\) and \(L(M) = L^1(w)\), then every module homomorphism from \(M\) into any Banach \(L^1(w)\)-module is continuous, where \(w\) is a weight function.
By using the results of Section 2 we try to extend some known results on the automatic continuity of module homomorphisms as well as homomorphisms for Fréchet algebras. We show that if \((A, \{p_n\})\) is a unital Fréchet algebra with a continued bisection of the identity, then \(A\) is functionally continuous.

**Lemma 3.1 ([3, 1.3.24]).** Let \((A, \{p_n\})\) be a unital Fréchet algebra with a continued bisection \((\{r_n\}, \{s_n\})\) of the identity. Then the set \(\{s_n : n \in \mathbb{N}\}\) is orthogonal.

**Lemma 3.2.** Let \((A, \{p_n\})\) be a unital Fréchet algebra with continued bisection \((\{r_n\}, \{s_n\})\) of the identity. Then \(p_n(r_m) \geq 1, p_n(s_m) \geq 1\) for all \(m, n \in \mathbb{N}\).

**Proof.** By the hypothesis, \(e_A = r_1 + s_1, r_1^2 = r_1\) and \(s_1^2 = s_1\). Hence

\[
1 = p_n(e_A) \leq p_n(r_1) + p_n(s_1), \quad p_n(r_1) \leq p_n(r_1)^2, \quad p_n(s_1) \leq p_n(s_1)^2.
\]

Thus \(p_n(r_1)\) and \(p_n(s_1)\) are either zero or \(p_n(r_1) \geq 1, p_n(s_1) \geq 1\). If \(p_n(r_1)\) and \(p_n(s_1)\) are zero, then \(p_n(e_A) = 0\), which is a contradiction. So we may suppose that \(p_n(r_1) \geq 1\). By the definition of the continued bisection of the identity, \(r_1 = as_1b\), for some \(a, b \in A\). So \(1 \leq p_n(r_1) \leq p_n(a)p_n(s_1)p_n(b)\), implying that \(p_n(s_1) \geq 1\). Now fix \(m, n \in \mathbb{N}\) and suppose that \(p_n(r_m) \geq 1\). Then \(1 \leq p_n(r_m) \leq p_n(r_{m+1}) + p_n(s_{m+1})\). So by the above discussion, we have \(p_n(r_{m+1}) \geq 1\) and \(p_n(s_{m+1}) \geq 1\).

**Theorem 3.3.** Let \((A, \{p_n\})\) be a unital Fréchet algebra with a continued bisection \((\{r_n\}, \{s_n\})\) of the identity. Let \((X, \{q_n\})\) be a Fréchet left A-module such that \(s_n \cdot X \subseteq \ker(q_{k_n})\) for some subsequence \(\{q_{k_n}\}\) and let \(Y\) be a Banach left A-module which is unital. Then every left A-module homomorphism \(\theta : X \to Y\) is automatically continuous. Moreover, for every bounded subset \(E\) of \(X\) there exists an \(M > 0\) such that

\[
\|s_n \cdot \theta(x)\| \leq Mp_{k_n}(s_n)^2,
\]

for all \(x \in E\) and for large enough \(n \in \mathbb{N}\).

**Proof.** Let \(I(s_n)\) be the two-sided ideal generated by \(s_n\) in \(A\), and let \(J(\theta)\) be the set of all \(a \in A\) such that the map \(x \to a\theta(x)\) is continuous. It is easy to see that \(I(s_n) = A\) for every \(n \geq 1\) and \(J(\theta)\) is a two-sided ideal of \(A\). Now by Lemma 3.2, \(p_m(s_n) \geq 1\) for all \(m, n \in \mathbb{N}\). Hence \(p_{k_n}(s_n) \geq 1\) for all \(n \in \mathbb{N}\). Let \(\theta : X \to Y\) be a left A-module homomorphism. One can check easily that \(s_n s_m = 0\) for all \(m \neq n\). By taking \(a_n = b_n = s_n\) in Proposition 2.2 (ii), the maps

\[
T_n : X \to Y
\]

\[
T_n(x) = s_n s_n \cdot \theta(x) = s_n \cdot \theta(x) = \theta(s_n \cdot x)
\]

are continuous for large enough \(n \in \mathbb{N}\) and hence \(s_m \in J(\theta)\) for some \(m \geq 1\). Since \(J(\theta)\) is a two-sided ideal, we have \(e_A \in I(s_m) \subseteq J(\theta)\). Therefore, \(\theta\) is
Let $\theta : X \to Y$ be a left $A$-module homomorphism and suppose that $Y_m$ is the completion of $Y/\ker(g_m)$ with respect to the norm $g'_m(y + \ker(g_m)) = g_m([y]_m) = g_m(y)$ for $y \in Y$, which is a unital Banach left $A$-module. Then, by Theorem 3.3, the left $A$-module homomorphism $\theta_m = [\theta]_m : X \to Y_m$ defined by $\theta_m(x) = \theta(x) + \ker(g_m)$ is continuous and if $E$ is a bounded subset of $X$, then there exists an $M > 0$ such that for all $x \in E$ we have

$$g_m(s_n \cdot \theta(x)) = g_m(\theta(s_n \cdot x)) = g'_m(\theta_m(s_n \cdot x)) = g'_m(s_n \cdot \theta_m(x)) \leq M p_k_n(s_n)^2,$$

for all large enough $n \in \mathbb{N}$. Now we suppose that $\{x_n\}$ is a sequence in $X$ such that $\lim_{k \to \infty} x_k = 0$ and $\lim_{k \to \infty} \theta(x_k) = y$ in $Y$. Since $\lim_{k \to \infty} \theta_m(x_k) = 0$ in $Y_m$, and on the other hand, $\lim_{k \to \infty} [y]_m = y$ in $Y_m$, it follows that $y \in \ker(g_m)$. Since $m$ is arbitrary, we conclude that $y \in \bigcap \ker(g_m)$ and Remark 2.1(ii) implies, $y = 0$. By the Closed Graph Theorem, the result follows.

Let $A$ and $B$ be Fréchet algebras and $\theta : A \to B$ be a homomorphism. Then the continuity ideal $\mathcal{I}(\theta)$ of the homomorphism $\theta$ is defined by

$$\mathcal{I}(\theta) = \{ a \in A : \theta(a)b = b\theta(a) = 0 \text{ for every } b \in \mathcal{S}(\theta) \}.$$ 

If $A$ is unital and $e_A \in \mathcal{I}(\theta)$, then $\theta$ is continuous on $A$.

**Lemma 3.5.** Let $A$ and $B$ be Fréchet algebras and $\theta : A \to B$ be a homomorphism. Then $\mathcal{I}(\theta)$ is a two sided ideal in $A$, and

$$\mathcal{I}(\theta) = \{ a \in A : \theta(a)b = \theta(ab) \text{ and } \theta(b) = \theta(ba) \text{ are both continuous mapping from } A \text{ to } B \}.$$ 

**Proof.** It is easy to see that $\mathcal{I}(\theta)$ is a linear subspace of $A$. Suppose that $b \in \mathcal{S}(\theta)$. So there exists a sequence $\{a_n\}$ in $A$ such that $a_n \to 0$ and $\theta(a_n) \to b$. If $a \in A$, then $a_n a \to 0$ and $\theta(a_n a) \to b\theta(a)$. Therefore for every $b \in \mathcal{S}(\theta)$
and $a \in A$, we have $b\theta(a) \in S(\theta)$. If $x \in \mathcal{I}(\theta)$ and $a \in A$, then $b\theta(ax) = (b\theta(a))\theta(x) = 0$, since $b\theta(a) \in S(\theta)$, and $\theta(ax)b = \theta(a)(\theta(x)b) = 0$. Hence $ax, xa \in \mathcal{I}(\theta)$, this means that $\mathcal{I}(\theta)$ is a two sided ideal in $A$. For the proof of the second part, we define continuous linear maps $s_\alpha, s^\alpha : B \to B$ by $s_\alpha(b) = \theta(\alpha)b$ and $s^\alpha(b) = b\theta(a)$ where $b \in B$ and $a \in A$. It easy to see that $s_\alpha \theta = \theta_\alpha$ and $s^\alpha \theta = \theta^\alpha$, by [3, 5.2.2(ii)], $\theta(\alpha)S(\theta) = s_\alpha[S(\theta)] = S(s_\alpha \theta) = S(\theta_\alpha)$ and $\overline{S(\theta)} \theta(a) = s^\alpha(\overline{S(\theta)} \theta(a)) = S(s^\alpha \theta) = S(\theta^\alpha)$. Thus $a \in \mathcal{I}(\theta)$ if and only if $\theta_\alpha$ and $\theta^\alpha$ are both continuous mapping from $A$ to $B$. □

**Theorem 3.6.** Let $(A, \{p_n\})$ be a unital Fréchet algebra with a continued bisection $(\{r_n\}, \{s_n\})$ of the identity and let $B$ be a unital Banach algebra. Then every unital homomorphism $\theta : A \to B$ is automatically continuous.

**Proof.** Let $\theta : A \to B$ be a unital homomorphism, i.e., $\theta(e_A) = e_B$. By Proposition 2.3(ii), the maps

$$T_n : A \to B$$

$$T_n(x) = \theta(s_n s_n x) = \theta(s_n x)$$

are continuous for large enough $n \in \mathbb{N}$, and so, by a similar argument as before, there exists $k \in \mathbb{N}$ with $s_k \in \mathcal{I}(\theta)$. Again by a successive argument as in the Theorem 3.3, we see that $r_k , r_{k-1}, s_{k-1}, \ldots, r_1, s_1, e_A$ belong to $\mathcal{I}(\theta)$, since $\theta$ is unital, so $\overline{S(\theta)}(\theta) = 0$ and hence $\theta$ is continuous. □

**Corollary 3.7.** Let $(A, \{p_n\})$ be a unital Fréchet algebra with a continued bisection of the identity. Then $A$ is functionally continuous.

**Corollary 3.8.** Let $(A, \{p_n\})$ be a unital Fréchet algebra with a continued bisection of the identity and $(B, \{q_n\})$ be a unital Fréchet algebra. Then every unital homomorphism from $A$ into $B$ is automatically continuous.

**Proof.** Let $\theta : A \to B$ be a homomorphism and let $B_m$ be the completion of $B / \ker(q_m)$ with respect to the norm $q'_m(b + \ker(q_m)) = q'_m([b]_m) = q_m(b)$ for $b \in B$, which is a unital Banach algebra. Then, by the Theorem 3.6, the homomorphism $\theta_m = [\theta]_m : A \to B_m$ is continuous.

Let $\{a_k\}$ be a sequence in $A$ such that $\lim_{k \to \infty} a_k = 0$ and $\lim_{k \to \infty} \theta(a_k) = b$ in $B$. Since $\lim_{k \to \infty} \theta_m(a_k) = 0$ in $B_m$ and, on the other hand, $\lim_{k \to \infty} \theta_m(a_k) = [b]_m$, it follows that $b \in \ker(q_m)$. Since $m$ is arbitrary, we conclude that $b \in \cap \ker(q_m)$ and hence $b = 0$, by Remark 2.1(ii). Now the result follows by the Closed Graph Theorem. □

**Remark 3.9.** Let $(A, \{p_n\})$ be a Fréchet algebra. If we define seminorms $P_n$ on $B(A)$, the algebra of all bounded linear operators on $A$, by

$$P_n(A) = \sup_{p_n(a) \leq 1} p_n(A(a)), \quad A \in B(A) \text{ and } a \in A,$$
then it is easy to see that \((B(A), \{P_n\})\) is a Fréchet algebra.

The following lemma has been proved in \([3, 2.5.11]\), for Banach algebras, but it is also valid for Fréchet algebras.

**Lemma 3.10.** Let \(A\) be a Fréchet algebra which is linearly homeomorphic to \(A \oplus A\). Then the Fréchet algebra \(B(A)\) has a continued bisection of the identity.

There are some examples of Fréchet algebra \(A\) such that \(A \simeq A \oplus A\). For example: (i) \(C(\Omega)\), where \(\Omega\) is an infinite, metrizable, compact space; (ii) \(C^n([0, 1])\) for \(n \in \mathbb{N}\); (iii) \(L^1([0, 1])\), \([2]\).

**Acknowledgements**

This research was partially supported by the Center of Excellence for Mathematics, University of Shahrekord. The author wishes to thank Prof. Taher Ghasemi Honary for guidance and help in preparing this paper.

**References**


(Hamid Shayanpour) DEPARTMENT OF PURE MATHEMATICS, FACULTY OF MATHEMATICAL SCIENCES, UNIVERSITY OF SHAHREKORD, P.O. BOX 88186-34141, SHAHREKORD, IRAN.

E-mail address: h.shayanpour@sci.sku.ac.ir