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MODULE HOMOMORPHISMS FROM FRÉCHET ALGEBRAS

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Dedicated to Prof. Taher Ghasemi Honary

Abstract. We first study some properties of A-module homomorphisms $\theta : X \to Y$, where X and Y are Fréchet A-modules and A is a unital Fréchet algebra. Then we show that if there exists a continued bisection of the identity for A, then θ is automatically continuous under certain condition on X. In particular, every homomorphism from A into certain Fréchet algebras (including Banach algebra) is automatically continuous. Finally, we show that every unital Fréchet algebra with a continued bisection of the identity, is functionally continuous.
Keywords: Automatic continuity, Fréchet algebras, module homomorphism, continued bisection of the identity, Fréchet A-module.
MSC(2010): Primary: 46H40; Secondary: 16D10, 46H05.

1. Introduction

In the following, we present the notations, definitions and known results, which are related to our work. For further details one can refer, for example, to [3].

A Fréchet algebra is a complete metrizable topological algebra whose topology is defined by a separating family $\mathcal{P} = (p_\alpha)$ of submultiplicative seminorms. Note that the topology of a Fréchet algebra A can be generated by a sequence $\{p_n\}_{n\in\mathbb{N}}$ of separating submultiplicative seminorms, i.e., $p_n(xy) \leq p_n(x)p_n(y)$ for all $n \in \mathbb{N}$ and every $x, y \in A$, such that $p_n(x) \leq p_{n+1}(x)$ for all $x \in A$ and $n \in \mathbb{N}$. The Fréchet algebra A with the above generating sequence of seminorms is denoted by $(A, \{p_n\})$. Note that a sequence $\{x_k\}$ in the Fréchet algebra $(A, \{p_n\})$ converges to $x \in A$ if and only if $p_n(x_k - x) \to 0$, for each $n \in \mathbb{N}$, as $k \to \infty$.

Let A be a complex algebra. A character on A is a non-zero homomorphism from A into $\mathbb{C}$. The set of all characters of A is denoted by S_A. If A is a complex topological algebra, then the set of continuous characters of A is denoted by M_A. If A is a complex topological algebra with the dual space A',

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then, $M_A \cup \{0\} \subseteq A'$. The relative w^*-topology on $M_A \cup \{0\}$ is the Gelfand topology. A topological algebra A is called functionally continuous if every character on A is continuous, i.e., $M_A = S_A$. It is known that every Banach algebra is functionally continuous. Let A be a complex algebra. A left [right] A-module is a complex linear space E together with a bilinear map $(a, x) \mapsto a \cdot x$ $[(a, x) \mapsto x \cdot a]$, $A \times E \to E$, such that

$$a \cdot (b \cdot x) = ab \cdot x \ [(x \cdot a) \cdot b = x \cdot ab] \quad (a, b \in A,\ x \in E).$$

An A-bimodule is a linear space E which is a left A-module and a right A-module such that

$$a \cdot (x \cdot b) = (a \cdot x) \cdot b \quad (a, b \in A,\ x \in E).$$

Let A be an algebra, and let E and F be left [right] A-modules. A linear map $T : E \to F$ is a left [right] A-module homomorphism if

$$T(a \cdot x) = a \cdot T(x) \ [T(x \cdot a) = T(x) \cdot a] \quad (a \in A,\ x \in E)$$

Let E and F be A-bimodules. A linear map $T : E \to F$ is an A-bimodule homomorphism if it is both a left and a right A-module homomorphism. For a Fréchet algebra A, a Fréchet (Banach) left A-module is a Fréchet (Banach) space X which is endowed with a structure of left A-module such that the binary action $\cdot : A \times X \to X$, $(a, x) \mapsto a \cdot x$ is continuous. When A is unital with the unit element e_A, then a left A-module X is called unital if $e_A \cdot x = x$ for every $x \in X$. Similarly, a Fréchet (Banach) right A-module is a Fréchet (Banach) space X which is endowed with a structure of right A-module such that the binary action $\cdot : A \times X \to X$, $(a, x) \mapsto x \cdot a$ is continuous. A linear space X is a Fréchet (Banach) A-bimodule if it is both left and right Fréchet (Banach) A-module and if

$$a \cdot (x \cdot b) = (a \cdot x) \cdot b \qquad (a, b \in A, x \in X).$$

Let A and B be topological linear spaces, and let $\theta : A \to B$ be a linear mapping. The separating space of θ is defined by

$$\mathfrak{S}(\theta) = \{b \in B : \text{there exists a net } (a_\delta) \text{ in } A \text{ such that } a_\delta \to 0 \text{ and } \theta(a_\delta) \to b\}.$$

The separating space $\mathfrak{S}(\theta)$ is a closed linear subspace of B and moreover, if A and B are F-spaces, by the Closed Graph Theorem, θ is continuous if and only if $\mathfrak{S}(\theta) = \{0\}$.

Let A be an algebra. Then the elements $a, b \in A$ are (mutually) orthogonal in A, written $a \perp b$, if $ab = 0 = ba$. Let A be a unital algebra. A continued bisection of the identity for A is a pair $(\{r_n\}, \{s_n\})$ of sequences of idempotents of A such that $e_A = r_1 + s_1$ and for each $n \in \mathbb{N}$ we have $r_n = r_{n+1} + s_{n+1}$ and $Ar_nA = As_nA$. By the definition of the continued bisection $(\{r_n\}, \{s_n\})$ of the identity, r_n, s_n and $r_n + s_n$ are idempotents and hence $(r_n + s_n)^2 = r_n + s_n$. Therefore,

$$r_n s_n + s_n r_n = 0. \tag{1.1}$$

Now we have

$$(1.2)\qquad \begin{aligned} r_ns_n + r_ns_nr_n = r_n(r_ns_n + s_nr_n) = 0 =&(r_ns_n + s_nr_n)r_n \\ =&r_ns_nr_n + s_nr_n, \end{aligned}$$

and so $r_ns_n = s_nr_n$. By (1.1) and (1.2), we have $r_ns_n = 0 = s_nr_n$ and hence $s_n \perp r_n$.

Continued bisections of the identity originate from Johnsons study of automatic continuity of homomorphisms from $B(X)$ for a Banach space X. In [5] Johnson proved $B(X)$ has a continued bisections of the identity whenever X is isomorphic to $X \oplus X$. However, there are operator algebras on some spaces without any continued bisection of the identity, for further details one can refer, for example, to [9] and [10]. Figiel in [4] gives an example of a reflexive space X with a continued bisections of the identity, which is not isomorphic to $X \oplus X$. Laustsen in [8] proved that every unital, properly infinite ring R has a continued bisection of the identity. Here a properly infinite ring R is a ring with idempotents elements P_1 and P_2 in R that are orthogonal and satisfy $P_n = ST$ and $e_R = TS$ (e_R is identity of R), for some elements S and T in R and for $n = 1, 2$.

In the first section, we prove some inequalities for A-module homomorphisms between Fréchet A-modules as well as homomorphisms between Fréchet algebras.

In the second section, we show that if $(A, \{p_n\})$ is a unital Fréchet algebra with a continued bisection $\{\{r_n\}, \{s_n\}\}$ of the identity, $(X, \{q_n\})$ is a Fréchet left A-module such that $s_nX \nsubseteq \ker(q_{k_n})$ for an increasing subsequence $\{k_n\}$ of $\mathbb{N}$, and Y is a Banach left A-module which is unital, then every left A-module homomorphism $\theta : X \to Y$ is automatically continuous. Moreover, for every bounded subset E of X, there exists an $M > 0$ such that $\|s_n\theta(x)\| \leq Mp_{k_n}(s_n)^2$, for all $x \in X$ and $n \in \mathbb{N}$. We also show that if $(A, \{p_n\})$ is a unital Fréchet algebra with a continued bisection of the identity, then every homomorphism $\theta : A \to B$, where B is a Banach algebra, is automatically continuous.

In particular, if $(A, \{p_n\})$ is a unital Fréchet algebra with a continued bisection of the identity, then A is functionally continuous.

2. Some inequalities for A-module homomorphisms

In 1974, Sinclair studied module homomorphisms between Banach A-modules, when A is a regular semisimple commutative unital Banach algebra [12]. Bade and Curtis in [1] showed that if θ is a homomorphism defined on a Banach algebra A, $\{a_n\}$ is a sequence of mutually orthogonal elements of A and if $\{b_n\}$ is an arbitrary sequence in A such that $b_na_n = b_n$, for all $n \in \mathbb{N}$, then there exists an $M > 0$ such that $\|\theta(b_n)\| \leq M\|a_n\|\|b_n\|$, for all $n \in \mathbb{N}$. Also, Johnson in [6] studied module homomorphisms between Banach G-modules, when G

is a locally compact abelian group. We now extend some of these results for Fréchet algebras and obtain some results on the automatic continuity of module homomorphisms on Fréchet A-modules.

First we present the following general results on Fréchet A-modules.

Remark 2.1. (i) Let $(A, \{p_n\})$ be a Fréchet algebra and $\{a_n\}$ be a sequence in A. Since $\{p_n\}$ is separating, there exists $k_1 \in \mathbb{N}$ such that $p_{k_1}(a_1) \neq 0$. Since $\{p_n\}$ is also an increasing sequence, we can choose $p_{k_2} \geq p_{k_1}$ such that $p_{k_2}(a_2) \neq 0$. By a successive argument, we have a subsequence $\{p_{k_n}\}$ of the sequence $\{p_n\}$ such that $p_{k_n}(a_n) \neq 0$.

(ii) Let $(A, \{p_n\})$ be a Fréchet algebra, $(X, \{q_n\})$ be a Fréchet left A-module. Since X is a Fréchet space, the sequence $\{q_n\}$ is separating. Therefore, $\bigcap_{n=1}^{\infty} \ker(q_n) = 0$.

The proof of the following proposition is inspired by the proof of the theorem 2.1 in [1].

Proposition 2.2. *Let $(A, \{p_n\})$ be a Fréchet algebra, $(X, \{q_n\})$ be a Fréchet left A-module, Y be a Banach left A-module, and $\theta : X \to Y$ be a left A-module homomorphism. Let $\{a_n\}$ be a sequence in A such that $a_n a_m = 0$ for $n \neq m$, and let $\{k_n\}$ be a sequence in $\mathbb{N}$ such that $p_{k_n}(a_n) \neq 0$ for all $n \in \mathbb{N}$.*

(i) If $\{x_n\}$ is a sequence in X such that $a_n \cdot x_m = 0$ for $n \neq m$, and if $\{r_n\}$ is a sequence in $\mathbb{N}$ such that $q_{r_n}(x_n) \neq 0$ for all $n \in \mathbb{N}$, then there is a constant $C > 0$ such that

$$\|\theta(a_n \cdot x_n)\| \leq C p_{k_n}(a_n) q_{r_n}(x_n), \tag{2.1}$$

for all $n \in \mathbb{N}$.

(ii) If $\{b_n\}$ is a sequence of elements of A such that $a_n b_m = 0$ for all $n \neq m$ and $b_n \cdot X \nsubseteq \ker(q_{r_n})$ for some subsequence $\{q_{r_n}\}$, then the linear operator $a_n b_n \cdot \theta(\cdot)$ is continuous for large enough $n \in \mathbb{N}$. Moreover, if $p_{r_n}(b_n) \neq 0$ for all $n \in \mathbb{N}$, then for every bounded subset $E \subseteq X$ there exists an $M > 0$ such that for every $x \in E$, we have

$$\|a_n b_n \cdot \theta(x)\| \leq M p_{k_n}(a_n) p_{r_n}(b_n), \tag{2.2}$$

for large enough $n \in \mathbb{N}$.

Proof. (i) By the hypothesis we may take a sequence $\{a_n'\}$ in A and a sequence $\{x_n'\}$ in X such that $p_{k_n}(a_n') = q_{r_n}(x_n') = 1$. Renaming the sequence $\{a_n'\}$ by $\{a_n\}$ and the sequence $\{x_n'\}$ by $\{x_n\}$, we may assume that $p_{k_n}(a_n) = q_{r_n}(x_n) = 1$, for all $n \in \mathbb{N}$. Assume towards a contradiction that there is no C for which (2.1) holds. Then there is a map

$$T : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad T((i,j)) = n(i,j),$$

such that it is increasing on both components, and moreover,

$$\|\theta(u_{(i,j)}.v_{(i,j)})\| \geq 4^{i+j}, \tag{2.3}$$

where $u_{(i,j)} = a_{n(i,j)}$ and $v_{(i,j)} = x_{n(i,j)}$. We set

$$S_n^i = \sum_{k=1}^{n} 2^{-k} u_{(i,k)}, \quad (i, n \in \mathbb{N}).$$

Fixing $i \in \mathbb{N}$, for each $m \in \mathbb{N}$ there exists k_m such that for every $j > k_m$, we have $m < k_{n(i,j)}$. On the other hand, we may assume that the sequence $\{p_n\}$ is increasing and hence for every $m \in \mathbb{N}$ and $j > k_m$ we have $p_m(\cdot) \leq p_{k_{n(i,j)}}(\cdot)$. By taking $p_{(i,j)} = p_{k_{n(i,j)}}$, for each $m \in \mathbb{N}$ and $n > r > k_m$ we have

$$p_m(S_n^i - S_r^i) \leq \sum_{k=r+1}^{n} \frac{p_m(u_{(i,k)})}{2^k} \leq \sum_{k=r+1}^{n} \frac{p_{(i,k)}(u_{(i,k)})}{2^k} = \sum_{k=r+1}^{n} \frac{1}{2^k},$$

which shows that $p_m(S_n^i)$ is a Cauchy sequence for each $m \in \mathbb{N}$. Therefore, for each i, the series

$$f_i = \sum_{k=1}^{\infty} 2^{-k} u_{(i,k)},$$

converges in A. Let L_i be the operation of left multiplication by f_i in Y. We have $f_i \cdot v_{(i,j)} = 2^{-j} u_{(i,j)} \cdot v_{(i,j)}$ and

$$\begin{aligned} L_i(\theta(v(i,j))) &= f_i \cdot \theta(v_{(i,j)}) = \theta(f_i \cdot v_{(i,j)}) \\ &= \theta(2^{-j} u_{(i,j)} \cdot v_{(i,j)}) = 2^{-j} \theta(u_{(i,j)} \cdot v_{(i,j)}). \end{aligned}$$

Since Y is a Banach left A-module, the operation of left multiplication in Y is continuous and so L_i is a non-zero continuous linear operator on Y. We now choose an integer $j(i) > i$ so that $\|L_i\| \leq 2^{j(i)}$ for each i, and set

$$S = \sum_{k=1}^{\infty} 2^{-k} v_{(k,j(k))}.$$

For each m, there exists r_m such that for $i > r_m$ we have $m < r_{n(i,1)} \leq r_{n(i,j(i))}$. On the other hand, since the sequence $\{q_n\}$ is increasing, for every $m \in \mathbb{N}$ and $i > r_m$ we have $q_m(\cdot) \leq q_{r_{n(i,j(i))}}(\cdot)$. By a similar method as in the proof of the convergence of f_i, S is convergent in X. Hence $f_i \cdot S = 2^{-i-j(i)} u_{(i,j(i))} \cdot v_{(i,j(i))}$ for each i. By the hypothesis and (2.3), we have

$$\|L_i(\theta(S))\| = \|f_i \theta(S)\| = \|2^{-i-j(i)} \theta(u_{(i,j(i))} \cdot v_{(i,j(i))})\| \geq 2^{i+j(i)}.$$

On the other hand,

$$\|L_i(\theta(S))\| \leq \|\theta(S)\| \|L_i\| \leq 2^{j(i)} \|\theta(S)\|.$$

Then we have $2^i \leq \|\theta(S)\|$ for every $i \in \mathbb{N}$. The resulting contradiction completes the proof of (i).

(ii) By the hypothesis, there exists a sequence $\{s_n\}$ in X such that $q_{r_n}(b_n \cdot s_n) = 1$. If $a_n b_n \cdot \theta$ is discontinuous for infinitely many $n \in \mathbb{N}$, without loss of generality, we may assume that for every $n \in \mathbb{N}$ there exists a sequence $\{x^n_m\}_m$ in X such that $\lim_{m\to\infty} x^n_m = 0$ and $\lim_{m\to\infty} a_n b_n \cdot \theta(x^n_m) = y_n$, but $y_n \neq 0$. Since for every n, $\lim_{m\to\infty} q_{r_n}(b_n \cdot x^n_m) = 0$, it follows that for large enough m we have

$$\|a_n b_n \cdot \theta(x^n_m)\| > np_{k_n}(a_n) q_{r_n}(b_n \cdot x^n_m).$$

Therefore, there exists a sequence $\{x_n\} \subseteq X$ such that

$$\|a_n b_n \cdot \theta(x_n)\| > np_{k_n}(a_n) q_{r_n}(b_n \cdot x_n).$$

So for each $n \in \mathbb{N}$, there exists $\varepsilon_n > 0$ such that

$$\|a_n b_n \cdot \theta(x_n)\| > \varepsilon_n,$$

also we can choose $\lambda_n \in \mathbb{N}$ such that

$$\delta_n = \varepsilon_n - \frac{1}{\lambda_n}\left(1 + \frac{np_{k_n}(a_n)}{1 + \|a_n b_n \cdot \theta(s_n)\|}\right) > 0.$$

If $q_{r_n}(b_n \cdot x_n) = 0$ for some n, by taking $z_n = x_n + \dfrac{s_n}{\lambda_n(1 + \|a_n b_n \cdot \theta(s_n)\|)}$, we have

$$\begin{aligned}\frac{1}{\lambda_n(1 + \|a_n b_n \cdot \theta(s_n)\|)} - q_{r_n}(b_n \cdot x_n) \le & q_{r_n}(b_n \cdot z_n) \\ \le & \frac{1}{\lambda_n(1 + \|a_n b_n \cdot \theta(s_n)\|)} + \\ & q_{r_n}(b_n \cdot x_n).\end{aligned}$$

Hence $q_{r_n}(b_n \cdot z_n) = \dfrac{1}{\lambda_n(1 + \|a_n b_n \cdot \theta(s_n)\|)} \neq 0$ and moreover,

$$np_{k_n}(a_n) q_{r_n}(b_n \cdot z_n) = \frac{np_{k_n}(a_n)}{\lambda_n(1 + \|a_n b_n \cdot \theta(s_n)\|)} = -\delta_n + \varepsilon_n - \frac{1}{\lambda_n}.$$

Thus

$$\begin{aligned}\|a_n b_n \cdot \theta(z_n)\| \ge & \|a_n b_n \cdot \theta(x_n)\| - \frac{\|a_n b_n \cdot \theta(s_n)\|}{\lambda_n(1 + \|a_n b_n \cdot \theta(s_n)\|)} \\ \ge & \varepsilon_n - \frac{1}{\lambda_n} = np_{k_n}(a_n) q_{r_n}(b_n \cdot z_n) + \delta_n \\ > & np_{k_n}(a_n) q_{r_n}(b_n \cdot z_n).\end{aligned}$$

Now by replacing x_n by z_n, we may always assume that there exists $\{x_n\} \subseteq X$ such that $q_{r_n}(b_n \cdot x_n) \neq 0$ and $\|a_n b_n \cdot \theta(x_n)\| > np_{k_n}(a_n) q_{r_n}(b_n \cdot x_n)$ for all $n \in \mathbb{N}$.

Now by applying (i) for the sequence $\{b_n \cdot x_n\}$ instead of $\{x_n\}$, there exists $C > 0$ such that

$$np_{k_n}(a_n)q_{r_n}(b_n \cdot x_n) < \|a_n b_n \cdot \theta(x_n)\| \leq Cp_{k_n}(a_n)q_{r_n}(b_n \cdot x_n),$$

which implies that $n < C$ for all $n \in \mathbb{N}$. By this contradiction we conclude that the linear operator $a_n b_n \cdot \theta(\cdot)$ is continuous for large enough $n \in \mathbb{N}$.

For the proof of the second part of (ii), note that the operator

$$\frac{a_n b_n \cdot \theta}{p_{k_n}(a_n)p_{r_n}(b_n)}$$

is continuous for large enough n. Hence it maps bounded sets into bounded sets, and the inverse images of open sets are open sets. If $B(0,1)$ is the open unit ball in Y, then there exists k such that

$$\frac{a_n b_n \cdot \theta(V_k)}{p_{k_n}(a_n)p_{r_n}(b_n)} \subseteq B(0,1),$$

where $V_k = \{x \in X : q_k(x) < \frac{1}{k}\}$. On the other hand, if E is a bounded set in X, there exists an $M > 0$ such that $E \subseteq MV_k$ and so $\|a_n b_n \cdot \theta(x)\| \leq Mp_{k_n}(a_n)p_{r_n}(b_n)$ for all $x \in E$ and so (2.2) holds. □

Proposition 2.3. *Let $(A, \{p_n\})$ be a Fréchet algebra, B be a Banach algebra and $\theta : A \to B$ be a homomorphism. Let $\{a_n\}$ be a sequence in A such that $a_n a_m = 0$ for $n \neq m$, and let $\{k_n\}$ be a sequence in $\mathbb{N}$ such that $p_{k_n}(a_n) \neq 0$ for all $n \in \mathbb{N}$.*

(i) *If $\{b_n\}$ is a sequence in A such that $a_n b_m = 0$ for $n \neq m$, and if $\{r_n\}$ is a sequence in $\mathbb{N}$ such that $p_{r_n}(b_n) \neq 0$ for all $n \in \mathbb{N}$, then there is a constant $C > 0$ such that*

$$\|\theta(a_n b_n)\| \leq Cp_{k_n}(a_n)p_{r_n}(b_n), \tag{2.4}$$

for all $n \in \mathbb{N}$.

(ii) *If $\{b_n\}$ is a sequence in A such that $a_n b_m = 0$ for all $n \neq m$ and $b_n A \nsubseteq \ker(p_{r_n})$ for some subsequence $\{p_{r_n}\}$, then the linear operator $T_n : A \to B$, $T_n(x) = \theta(a_n b_n x)$ is continuous for large enough $n \in \mathbb{N}$. Moreover, for every bounded subset $E \subseteq A$ there exists an $M > 0$ such that for every $x \in E$, we have*

$$\|T_n(x)\| \leq Mp_{k_n}(a_n)p_{r_n}(b_n), \tag{2.5}$$

for large enough n.

Proof. (i) By the hypothesis we may assume that $p_{k_n}(a_n) = p_{r_n}(b_n) = 1$. Assume towards a contradiction that there is no C for which (2.4) holds. Then, by following a similar method as in the previous proposition, there is a map

$$T : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad T((i,j)) = n(i,j)$$

such that it is increasing on both components, and moreover,

$$\|\theta(u_{(i,j)}v_{(i,j)})\| \geq 4^{i+j}, \tag{2.6}$$

where $u_{(i,j)} = a_{n(i,j)}$ and $v_{(i,j)} = b_{n(i,j)}$. By the same argument as before, for each $i \in \mathbb{N}$, the series $f_i = \sum_{k=1}^{\infty} 2^{-k} u_{(i,k)}$ is convergent and so $f_i \in A$. For each $i \in \mathbb{N}$ choose an integer $j(i) > i$ so that $\|\theta(f_i)\| \leq 2^{j(i)}$, and set

$$S = \sum_{k=1}^{\infty} 2^{-k} v_{(k,j(k))},$$

which is a convergent series in A, by a similar argument as in the previous proposition. Hence $f_i S = 2^{-i-j(i)} u_{(i,j(i))} v_{(i,j(i))}$ for each i and so by the hypothesis and (2.6) we have

$$\|\theta(f_i S)\| = \|2^{-i-j(i)} \theta(u_{(i,j(i))} v_{(i,j(i))})\| \geq 2^{i+j(i)}.$$

On the other hand,

$$\|\theta(f_i S)\| \leq \|\theta(S)\| \|\theta(f_i)\| \leq 2^{j(i)} \|\theta(S)\|,$$

and hence $2^i \leq \|\theta(S)\|$ for each $i \in \mathbb{N}$, which is a contradiction. This completes the proof of (i).

(ii) By the hypothesis, there exists a sequence $\{s_n\}$ in A such that $p_{r_n}(b_n s_n) = 1$. If T_n is discontinuous for infinitely many $n \in \mathbb{N}$, by a similar argument as in the previous proposition, there exists a sequence $\{x_n\}$ in A such that $p_{r_n}(b_n x_n) \neq 0$ for all $n \in \mathbb{N}$ and $\|\theta(a_n b_n x_n)\| = \|T_n(x_n)\| > n p_{k_n}(a_n) p_{r_n}(b_n x_n)$ for all $n \in \mathbb{N}$. Now by applying (i) with the sequence $\{b_n x_n\}$ instead of $\{x_n\}$, there exists $C > 0$ such that

$$\|\theta(a_n b_n x_n)\| \leq C p_{k_n}(a_n) p_{r_n}(b_n x_n),$$

which implies that $n < C$ for all $n \in \mathbb{N}$. By this contradiction we conclude that the linear operator T_n is continuous for large enough $n \in \mathbb{N}$.

Since the operator

$$\frac{T_n}{p_{k_n}(a_n) p_{r_n}(b_n)}$$

is continuous, it maps bounded sets into bounded sets and the inverse images of open sets are open sets. By a similar argument as before, there exists an $M > 0$ such that $\|T_n(x)\| \leq M p_{k_n}(a_n) p_{r_n}(b_n)$ for all $x \in E$ and for large enough n. □

3. Automatic continuity of module homomorphisms from Fréchet algebras

In [7] the authors showed that if $M = \{(x_n)_{n=0}^{\infty} \in l^1(w) : \ x_0 = 0\}$ and $L(M) = l^1(w)$, then every module homomorphism from M into any Banach $l^1(w)$-module is continuous, where w is a weight function.

By using the results of Section 2 we try to extend some known results on the automatic continuity of module homomorphisms as well as homomorphisms for Fréchet algebras. We show that if $(A, \{p_n\})$ is a unital Fréchet algebra with a continued bisection of the identity, then A is functionally continuous.

Lemma 3.1 ([3, 1.3.24])**.** *Let $(A, \{p_n\})$ be a unital Fréchet algebra with a continued bisection $(\{r_n\}, \{s_n\})$ of the identity. Then the set $\{s_n : n \in \mathbb{N}\}$ is orthogonal.*

Lemma 3.2. *Let $(A, \{p_n\})$ be a unital Fréchet algebra with continued bisection $(\{r_n\}, \{s_n\})$ of the identity. Then $p_n(r_m) \geq 1, p_n(s_m) \geq 1$ for all $m, n \in \mathbb{N}$.*

Proof. By the hypothesis, $e_A = r_1 + s_1$, $r_1^2 = r_1$ and $s_1^2 = s_1$. Hence

$$1 = p_n(e_A) \leq p_n(r_1) + p_n(s_1), \quad p_n(r_1) \leq p_n(r_1)^2, \quad p_n(s_1) \leq p_n(s_1)^2.$$

Thus $p_n(r_1)$ and $p_n(s_1)$ are either zero or $p_n(r_1) \geq 1$, $p_n(s_1) \geq 1$. If $p_n(r_1)$ and $p_n(s_1)$ are zero, then $p_n(e_A) = 0$, which is a contradiction. So we may suppose that $p_n(r_1) \geq 1$. By the definition of the continued bisection of the identity, $r_1 = as_1b$, for some $a, b \in A$. So $1 \leq p_n(r_1) \leq p_n(a)p_n(s_1)p_n(b)$, implying that $p_n(s_1) \geq 1$. Now fix $m, n \in \mathbb{N}$ and suppose that $p_n(r_m) \geq 1$. Then $1 \leq p_n(r_m) \leq p_n(r_{m+1}) + p_n(s_{m+1})$. So by the above discussion, we have $p_n(r_{m+1}) \geq 1$ and $p_n(s_{m+1}) \geq 1$.

□

Theorem 3.3. *Let $(A, \{p_n\})$ be a unital Fréchet algebra with a continued bisection $(\{r_n\}, \{s_n\})$ of the identity. Let $(X, \{q_n\})$ be a Fréchet left A-module such that $s_n \cdot X \nsubseteq \ker(q_{k_n})$ for some subsequence $\{q_{k_n}\}$ and let Y be a Banach left A-module which is unital. Then every left A-module homomorphism $\theta : X \to Y$ is automatically continuous. Moreover, for every bounded subset E of X there exists an $M > 0$ such that*

$$\|s_n \cdot \theta(x)\| \leq M p_{k_n}(s_n)^2,$$

for all $x \in E$ and for large enough $n \in \mathbb{N}$.

Proof. Let $I(s_n)$ be the two-sided ideal generated by s_n in A, and let $J(\theta)$ be the set of all $a \in A$ such that the map $x \to a\theta(x)$ is continuous. It is easy to see that $I(s_n) = A$ for every $n \geq 1$ and $J(\theta)$ is a two-sided ideal of A. Now by Lemma 3.2, $p_m(s_n) \geq 1$ for all $m, n \in \mathbb{N}$. Hence $p_{k_n}(s_n) \geq 1$ for all $n \in \mathbb{N}$. Let $\theta : X \to Y$ be a left A-module homomorphism. One can check easily that $s_n s_m = 0$ for all $m \neq n$. By taking $a_n = b_n = s_n$ in Proposition 2.2 (ii), the maps

$$T_n : X \to Y$$
$$T_n(x) = s_n s_n \cdot \theta(x) = s_n \cdot \theta(x) = \theta(s_n \cdot x)$$

are continuous for large enough $n \in \mathbb{N}$ and hence $s_m \in J(\theta)$ for some $m \geq 1$. Since $J(\theta)$ is a two-sided ideal, we have $e_A \in I(s_m) \subset J(\theta)$. Therefore, θ is continuous. Moreover, by Proposition 2.2(ii) for a bounded subset E of X, there exists $M>0$ such that

$$\|s_n \cdot \theta(x)\| \leq M p_{k_n}(s_n) p_{k_n}(s_n) = M p_{k_n}(s_n)^2,$$

for large enough n and all $x \in E$. □

Theorem 3.4. *Let $(A, \{p_n\})$ be a unital Fréchet algebra with a continued bisection $(\{r_n\}, \{s_n\})$ of the identity. Let $(X, \{q_n\})$ be a Fréchet left A-module such that $s_n \cdot X \nsubseteq \ker(q_{k_n})$ for some subsequence $\{q_{k_n}\}$, and let $(Y, \{g_n\})$ be a unital Fréchet left A-module. Then every left A-module homomorphism $\theta : X \to Y$ is automatically continuous. Moreover, for every bounded subset E of X and $m \in \mathbb{N}$ there exists an $M > 0$ such that for all $x \in E$ we have $g_m(s_n \cdot \theta(x)) \leq M p_{k_n}(s_n)^2$ for all large enough $n \in \mathbb{N}$.*

Proof. Let $\theta : X \to Y$ be a left A-module homomorphism and suppose that Y_m is the completion of $Y/\ker(g_m)$ with respect to the norm $g'_m(y + \ker(g_m)) = g'_m([y]_m) = g_m(y)$ for $y \in Y$, which is a unital Banach left A-module. Then, by Theorem 3.3, the left A-module homomorphism $\theta_m = [\theta]_m : X \to Y_m$ defined by $\theta_m(x) = \theta(x) + \ker(g_m)$ is continuous and if E is a bounded subset of X, then there exists an $M > 0$ such that for all $x \in E$ we have

$$\begin{aligned} g_m(s_n \cdot \theta(x)) = g_m(\theta(s_n \cdot x)) &= g'_m([\theta]_m(s_n \cdot x)) \\ &= g'_m(\theta_m(s_n \cdot x)) = g'_m(s_n \cdot \theta_m(x)) \\ &\leq M p_{k_n}(s_n)^2, \end{aligned}$$

for all large enough $n \in \mathbb{N}$. Now we suppose that $\{x_k\}$ is a sequence in X such that $\lim_{k\to\infty} x_k = 0$ and $\lim_{k\to\infty} \theta(x_k) = y$ in Y. Since $\lim_{k\to\infty} \theta_m(x_k) = 0$ in Y_m, and on the other hand, $\lim_{k\to\infty} \theta_m(x_k) = [y]_m$, it follows that $y \in \ker(g_m)$. Since m is arbitrary, we conclude that $y \in \cap \ker(g_m)$ and Remark 2.1(ii) implies, $y = 0$. By the Closed Graph Theorem, the result follows. □

Let A and B be Fréchet algebras and $\theta : A \to B$ be a homomorphism. Then the continuity ideal $\mathcal{I}(\theta)$ of the homomorphism θ is defined by

$$\mathcal{I}(\theta) = \{a \in A : \theta(a)b = b\theta(a) = 0 \text{ for every } b \in \mathfrak{S}(\theta)\}.$$

If A is unital and $e_A \in \mathcal{I}(\theta)$, then θ is continuous on A.

Lemma 3.5. *Let A and B be Fréchet algebras and $\theta : A \to B$ be a homomorphism. Then $\mathcal{I}(\theta)$ is a two sided ideal in A, and*

$$\mathcal{I}(\theta) = \{a \in A : \theta_a(b) = \theta(ab) \text{ and } \theta^a(b) = \theta(ba) \text{ are both continuous mapping from } A \text{ to } B\}.$$

Proof. It is easy to see that $\mathcal{I}(\theta)$ is a linear subspace of A. Suppose that $b \in \mathfrak{S}(\theta)$. So there exists a sequence $\{a_n\}$ in A such that $a_n \to 0$ and $\theta(a_n) \to b$. If $a \in A$, then $a_n a \to 0$ and $\theta(a_n a) \to b\theta(a)$. Therefore for every $b \in \mathfrak{S}(\theta)$

and $a \in A$, we have $b\theta(a) \in \mathfrak{S}(\theta)$. If $x \in \mathcal{I}(\theta)$ and $a \in A$, then $b\theta(ax) = (b\theta(a))\theta(x) = 0$, since $b\theta(a) \in \mathfrak{S}(\theta)$, and $\theta(ax)b = \theta(a)(\theta(x)b) = 0$. Hence $ax, xa \in \mathcal{I}(\theta)$, this means that $\mathcal{I}(\theta)$ is a two sided ideal in A. For the proof of the second part, we define continuous linear maps $s_a, s^a : B \to B$ by $s_a(b) = \theta(a)b$ and $s^a(b) = b\theta(a)$ where $b \in B$ and $a \in A$. It easy to see that $s_a\theta = \theta_a$ and $s^a\theta = \theta^a$, by [3, 5.2.2(ii)], $\overline{\theta(a)\mathfrak{S}(\theta)} = \overline{s_a(\mathfrak{S}(\theta))} = \mathfrak{S}(s_a\theta) = \mathfrak{S}(\theta_a)$ and $\overline{\mathfrak{S}(\theta)\theta(a)} = \overline{s^a(\mathfrak{S}(\theta))} = \mathfrak{S}(s^a\theta) = \mathfrak{S}(\theta^a)$. Thus $a \in \mathcal{I}(\theta)$ if and only if θ_a and θ^a are both continuous mapping from A to B. □

Theorem 3.6. *Let $(A, \{p_n\})$ be a unital Fréchet algebra with a continued bisection $(\{r_n\}, \{s_n\})$ of the identity and let B be a unital Banach algebra. Then every unital homomorphism $\theta : A \to B$ is automatically continuous.*

Proof. Let $\theta : A \to B$ be a unital homomorphism, i.e., $\theta(e_A) = e_B$. By Proposition 2.3(ii), the maps

$$T_n : A \to B$$

$$T_n(x) = \theta(s_n s_n x) = \theta(s_n x)$$

are continuous for large enough $n \in \mathbb{N}$, and so, by a similar argument as before, there exists $k \in \mathbb{N}$ with $s_k \in \mathcal{I}(\theta)$. Again by a successive argument as in the Theorem 3.3, we see that $r_k, r_{k-1}, s_{k-1}, ..., r_1, s_1, e_A$ belong to $\mathcal{I}(\theta)$, since θ is unital, so $\mathfrak{S}(\theta) = 0$ and hence θ is continuous. □

Corollary 3.7. *Let $(A, \{p_n\})$ be a unital Fréchet algebra with a continued bisection of the identity. Then A is functionally continuous.*

Corollary 3.8. *Let $(A, \{p_n\})$ be a unital Fréchet algebra with a continued bisection of the identity and $(B, \{q_n\})$ be a unital Fréchet algebra. Then every unital homomorphism from A into B is automatically continuous.*

Proof. Let $\theta : A \to B$ be a homomorphism and let B_m be the completion of $B/\ker(q_m)$ with respect to the norm $q'_m(b + \ker(q_m)) = q'_m([b]_m) = q_m(b)$ for $b \in B$, which is a unital Banach algebra. Then, by the Theorem 3.6, the homomorphism $\theta_m = [\theta]_m : A \to B_m$ is continuous.

Let $\{a_k\}$ be a sequence in A such that $\lim_{k\to\infty} a_k = 0$ and $\lim_{k\to\infty} \theta(a_k) = b$ in B. Since $\lim_{k\to\infty} \theta_m(a_k) = 0$ in B_m and, on the other hand, $\lim_{k\to\infty} \theta_m(a_k) = [b]_m$, it follows that $b \in \ker(q_m)$. Since m is arbitrary, we conclude that $b \in \cap \ker(q_m)$ and hence $b = 0$, by Remark 2.1(ii). Now the result follows by the Closed Graph Theorem. □

Remark 3.9. Let $(A, \{p_n\})$ be a Fréchet algebra. If we define seminorms P_n on $B(A)$, the algebra of all bounded linear operators on A, by

$$P_n(\Lambda) = \sup_{p_n(a)\leq 1} p_n(\Lambda(a)), \quad \Lambda \in B(A) \text{ and } a \in A,$$

then it is easy to see that $(B(A), \{P_n\})$ is a Fréchet algebra.

The following lemma has been proved in [3, 2.5.11], for Banach algebras, but it is also valid for Fréchet algebras.

Lemma 3.10. *Let A be a Fréchet algebra which is linearly homeomorphic to $A \oplus A$. Then the Fréchet algebra $B(A)$ has a continued bisection of the identity.*

There are some examples of Fréchet algebra A such that $A \simeq A \oplus A$. For example: (i) $C(\Omega)$, where Ω is an infinite, metrizable, compact space; (ii) $C^n([0,1])$ for $n \in \mathbb{N}$; (iii) $L^1([0,1])$, [2].

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References

[1] W.G. Bade and P.C. Curtis Jr., Homomorphisms of commutative Banach algebras, *Amer. J. Math.* **82** (1960) 589–608.
[2] S. Banach, Théorie des Opérations Linéaires, Monografje Matematyczne 1, Polish Scientific Publishers, Warsaw ,1932, Reprinted Chelsea, New York, 1955.
[3] H.G. Dales, Banach Algebras and Automatic Continuity, London Math. Soc. Monogr. Ser. 24, Clarendon Press, Oxford, 2000.
[4] T. Figiel, An example of infinite dimensional reflexive Banach space non-isomorphic to its Cartesian square, *Studia Math.* **42** (1972) 295–306.
[5] B.E. Johnson, Continuity of homomorphisms of algebras of operators, *J. Lond. Math. Soc.* **42** (1967) 537–541.
[6] B.E. Johnson, Continuity of homomorphisms of Banach G-modules, *Pacific J. Math.* **120** (1985), no. 1, 111–121.
[7] K.W. Jun, Y.W. Lee and D.W. Park, The continuity of derivations and module homomorphisms, *Bull. Korean Math. Soc.* **27** (1990), no. 2, 197–206.
[8] N.J. Laustsen, On ring-theoretic (in)finiteness of Banach algebras of operators on Banach spaces, *Glasg. Math. J.* **45** (2003), no. 1, 11–19.
[9] R.J. Loy and G.A. Willis, Continuity of derivations on $B(E)$ for certain Banach spaces E, *J. Lond. Math. Soc. (2)* **40** (1989), no. 2, 327–346.
[10] C.P. Ogden, Homomorphisms from $\mathfrak{B}(\ell(\omega_\eta))$, *J. Lond. Math. Soc. (2)* **54** (1996), no. 2, 346–358.
[11] W. Rudin, Functional Analysis, McGraw-Hill, 2nd edition, New York, 1991.
[12] A.M. Sinclair, Homomorphisms from C^*-algebras, *Proc. Lond. Math. Soc. (3)* **29** (1974), no. 3, 435–452.

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