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DIGITAL BORSUK-ULAM THEOREM

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ABSTRACT. The aim of this paper is to compute a simplicial cohomology group of some specific digital images. Then we define ring and algebra structures of a digital cohomology with the cup product. Finally, we prove a special case of the Borsuk-Ulam theorem for digital images.
Keywords: Digital simplicial cohomology group, cup product, cohomology ring, cohomology algebra.
MSC(2010): Primary: 55U20; Secondary: 55N99, 68U05, 68U10.

1. **Introduction**

Digital topology [32] is concerned with developing image analysis and computer graphics. The digital simplicial homology groups [16] and cohomology groups are major tools for image analysis because a general algorithm to determine whether two different objects have isomorphic homology groups or cohomology groups could be very effective tools for image analysis. As a result, digital homology and cohomology are significant fields for researchers.

Simplicial homology groups of digital images have been studied by several researchers [2, 11, 13, 17, 37]. Arslan et al. [2] define the simplicial homology groups of n-dimensional digital images which are based on the simplicial homology groups of topological spaces in algebraic topology. They also compute simplicial homology groups of MSS_{18}. Boxer et al. [11] improve knowledge that are related to simplicial homology groups of digital images. Demir and Karaca [37] introduce the simplicial homology groups of a connected sum of digital closed κ-surfaces. They give theorems about computing the digital simplicial homology groups of $MSS_{18}\sharp MSS_{18}$, MSS_6 and $MSS_6\sharp MSS_6$. Ege and Karaca [17] present some fundamental properties and definitions with respect

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to digital simplicial homology groups. They give the Eilenberg-Steenrod axioms for digital images, Universal Coefficient Theorem for digital images and show that none of excision axiom, K$\ddot{u}$nnneth formula and Hurewicz theorem does not hold in digital images.

Karaca and Burak [25] propose a method for calculating the cohomology ring of digital images. They compute cohomology ring of MSS'_{18} and MSS_{18}. Also they give definitions and theorems that are related to relative cohomology groups of digital images.

Furthermore the cup product makes the cohomology of a topological pair into a graded algebra. In this work we show that $H^{*,\kappa}(X,G)$ is a graded G-algebra with cup product.

Borsuk [5] presented a proof of a conjecture of Ulam that has become known as the Borsuk-Ulam theorem. Crabb and Jaworowski [15] state a largely expository account of various aspects of the Borsuk-Ulam theorem, including extension of the classical theorem to families of maps parametrized by a base space and to multivalued maps. Roy and Steiger [35] determine some combinatorial consequences, typically asserting the existence of a certain combinatorial object. They state algorithmic issues about the computational complexity of finding the asserted combinatorial object.

In Section 2, we present some general notions of digital images. In next section we give definitions and theorems with respect to cohomology groups of digital images and compute the simplicial cohomology groups of $MSS_{18}\sharp MSS_{18}$ and $MSS_6\sharp MSS_6$. In the last section, we define the simplicial cup product and its general properties. Moreover, we give an example about computing the cohomology ring of $MSS_{18}\sharp MSS_{18}$. Then we present algebra structures of digital cohomology with the cup product. Finally, we prove a digital Borsuk-Ulam theorem and give some examples about these concepts [15].

2. **Preliminaries**

Let $\mathbb{Z}$ be the set of integers. Then $\mathbb{Z}^n$ is the set of lattice points in the n-dimensional Euclidean space. A (binary) digital image is a pair (X,κ), where $X \subset \mathbb{Z}^n$ for some positive integer n and κ represents certain adjacency relation for the members of X. We use a variety of adjacency relations in the study of digital images.

Let l, n be positive integers, $1 \leq l \leq n$ and distinct two points

$$p = (p_1, p_2, ..., p_n),\ q = (q_1, q_2, ..., q_n) \in \mathbb{Z}^n,$$

p and q are k_l-*adjacent* [7] if there are at most l indices i such that $|p_i - q_i| = 1$ and for all other indices j such that $|p_j - q_j| \neq 1$, $p_j = q_j$.

The notation k_l is sometimes also understood as the number of points $q \in \mathbb{Z}^n$

that are k_l-adjacent to a given point $p \in \mathbb{Z}^n$. Thus, in $\mathbb{Z}$ we have $k_1 = 2$; in $\mathbb{Z}^2$ we have $k_1 = 4$ and $k_2 = 8$; in $\mathbb{Z}^3$ we have $k_1 = 6$, $k_2 = 18$ and $k_3 = 26$.
Let κ be an adjacency relation on $\mathbb{Z}^n$. A *κ-neighbor* [7] of $p \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ that is κ-adjacent to p. The *κ-neighborhood* of p is defined to be the set

$$N_\kappa(p) = \{q \mid q \,\text{is}\, \kappa\text{−adjacent to } p\}.$$

Let $a, b \in \mathbb{Z}$ with $a < b$. A set of the form

$$[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} | a \leq z \leq b\}$$

is called a *digital interval* [6].

Let $X \subset \mathbb{Z}^n$ be a digital image with κ-adjacency. A digital image X is *κ-connected* [24] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, ..., x_r\}$ of points of a digital image X such that $x = x_0$, $y = x_r$ and x_i and x_{i+1} are κ-neighbors where $i = 0, 1, ..., r - 1$. A κ-component of a digital image X is a maximal κ-connected subset of X.

Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0-adjacency and κ_1-adjacency, respectively. Then the function $f : X \to Y$ is said to be (κ_0, κ_1)-*continuous* [7] if for every κ_0-connected subset U of X, $f(U)$ is a κ_1-connected subset of Y. We say that such a function is digitally continuous. Similar concepts are determined on discrete manifolds in [13]: Let D_1 and D_2 be two discrete manifolds and $f : D_1 \to D_2$ be a mapping. The function f is said to be an *immersion* from D_1 to D_2 or a *gradually varied operator* if x and y are adjacent in D_1 implies either $f(x) = f(y)$ or $f(x)$, $f(y)$ are adjacent in D_2.

Proposition 2.1 ([7]). *Let $(X,\kappa_0) \subset \mathbb{Z}^{n_0}$ and $(Y,\kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images. Then the function $f : X \to Y$ is said to be (κ_0, κ_1)-continuous if and only if for every pair of κ_0-adjacent points $\{x_0, x_1\}$ of X, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are κ_1-adjacent in Y.*

A $(2, \kappa)$-continuous function $f : [0, m]_{\mathbb{Z}} \to X$ such that $f(0) = x$ and $f(m) = y$ is called *a digital κ-path* [7] from x to y in a digital image X. A digital image X is digital κ-path connected, if for every $x, y \in X$, there exists a κ-path in X from x to y.

Definition 2.2 ([7]). Let $(X,\kappa_0) \subset \mathbb{Z}^{n_0}$ and $(Y,\kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images. Two (κ_0,κ_1)-continuous functions $f, g : X \to Y$ are said to be digitally (κ_0, κ_1)-homotopic in Y, if there is a positive integer m and a function $H : X \times [0, m]_{\mathbb{Z}} \to Y$ such that for all $x \in X$, $H(x, 0) = f(x)$ and $H(x, m) = g(x)$; for all $x \in X$, the induced function $H_x : [0, m]_{\mathbb{Z}} \to Y$ defined by

$$H_x(t) = H(x, t) \text{ for all } t \in [0, m]_{\mathbb{Z}},$$

is $(2, \kappa_1)$-continuous; and for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $H_t : X \to Y$ defined by

$$H_t(x) = H(x,t) \text{ for all } x \in X,$$

is (κ_0, κ_1)-continuous. The function H is called a digital (κ_0, κ_1)-homotopy [2] between f and g. A digital image (X, κ) is said to be κ-contractible if its identity map is (κ, κ)-homotopic to a constant function $\bar{c}$ for some $c \in X$, where the constant function $\bar{c} : X \to X$ is defined by $\bar{c}(x) = c$ for all $x \in X$.

For a digital image (X, κ) and its subset (A, κ), we call (X, A) *a digital image pair with κ-adjacency.* Moreover, if A is a singleton set x_0, then (X, x_0) is called *a pointed digital image.*

A simple closed κ-curve [10] of $m \geq 4$ points in a digital image X is a sequence

$$\{f(0), f(1), ..., f(m-1)\}$$

of images of the κ-path $f : [0, m-1]_{\mathbb{Z}} \to X$ such that $f(i)$ and $f(j)$ are κ-adjacent if and only if $j = i \pm 1 \bmod m$.

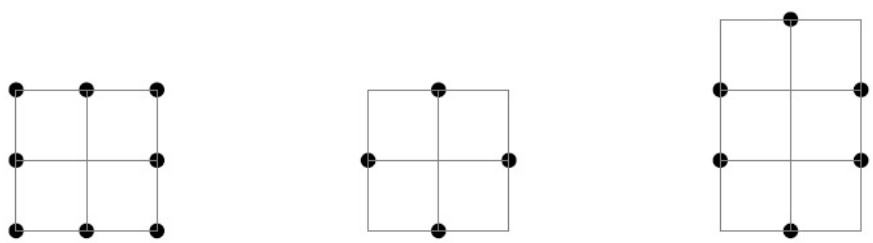

FIGURE 1. Minimal simple closed curves MSC_4, MSC'_8 and MSC_8.

A point $x \in X$ is called a *κ-corner* [4] if x is κ-adjacent to two and only two points $y, z \in X$ such that y and z are κ-adjacent to each other. The κ-corner x is called *simple* [3] if y, z are not κ-corners and if x is the only point κ-adjacent to both y, z. X is called a *generalized simple closed κ-curve* [29] if what is obtained by removing all simple κ-corners of X is a simple closed κ-curve. If (X, κ) is a κ-connected digital image in $\mathbb{Z}^3$, $|X|^x = N^*_{26}(x) \cap X$, where $N^*_{26}(x) = \{x' : x \text{ and } x' \text{ are } 26\text{−adjacent}\}$ [3,4]. Generally, if (X, κ) is a κ-connected digital image in $\mathbb{Z}^n$, $n \geq 3$, $|X|^x = N^*_{3^n-1}(x) \cap X$, where

$$N^*_{3^n-1}(x) = \{x' : x \text{ and } x' \text{ are } (3^n-1)\text{−adjacent}\} \text{ [20]}.$$

Let $(X, \kappa_0) \subset \mathbb{Z}^{n_0}$ and $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images. A function $f : X \to Y$ is (κ_0, κ_1)-*isomorphism* [9] if f is (κ_0, κ_1)-continuous and bijective and further $f^{-1} : Y \to X$ is (κ_1, κ_0)-continuous, in which case we denote $X \approx_{(\kappa_0, \kappa_1)} Y$.

Definition 2.3 ([20])**.** Let $c^* = (x_0, x_1, ..., x_n)$ be a closed κ-curve in $\mathbb{Z}^2$ where $\{\kappa, \overline{\kappa}\} = \{4, 8\}$. A point x of the complement $\overline{c^*}$ of a closed κ-curve c^* in $\mathbb{Z}^2$ is said to be interior of c^* if it belongs to the bounded $\overline{\kappa}$-connected component of $\overline{c^*}$. The set of all interior points of c^* is denoted by $Int(c^*)$.

For a closed κ-surface S_κ, we denote by $\overline{S_\kappa}$ the complement of S_κ in $\mathbb{Z}^n$. Then a point x of $\overline{S_\kappa}$ is said to be *interior* of S_κ if it belongs to the bounded $\overline{\kappa}$-connected component of S_κ. The set of all interior points of S_κ is denoted by $int(S_\kappa)$.

Definition 2.4 ([21]). Let (X, κ) be a digital image in $\mathbb{Z}^n$, $n \geq 3$, and $\overline{X} = \mathbb{Z}^n - X$. Then X is called a *closed κ-surface* if it satisfies the following.

(1) In the case that $(\kappa, \overline{\kappa}) \in \{(\kappa, 2n), (2n, 3^n - 1)\}$, where the κ-adjacency is taken from Definition 2.3. with $\kappa \neq 3^n - 2^n - 1$,
- for each point $x \in X$, $|X|^x$ has exactly one κ-component κ-adjacent to x,
- $|\overline{X}|^x$ has exactly two $\overline{\kappa}$-components $\overline{\kappa}$-adjacent to x; we denote by C^{xx} and D^{xx} these two components; and
- for any point $y \in N_\kappa(x) \cap X$, $N_{\overline{\kappa}}(y) \cap C^{xx} \neq \varnothing$ and $N_{\overline{\kappa}}(y) \cap D^{xx} \neq \varnothing$, where $N_\kappa(x)$ means the κ-neighbors of x.

Furthermore, if a closed κ-surface X does not have a simple κ-point, then X is called simple.

(2) In the case that $(\kappa, \overline{\kappa}) = (3^n - 2^n - 1, 2n)$,
- X is κ-connected,
- for each point $x \in X$, $|X|^x$ is a generalized simple closed κ-curve.

Furthermore, if the image $|X|^x$ is a simple closed κ-curve, then the closed κ-surface X is called simple.

Example 2.5. MSS_{18} and MSS'_{18} are minimal simple closed 18-surfaces.

The following digital images MSC_4^*, $MSC_8'^*$ and MSC_8^* which come from the minimal simple closed curves MSC_4, MSC'_8 and MSC_8 in $\mathbb{Z}^2$, respectively, play important roles in establishing a connected sum of closed k-surfaces [20]:
- $MSC_4^* = MSC_4 \cup Int(MSC_4)$,
- $MSC_8'^* = MSC'_8 \cup Int(MSC'_8)$,
- $MSC_8^* = MSC_8 \cup Int(MSC_8)$.

The digital images MSS_{18}^* and MSS_6^* are in $\mathbb{Z}^3$. They are obtained from the minimal simple closed curves MSC_8 and MSC_4 in $\mathbb{Z}^2$, respectively, and essentially used in generating the notion of connected sum [20],

- $MSS_6^* = MSS_6 \cup Int(MSS_6)$ where
$$MSS_6 \approx_{(6,6)} (MSC_4 \times [0,2]_{\mathbb{Z}}) \cup (Int(MSC_4) \times \{0,2\})$$
and MSC_4 is 4-isomorphic to the set
$$\{(1,0),(1,1),(0,1),(-1,1),(-1,0),(-1,-1),(0,-1),(1,-1)\}.$$
- $MSS_{18}^* = MSS_{18} \cup Int(MSS_{18})$ where
$$MSS_{18} \approx_{(18,18)} (MSC_8 \times \{1\}) \cup (Int(MSC_8) \times \{0,2\})$$
and MSC_8 is 8-isomorphic to the set

{(0,0),(-1,1),(-2,0),(-2,-1),(-1,-2),(0,-1)}.

Definition 2.6 ([20]). Let S_{κ_0} be a closed κ_0-surface in $\mathbb{Z}^{n_0}$ and S_{κ_1} be a closed κ_1-surface in $\mathbb{Z}^{n_1}$ for $n_0, n_1 \geq 3$. Consider $A'_{\kappa_0} \subset A_{\kappa_0} \subset S_{\kappa_0}$ such that
$A'_{\kappa_0} \approx_{(\kappa_0,8)} Int(MSC^*_8)$, $A'_{\kappa_0} \approx_{(\kappa_0,4)} Int(MSC^*_4)$ or $A'_{\kappa_0} \approx_{(\kappa_0,8)} Int(MSC'^*_8)$.
Let $f : A_{\kappa_0} \to f(A_{\kappa_0}) \subset S_{\kappa_1}$ be a (κ_0, κ_1)-isomorphism and let

$$S'_{\kappa_1} = S_{\kappa_1} - f(A'_{\kappa_0}) \text{ and } S'_{\kappa_0} = S_{\kappa_0} - A'_{\kappa_0}.$$

Then the connected sum, denoted by $S_{\kappa_0} \sharp S_{\kappa_1}$, is the quotient space $S_{\kappa_0} \cup S_{\kappa_1} / \sim$, where $i : A_{\kappa_0} - A'_{\kappa_0} \to S'_{\kappa_0}$ is the inclusion map and $i(x) \sim f(x)$ for $x \in A_{\kappa_0} - A'_{\kappa_0}$.

Example 2.7. Consider $MSS_{18} \sharp MSS_{18}$.

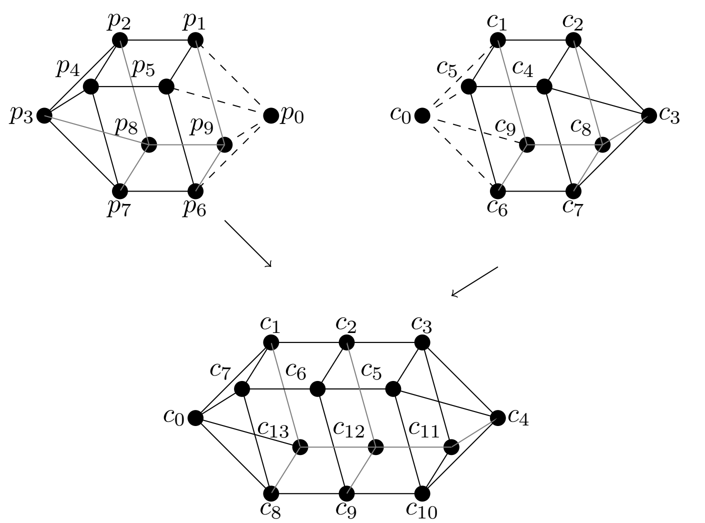

FIGURE 2. $MSS_{18} \sharp MSS_{18}$

Definition 2.8 ([36]). Let S be a set of nonempty subset of a digital image (X, κ). Then the members of S are called simplexes of (X, κ), if the following hold:

- If p and q are distinct points of $s \in S$, then p and q are κ-adjacent.
- If $s \in S$ and $\varnothing \neq t \subset s$, then $t \in S$.

A m-simplex is a simplex S such that $|S| = m + 1$. Let P be a digital m-simplex. If P' is a nonempty proper subset of P, then P' is called *a face* of P. We write $Vert(P)$ to denote the vertex set of P, namely, the set of all digital 0-simplexes in P.

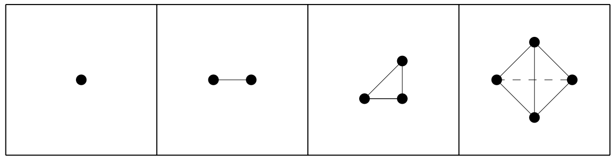

FIGURE 3. (2,0), (2,1), (8,2) and (26,3)-simplexes

Let (X,κ) be a finite collection of digital m-simplices, $0 \leq m \leq d$ for some non-negative integer d. (X,κ) is called *a finite digital simplicial complex* [2] if the following statements hold:

- If P belongs to X, then every face of P also belongs to X.
- If $P, Q \in X$, then $P \cap Q$ is either empty or a common face of P and Q.

Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex. (X,κ) called digital oriented simplicial complex if there is an ordering on the vertex set of (X,κ) [2]. The dimension of X is the biggest integer m such that X has an m-simplex [2]. $C_q^\kappa(X)$ is a free abelian group with basis all digital (κ,q)-simplices in X [2].

Proposition 2.9 ([11]). *Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex with m-dimension. Then for all $q > m$, $C_q^\kappa(X)$ is a trivial group.*

Definition 2.10 ([2]). Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital oriented simplicial complex with m-dimension. A homomorphism

$$\partial_q : C_q^\kappa(X) \to C_{q-1}^\kappa(X)$$

called the boundary operator. If $\sigma = [v_0, ..., v_q]$ is an oriented simplex with $0 < q \leq m$, we define

$$\partial_q \sigma = \partial_q [v_0, ..., v_q] = \sum_{i=0}^{q} (-1)^i [v_0, ..., \widehat{v_i}, ..., v_q],$$

where the symbol $\widehat{v_i}$ means that the vertex v_i is to be deleted from the array.

Proposition 2.11 ([2]). *For $m \geq q$, we have $\partial_{q-1} \circ \partial_q = 0$.*

Theorem 2.12 ([2]). *Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex of dimension m. Then*

$$C_*^\kappa(X) : 0 \xrightarrow{\partial_{m+1}} C_m^\kappa(X) \xrightarrow{\partial_m} C_{m-1}^\kappa(X) \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_1} C_0^\kappa(X) \xrightarrow{\partial_0} 0$$

is a chain complex.

Definition 2.13 ([11]). Let $(X,\kappa) \subset \mathbb{Z}^n$ be a digital oriented simplicial complex with m-dimension. The kernel of $\partial_q : C_q^\kappa(X) \to C_{q-1}^\kappa(X)$ is called the group of q-cycles and denoted by $Z_q^\kappa(X)$. The image of $\partial_{q+1} : C_{q+1}^\kappa(X) \to C_q^\kappa(X)$ is called the group of q-boundaries and is denoted by $B_q^\kappa(X)$. We define the q th simplicial homology group of X by

$$H_q^\kappa(X) = Z_q^\kappa(X)/B_q^\kappa(X).$$

Theorem 2.14 ([2]). *If $f : X \to Y$ is a digital (κ_0, κ_1)-isomorphism, then for all $q \leq m$*

$$H_q^{\kappa_0}(X) \cong H_q^{\kappa_1}(Y).$$

Theorem 2.15 ([2]). *If (X, κ) is a single vertex, then*

$$H_q^\kappa(X) = \begin{cases} \mathbb{Z} & , \quad q = 0 \\ 0 & , \quad q \neq 0. \end{cases}$$

Definition 2.16 ([30]). Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex; let G be an abelian group. The digital simplicial cochain complex $(C^*(X),\delta)$ is defined as follows: for any $q \in \mathbb{Z}$, the q-dimensional digital cochain group with coefficients in G, is the group

$$C^{q,\kappa}(X; G) = Hom(C_q^\kappa(X), G).$$

The coboundary operator δ is defined to be the dual of the boundary operator $\partial : C_{q+1}^\kappa(X) \to C_q^\kappa(X)$. Thus

$$C^{q+1,\kappa}(X; G) \xleftarrow{\delta} C^{q,\kappa}(X; G)$$

so that δ raises dimension by one. The abelian group G is omitted from the notation when it equals the group of integers. Elements of $C^{q,\kappa}(X)$ are called *digital cochains* and denoted either by c^q or by c^*, if we don't need to specify their dimension q. The value of a digital cochain c^q on a chain d_q is denoted by $< c^q, d_q >$. The *q-th coboundary map*

$$\delta^q : C^{q,\kappa}(X) \to C^{q+1,\kappa}(X)$$

is the dual homomorphism of ∂_{q+1} defined by

$$< \delta^q c^q, d_{q+1} > = < c^q, \partial_{q+1} d_{q+1} >.$$

Definition 2.17 ([30]). The kernel of δ is called the group of cocycles and denoted by $Z^{q,\kappa}(X; G)$, its image is called the group of coboundaries and denoted by $B^{q,\kappa}(X; G)$. The cohomology group of a digital image (X, κ) with coefficients in G is the group

$$H^{q,\kappa}(X; G) = Z^{q,\kappa}(X; G)/B^{q,\kappa}(X; G).$$

Theorem 2.18 ([18]). *If (X, κ) is a single vertex, then*

$$H^{q,\kappa}(X) = \begin{cases} \mathbb{Z} & , \quad q = 0 \\ 0 & , \quad q \neq 0. \end{cases}$$

Example 2.19. Let MSS'_{18} be a digital surface with 18-adjacency in $\mathbb{Z}^3$. Karaca and Burak [25], show that the digital cohomology groups of MSS'_{18} are

$$H^{q,18}(MSS'_{18}) = \begin{cases} \mathbb{Z} & , \quad q = 0, 2 \\ 0 & , \quad q \neq 0, 2. \end{cases}$$

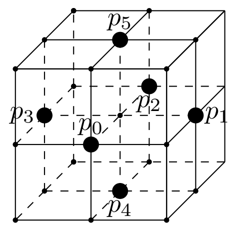

FIGURE 4. MSS'_{18}

Example 2.20. Let MSS_{18} be a digital surface with 18-adjacency. The following result is given in [25]:

$$H^{q,18}(MSS_{18}) = \begin{cases} \mathbb{Z} & , \quad q=0 \\ \mathbb{Z}^3 & , \quad q=1 \\ 0 & , \quad q \geq 2. \end{cases}$$

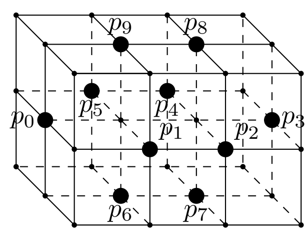

FIGURE 5. MSS_{18}

3. Simplicial cohomology groups of $MSS_{18}\sharp MSS_{18}$ and $MSS_6\sharp MSS_6$

Theorem 3.1. *The digital simplicial cohomology groups of* $MSS_{18}\sharp MSS_{18}$ *are*

$$H^{q,18}(MSS_{18}\sharp MSS_{18}) = \begin{cases} \mathbb{Z} & , \quad q=0 \\ \mathbb{Z}^7 & , \quad q=1 \\ 0 & , \quad q \geq 2. \end{cases}$$

Proof. Let $MSS_{18}\sharp MSS_{18} = \{p_0=(1,1,1), p_1=(1,2,1), p_2=(1,3,1),$
$p_3=(0,4,1), p_4=(-1,3,1), p_5=(-1,2,1), p_6=(-1,1,1), p_7=(0,0,1),$
$p_8=(0,3,0), p_9=(0,2,0), p_{10}=(0,1,0), p_{11}=(0,3,2), p_{12}=(0,2,2),$
$p_{13}=(0,1,2)\} \subset \mathbb{Z}^3$, where

$p_6 < p_5 < p_4 < p_7 < p_{13} < p_{10} < p_{12} < p_9 < p_{11} < p_8 < p_3 < p_0 < p_1 < p_2$.

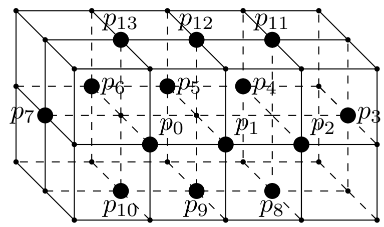

FIGURE 6. $MSS_{18}\sharp MSS_{18}$

$C_0^{18}(MSS_{18}\sharp MSS_{18})$, $C_1^{18}(MSS_{18}\sharp MSS_{18})$ and $C_2^{18}(MSS_{18}\sharp MSS_{18})$ are free abelian groups with bases, respectively,

$$\langle p_0\rangle, \langle p_1\rangle, \langle p_2\rangle, \langle p_3\rangle, \langle p_4\rangle, \langle p_5\rangle, \langle p_6\rangle, \langle p_7\rangle, \langle p_8\rangle, \langle p_9\rangle, \langle p_{10}\rangle, \langle p_{11}\rangle, \langle p_{12}\rangle, \langle p_{13}\rangle,$$

1-simplexes

$$e_0 = \langle p_0p_1\rangle, e_1 = \langle p_1p_2\rangle, e_2 = \langle p_3p_2\rangle, e_3 = \langle p_4p_3\rangle, e_4 = \langle p_5p_4\rangle, e_5 = \langle p_6p_5\rangle,$$
$$e_6 = \langle p_6p_7\rangle, e_7 = \langle p_7p_0\rangle, e_8 = \langle p_7p_{10}\rangle, e_9 = \langle p_{10}p_9\rangle, e_{10} = \langle p_9p_8\rangle,$$
$$e_{11} = \langle p_8p_3\rangle, e_{12} = \langle p_{11}p_3\rangle, e_{13} = \langle p_{12}p_{11}\rangle, e_{14} = \langle p_{13}p_{12}\rangle, e_{15} = \langle p_7p_{13}\rangle,$$
$$e_{16} = \langle p_{13}p_0\rangle, e_{17} = \langle p_{10}p_0\rangle, e_{18} = \langle p_6p_{10}\rangle, e_{19} = \langle p_6p_{13}\rangle, e_{20} = \langle p_9p_1\rangle,$$
$$e_{21} = \langle p_5p_9\rangle, e_{22} = \langle p_5p_{12}\rangle, e_{23} = \langle p_{12}p_1\rangle, e_{24} = \langle p_8p_2\rangle, e_{25} = \langle p_4p_8\rangle,$$
$$e_{26} = \langle p_4p_{11}\rangle, e_{27} = \langle p_{11}p_2\rangle$$

and
2-simplexes

$$\sigma_0 = \langle p_7p_{10}p_0\rangle, \sigma_1 = \langle p_7p_{13}p_0\rangle, \sigma_2 = \langle p_6p_7p_{10}\rangle, \sigma_3 = \langle p_6p_7p_{13}\rangle,$$
$$\sigma_4 = \langle p_8p_3p_2\rangle, \sigma_5 = \langle p_{11}p_3p_2\rangle, \sigma_6 = \langle p_4p_{11}p_3\rangle, \sigma_7 = \langle p_4p_8p_3\rangle.$$

Since $C_q^{18}(MSS_{18}\sharp MSS_{18})$ is a trivial group for $q \geq 3$, we have

$$0 \xrightarrow{\partial_3} C_2^{18}(MSS_{18}\sharp MSS_{18}) \xrightarrow{\partial_2} C_1^{18}(MSS_{18}\sharp MSS_{18}) \xrightarrow{\partial_1} C_0^{18}(MSS_{18}\sharp MSS_{18}) \xrightarrow{\partial_0} 0.$$

By the definition of cochain, we obtain

$$C^{0,18}(MSS_{18}\sharp MSS_{18}) \cong Hom(C_0^{18}(MSS_{18}\sharp MSS_{18}), \mathbb{Z}),$$
$$C^{1,18}(MSS_{18}\sharp MSS_{18}) \cong Hom(C_1^{18}(MSS_{18}\sharp MSS_{18}), \mathbb{Z}),$$
$$C^{2,18}(MSS_{18}\sharp MSS_{18}) \cong Hom(C_2^{18}(MSS_{18}\sharp MSS_{18}), \mathbb{Z}).$$

Hence we get

$$0 \xrightarrow{\delta^{-1}} C^{0,18}(MSS_{18}\sharp MSS_{18}) \xrightarrow{\delta^0} C^{1,18}(MSS_{18}\sharp MSS_{18}) \xrightarrow{\delta^1} C^{2,18}(MSS_{18}\sharp MSS_{18}) \xrightarrow{\delta^2} 0.$$

It is easy to see that

$$\begin{aligned}
&\partial_1(e_0) = p_1 - p_0, && \partial_1(e_{14}) = p_{12} - p_{13},\\
&\partial_1(e_1) = p_2 - p_1, && \partial_1(e_{15}) = p_{13} - p_7,\\
&\partial_1(e_2) = p_2 - p_3, && \partial_1(e_{16}) = p_0 - p_{13},\\
&\partial_1(e_3) = p_3 - p_4, && \partial_1(e_{17}) = p_0 - p_{10},\\
&\partial_1(e_4) = p_4 - p_5, && \partial_1(e_{18}) = p_{10} - p_6,\\
&\partial_1(e_5) = p_5 - p_6, && \partial_1(e_{19}) = p_{13} - p_6,\\
&\partial_1(e_6) = p_7 - p_6, && \partial_1(e_{20}) = p_1 - p_9,\\
&\partial_1(e_7) = p_0 - p_7, && \partial_1(e_{21}) = p_9 - p_5,\\
&\partial_1(e_8) = p_{10} - p_7, && \partial_1(e_{22}) = p_{12} - p_5,\\
&\partial_1(e_9) = p_9 - p_{10}, && \partial_1(e_{23}) = p_1 - p_{12},\\
&\partial_1(e_{10}) = p_8 - p_9, && \partial_1(e_{24}) = p_2 - p_8,\\
&\partial_1(e_{11}) = p_3 - p_8, && \partial_1(e_{25}) = p_8 - p_4,\\
&\partial_1(e_{12}) = p_3 - p_{11}, && \partial_1(e_{26}) = p_{11} - p_4,\\
&\partial_1(e_{13}) = p_{11} - p_{12}, && \partial_1(e_{27}) = p_2 - p_{11}.
\end{aligned}$$

So we find 0-cochains,

$$\begin{aligned}
\delta^0 p_0^* &= -e_0 + e_7 + e_{16} + e_{17},\\
\delta^0 p_1^* &= e_0 - e_1 + e_{20} + e_{23},\\
\delta^0 p_2^* &= e_1 + e_2 + e_{24} + e_{27},\\
\delta^0 p_3^* &= -e_2 + e_3 + e_{11} + e_{12},\\
\delta^0 p_4^* &= -e_3 + e_4 - e_{25} - e_{26},\\
\delta^0 p_5^* &= -e_4 + e_5 - e_{21} - e_{22},\\
\delta^0 p_6^* &= -e_5 - e_6 - e_{18} - e_{19},\\
\delta^0 p_7^* &= e_6 - e_7 - e_8 - e_{15},\\
\delta^0 p_8^* &= e_{10} - e_{11} - e_{24} + e_{25},\\
\delta^0 p_9^* &= e_9 - e_{10} - e_{20} + e_{21},\\
\delta^0 p_{10}^* &= e_8 - e_9 + e_{18} - e_{17},\\
\delta^0 p_{11}^* &= -e_{12} + e_{13} + e_{26} - e_{27},\\
\delta^0 p_{12}^* &= -e_{13} + e_{14} + e_{22} - e_{23},\\
\delta^0 p_{13}^* &= -e_{14} + e_{15} - e_{16} + e_{19}.
\end{aligned}$$

From the definition of ∂_2, we can easily obtain

$$\begin{aligned}
\partial_2(\sigma_0) &= \langle p_{10}p_0\rangle - \langle p_7p_0\rangle + \langle p_7p_{10}\rangle = e_{17} - e_7 + e_8,\\
\partial_2(\sigma_1) &= \langle p_{13}p_0\rangle - \langle p_7p_0\rangle + \langle p_7p_{13}\rangle = e_{16} - e_7 + e_{15},\\
\partial_2(\sigma_2) &= \langle p_7p_{10}\rangle - \langle p_6p_{10}\rangle + \langle p_6p_7\rangle = e_8 - e_{18} + e_6,\\
\partial_2(\sigma_3) &= \langle p_7p_{13}\rangle - \langle p_6p_{13}\rangle + \langle p_6p_7\rangle = e_{15} - e_{19} + e_6,\\
\partial_2(\sigma_4) &= \langle p_3p_2\rangle - \langle p_8p_2\rangle + \langle p_8p_3\rangle = e_2 - e_{24} + e_{11},\\
\partial_2(\sigma_5) &= \langle p_3p_2\rangle - \langle p_{11}p_2\rangle + \langle p_{11}p_3\rangle = e_2 - e_{27} + e_{12},\\
\partial_2(\sigma_6) &= \langle p_{11}p_3\rangle - \langle p_4p_3\rangle + \langle p_4p_{11}\rangle = e_{12} - e_3 + e_{26},\\
\partial_2(\sigma_7) &= \langle p_8p_3\rangle - \langle p_4p_3\rangle + \langle p_4p_8\rangle = e_{11} - e_3 + e_{25}.
\end{aligned}$$

Thus, we get 1-cochains,

$$\begin{array}{ll}
\delta^1 e_0^* = \{0\}, & \delta^1 e_{14}^* = \{0\}, \\
\delta^1 e_1^* = \{0\}, & \delta^1 e_{15}^* = \sigma_1 + \sigma_3, \\
\delta^1 e_2^* = \sigma_4 + \sigma_5, & \delta^1 e_{16}^* = \sigma_1, \\
\delta^1 e_3^* = -\sigma_6 - \sigma_7, & \delta^1 e_{17}^* = \sigma_0, \\
\delta^1 e_4^* = \{0\}, & \delta^1 e_{18}^* = -\sigma_2, \\
\delta^1 e_5^* = \{0\}, & \delta^1 e_{19}^* = -\sigma_3, \\
\delta^1 e_6^* = \sigma_2 + \sigma_3, & \delta^1 e_{20}^* = \{0\}, \\
\delta^1 e_7^* = -\sigma_0 - \sigma_1, & \delta^1 e_{21}^* = \{0\}, \\
\delta^1 e_8^* = \sigma_0 + \sigma_2, & \delta^1 e_{22}^* = \{0\}, \\
\delta^1 e_9^* = \{0\}, & \delta^1 e_{23}^* = \{0\}, \\
\delta^1 e_{10}^* = \{0\}, & \delta^1 e_{24}^* = -\sigma_4, \\
\delta^1 e_{11}^* = \sigma_4 + \sigma_7, & \delta^1 e_{25}^* = \sigma_7, \\
\delta^1 e_{12}^* = \sigma_5 + \sigma_6, & \delta^1 e_{26}^* = \sigma_6, \\
\delta^1 e_{13}^* = \{0\}, & \delta^1 e_{27}^* = -\sigma_5.
\end{array}$$

Let's find the kernel of δ^0. By the definition of δ^0, we see that

$$\begin{aligned}
\delta^0(\sum_{i=0}^{13} n_i p_i^*) &= n_0(-e_0 + e_7 + e_{16} + e_{17}) + n_1(e_0 - e_1 + e_{20} + e_{23}) \\
&+ n_2(e_1 + e_2 + e_{24} + e_{27}) + n_3(-e_2 + e_3 + e_{11} + e_{12}) \\
&+ n_4(-e_3 + e_4 - e_{25} - e_{26}) + n_5(-e_4 + e_5 - e_{21} - e_{22}) \\
&+ n_6(-e_5 - e_6 - e_{18} - e_{19}) + n_7(e_6 - e_7 - e_8 - e_{15}) \\
&+ n_8(e_{10} - e_{11} - e_{24} + e_{25}) + n_9(e_9 - e_{10} - e_{20} + e_{21}) \\
&+ n_{10}(e_8 - e_9 + e_{18} - e_{17}) + n_{11}(-e_{12} + e_{13} + e_{26} - e_{27}) \\
&+ n_{12}(-e_{13} + e_{14} + e_{22} - e_{23}) + n_{13}(-e_{14} + e_{15} - e_{16} + e_{19}).
\end{aligned}$$

Solving the equation

$$\begin{aligned}
&e_0(-n_0 + n_1) + e_1(-n_1 + n_2) + e_2(n_2 - n_3) + e_3(n_3 - n_4) + e_4(n_4 - n_5) \\
&+ e_5(n_5 - n_6) + e_6(-n_6 + n_7) + e_7(n_0 - n_7) + e_8(-n_7 + n_{10}) + e_9(n_9 - n_{10}) \\
&+ e_{10}(-n_9 + n_8) + e_{11}(n_3 - n_8) + e_{12}(n_3 - n_{11}) + e_{13}(n_{11} - n_{12}) + e_{14}(n_{12} - n_{13}) \\
&+ e_{15}(n_{13} - n_7) + e_{16}(n_0 - n_{13}) + e_{17}(n_0 - n_{10}) + e_{18}(-n_6 + n_{10}) + e_{19}(-n_6 + n_{13}) \\
&+ e_{20}(n_1 - n_9) + e_{21}(-n_5 + n_9) + e_{22}(-n_5 + n_{12}) + e_{23}(n_1 - n_{12}) + e_{24}(n_2 - n_8) \\
&+ e_{25}(-n_4 + n_8) + e_{26}(-n_4 + n_{11}) + e_{27}(n_2 - n_{11}) = 0,
\end{aligned}$$

we find

$$n_0 = n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = n_7 = n_8 = n_9 = n_{10} = n_{11} = n_{12} = n_{13} = n.$$

Hence, we get the group of zero dimensional cocycles

$$\begin{aligned}
Z^{0,18}(MSS_{18}) = \{&n(p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 + p_{10} \\
&+ p_{11} + p_{12} + p_{13}) \mid n \in \mathbb{Z}\}
\end{aligned}$$

$$\cong \mathbb{Z}.$$

Since $B^{0,18}(MSS_{18}\sharp MSS_{18}) \cong 0$, we obtain

$$H^{0,18}(MSS_{18}\sharp MSS_{18}) \cong \mathbb{Z}.$$

Let

$$\begin{aligned}\delta^1(\sum_{i=0}^{27} k_i e_i^*) = {} & k_0(\{0\}) + k_1(\{0\}) + k_2(\sigma_4+\sigma_5) + k_3(-\sigma_6-\sigma_7) + k_4(\{0\}) \\ & +k_5(\{0\}) + k_6(\sigma_2+\sigma_3) + k_7(-\sigma_0-\sigma_1) + k_8(\sigma_0+\sigma_2) + k_9(\{0\}) \\ & +k_{10}(\{0\}) + k_{11}(\sigma_4+\sigma_7) + k_{12}(\sigma_5+\sigma_6) + k_{13}(\{0\}) + k_{14}(\{0\}) \\ & +k_{15}(\sigma_1+\sigma_3) + k_{16}(\sigma_1) + k_{17}(\sigma_0) + k_{18}(-\sigma_2) + k_{19}(-\sigma_3) \\ & +k_{20}(\{0\}) + k_{21}(\{0\}) + k_{22}(\{0\}) + k_{23}(\{0\}) + k_{24}(-\sigma_4) \\ & +k_{25}(\sigma_7) + k_{26}(\sigma_6) + k_{27}(-\sigma_5).\end{aligned}$$

We find the kernel of δ^1 and we have

$$\begin{aligned}& \sigma_0(-k_7+k_8+k_{17}) + \sigma_1(-k_7+k_{15}+k_{16}) + \sigma_2(k_6-k_{18}+k_8) \\ & \quad +\sigma_3(k_6-k_{19}+k_{15}) + \sigma_4(k_2+k_{11}-k_{24}) + \sigma_5(k_2+k_{12}-k_{27}) \\ & \quad +\sigma_6(-k_3+k_{12}+k_{26}) + \sigma_7(-k_3+k_{11}+k_{25}) = 0\end{aligned}$$

Solving the equation above, we get

$$\begin{aligned}k_3 &= k_{12}+k_{26}, \\ k_6 &= k_8+k_{18}, \\ k_7 &= k_8+k_{17}, \\ k_{16} &= k_8-k_{15}+k_{17}, \\ k_{19} &= k_8+k_{15}+k_{18}, \\ k_{24} &= k_2+k_{11}, \\ k_{25} &= -k_{11}+k_{12}+k_{26}, \\ k_{27} &= k_2+k_{12}.\end{aligned}$$

Hence, we conclude that

$$\begin{aligned}Z^{1,18}(MSS_{18}\sharp MSS_{18}) = {} & \{k_0e_0^* + k_1e_1^* + k_2e_2^* + (k_{12}+k_{26})e_3^* + k_4e_4^* \\ & +k_5e_5^* + (k_8+k_{18})e_6^* + (k_8+k_{17})e_7^* + k_8e_8^* + k_9e_9^* \\ & +k_{10}e_{10}^* + k_{11}e_{11}^* + k_{12}e_{12}^* + k_{13}e_{13}^* + k_{14}e_{14}^* + k_{15}e_{15}^* \\ & +(k_8-k_{15}+k_{17})e_{16}^* + k_{17}e_{17}^* + k_{18}e_{18}^* \\ & +(k_8+k_{15}+k_{18})e_{19}^* + k_{20}e_{20}^* + k_{21}e_{21}^* + k_{22}e_{22}^* + k_{23}e_{23}^* \\ & +(k_2+k_{11})e_{24}^* + (-k_{11}+k_{12}+k_{26})e_{25}^* + k_{26}e_{26}^* \\ & +(k_2+k_{12})e_{27}^* \mid k_i \in \mathbb{Z}, i = 0,1,2,4,5,8,9,10,11, \\ & 12,13,14,15,17,18,20,21,22,23,26\}\end{aligned}$$

$\cong \mathbb{Z}^{20}.$

On the other hand, we obtain

$$\begin{aligned} B^{1,18}(MSS_{18}\sharp MSS_{18}) = \{ & t_0e_0 + t_1e_1 + t_2e_2 + t_3e_3 + t_4e_4 + t_5e_5 + t_6e_6 \\ & +(-t_0-t_1+t_2+t_3+t_4+t_5-t_6)e_7+t_7e_8+t_8e_9+t_9e_{10} \\ & +(t_3+t_4+t_5+t_6+t_7+t_8-t_9)e_{11}+t_{10}e_{12}+t_{11}e_{13}+t_{12}e_{14} \\ & +(t_3+t_4+t_5+t_6+2t_7+2t_8-t_{10}-t_{11}-t_{12})e_{15} \\ & +(-t_0-t_1+t_2+t_{10}+t_{11}+t_{12})e_{16} \\ & +(-t_0-t_1+t_2+t_3+t_4+t_5-t_6-t_7)e_{17}+(t_6+t_7)e_{18} \\ & +(t_3+t_4+t_5+2t_6+2t_7+2t_8-t_{10}-t_{11}-t_{12})e_{19} \\ & +(-t_1+t_2+t_3+t_4+t_5-t_6-t_7-t_8)e_{20} \\ & +(-t_5+t_6+t_7+t_8)e_{21} \\ & +(t_3+t_4-t_{10}-t_{11})e_{22}+(-t_1+t_2+t_{10}+t_{11})e_{23} \\ & +(t_2+t_3+t_4+t_5+t_6+t_7+t_8-t_9)e_{24} \\ & +(-t_5-t_6-t_7-t_8)e_{25}+(t_3-t_{10})e_{26}+(t_2+t_{10})e_{27} \\ & \mid t_i \in \mathbb{Z},\ i=0,1,2,3,4,5,6,7,8,9,10,11,12\} \cong \mathbb{Z}^{13}. \end{aligned}$$

So we have

$$H^{1,18}(MSS_{18}\sharp MSS_{18}) \cong \mathbb{Z}^7.$$

Therefore,

$$\begin{aligned} B^{2,18}(MSS_{18}\sharp MSS_{18}) = \{ & h_0\sigma_0+h_1\sigma_1+h_2\sigma_2+h_3\sigma_3+h_4\sigma_4+h_5\sigma_5+h_6\sigma_6 \\ & +h_7\sigma_7 \mid h_i\in\mathbb{Z},\ i=0,1,2,3,4,5,6,7\} \cong \mathbb{Z}^8. \end{aligned}$$

Since $Z^{2,18}(MSS_{18}\sharp MSS_{18}) \cong \mathbb{Z}^8$, we have

$$H^{2,18}(MSS_{18}\sharp MSS_{18}) \cong \{0\}. \qquad \square$$

Theorem 3.2. *The digital simplicial cohomology groups of $MSS_6\sharp MSS_6$ are*

$$H^{q,6}(MSS_6\sharp MSS_6) = \begin{cases} \mathbb{Z} & , \quad q=0 \\ \mathbb{Z}^{39} & , \quad q=1 \\ 0 & , \quad q\geq 2. \end{cases}$$

4. Simplicial cohomology ring of $MSS_{18}\sharp MSS_{18}$

Definition 4.1 ([31])**.** Let (X,κ) be a digital simplicial complex. Suppose that the coefficient group G is the additive group of a commutative ring with identity. The digital simplicial cup product

$$\smile : C^{p,\kappa}(X,G)\times C^{q,\kappa}(X,G)\to C^{p+q,\kappa}(X,G)$$

of cochains c^p and c^q is defined by the formula

$$< c^p \smile c^q, [v_0,...,v_{p+q}] > = < c^p, [v_0,...,v_p] > . < c^q, [v_p,...,v_{p+q}] >,$$

where $v_0 < ... < v_{p+q}$ in the given ordering and "." is the product in G.

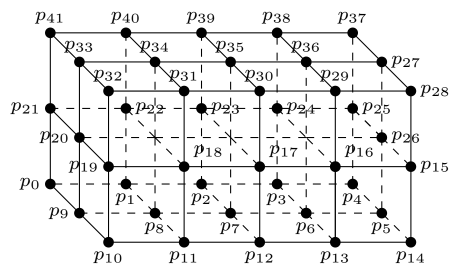

FIGURE 7. $MSS_6 \sharp MSS_6$

Theorem 4.2 ([31]). *Let* $\alpha, \alpha_1, \alpha_2 \in H^{p,\kappa}(X, G_1)$ *and* $\beta, \beta_1, \beta_2 \in H^{q,\kappa}(X, G_2)$. *Then we get*

$$(\alpha_1 + \alpha_2) \smile \beta = \alpha_1 \smile \beta + \alpha_2 \smile \beta$$

and

$$\alpha \smile (\beta_1 + \beta_2) = \alpha \smile \beta_1 + \alpha \smile \beta_2.$$

Proof. Let $\alpha, \alpha_1, \alpha_2 \in H^{p,\kappa}(X, G_1)$ and $\beta, \beta_1, \beta_2 \in H^{q,\kappa}(X, G_2)$. Since
$< (\alpha_1+\alpha_2) \smile \beta, [v_0, ..., v_{p+q}] >=< (\alpha_1+\alpha_2), [v_0, ..., v_p] > . < \beta, [v_p, ..., v_{p+q}] >$

$$= (< \alpha_1, [v_0, ..., v_p] > + < \alpha_2, [v_0, ..., v_p] >). < \beta, [v_p, ..., v_{p+q}] >$$

$$=< \alpha_1, [v_0, ..., v_p] > . < \beta, [v_p, ..., v_{p+q}] >$$

$$+ < \alpha_2, [v_0, ..., v_p] > . < \beta, [v_p, ..., v_{p+q}] >$$

$$=< \alpha_1 \smile \beta, [v_0, ..., v_{p+q}] > + < \alpha_2 \smile \beta, [v_0, ..., v_{p+q}] >$$

$$=< \alpha_1 \smile \beta + \alpha_2 \smile \beta, [v_0, ..., v_{p+q}] >$$

and
$< \alpha \smile (\beta_1+\beta_2), [v_0, ..., v_{p+q}] >=< \alpha, [v_0, ..., v_p] > . < (\beta_1+\beta_2), [v_p, ..., v_{p+q}] >$

$$=< \alpha, [v_0, ..., v_p] > .(< \beta_1, [v_p, ..., v_{p+q}] > + < \beta_2, [v_p, ..., v_{p+q}] >)$$

$$=< \alpha, [v_0, ..., v_p] > . < \beta_1, [v_p, ..., v_{p+q}] >$$

$$+ < \alpha, [v_0, ..., v_p] > . < \beta_2, [v_p, ..., v_{p+q}] >$$

$$=< \alpha \smile \beta_1, [v_0, ..., v_{p+q}] > + < \alpha \smile \beta_2, [v_0, ..., v_{p+q}] >$$

$=< \alpha \smile \beta_1 + \alpha \smile \beta_2, [v_0, ..., v_{p+q}] >$,
we obtain
$(\alpha_1 + \alpha_2) \smile \beta = \alpha_1 \smile \beta + \alpha_2 \smile \beta$
and
$\alpha \smile (\beta_1 + \beta_2) = \alpha \smile \beta_1 + \alpha \smile \beta_2$. □

Theorem 4.3 ([31]). $\delta(c^p \smile c^q) = \delta c^p \smile c^q + (-1)^p c^p \smile \delta c^q$.

Proof. The values of the digital simplicial cochains $\delta c^p \smile c^q$ and $(-1)^p c^p \smile \delta c^q$ at $[v_0, ..., v_{p+q+1}]$ are equal to

$$\sum_{0 \leq i \leq p+1} (-1)^i \, c^p \, [v_0, ..., \widehat{v_i}, ..., v_{p+1}] \, c^q \, [\, v_{p+1}, ..., v_{p+q+1} \,] \qquad (1)$$

and

$$(-1)^p \sum_{p \leq i \leq p+q+1} (-1)^{i-p} c^p [v_0, ..., v_p] c^q [v_p, ..., \widehat{v_i}, ..., v_{p+q+1}], \qquad (2)$$

respectively. The first term in (2) removes the last term in (1). The sum of the other terms in these sums equals the value of the digital simplicial cochain $\delta(c^p \smile c^q)$ at $[v_0, ..., v_{p+q+1}]$. □

Theorem 4.4 ([31]). *Let (X, κ) be a digital simplicial complex. The cup product on digital simplicial cochains is associative, that is,*

$$(c^p \smile c^q) \smile c^r = c^p \smile (c^q \smile c^r).$$

The digital simplicial cochain given by 1_X is the unit element, that is,

$$1_X \smile c^p = c^p \smile 1_X = c^p.$$

Proof. Let $c^p \in H^{p,\kappa}(X, G_1)$, $c^q \in H^{q,\kappa}(X, G_2)$ and $c^r \in H^{r,\kappa}(X, G_3)$. Then

$$\begin{aligned}
&< (c^p \smile c^q) \smile c^r, [v_0, ..., v_{p+q+r}] > = < (c^p \smile c^q), [v_0, ..., v_{p+q}] > \\
&\qquad . < c^r, [v_{p+q}, ..., v_{p+q+r}] > \\
&= (< c^p, [v_0, ..., v_p] > . < c^q, [v_p, ..., v_{p+q}] >). < c^r, [v_{p+q}, ..., v_{p+q+r}] > \\
&= < c^p, [v_0, ..., v_p] > .(< c^q, [v_p, ..., v_{p+q}] > . < c^r, [v_{p+q}, ..., v_{p+q+r}] >) \\
&= < c^p, [v_0, ..., v_p] > .(< c^q \smile c^r, [v_p, ..., v_{p+q+r}] >) \\
&= < c^p \smile (c^q \smile c^r), [v_0, ..., v_{p+q+r}] >.
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
< 1_X \smile c^p, [v_0, ..., v_p] > &= < 1_X, [v_0, ..., v_p] > . < c^p, [v_0, ..., v_p] > \\
&= < c^p, [v_0, ..., v_p] >
\end{aligned}$$

and

$$\begin{aligned}
< c^p \smile 1_X, [v_0, ..., v_p] > &= < c^p, [v_0, ..., v_p] > . < 1_X, [v_0, ..., v_p] > \\
&= < c^p, [v_0, ..., v_p] >.
\end{aligned}$$

□

Theorem 4.5 ([31]). *If $c^p \in H^{p,\kappa}(X, G_1)$ and $c^q \in H^{q,\kappa}(X, G_2)$ are digital cocycles, then*

$$c^p \smile c^q = (-1)^{pq} c^q \smile c^p.$$

Proof. By Definition 4.1, we have
$< c^p \smile c^q, [v_0, ..., v_{p+q}] >=< c^p, [v_0, ..., v_p] > . < c^q, [v_p, ..., v_{p+q}] >$
and
$< c^q \smile c^p, [v_{p+q}, ..., v_0] >=< c^q, [v_{p+q}, ..., v_p] > . < c^p, [v_p, ..., v_0] > .$
Since $[v_r, ..., v_0] = (-1)^{r(r+1)/2}[v_0, ..., v_r]$, we find

$$(p + q)(p + q + 1) - p(p + 1) - q(q + 1) = 2pq.$$ □

Theorem 4.6 ([31])**.** *Let $(X, \kappa_1) \subset \mathbb{Z}^{n_1}$ and $(Y, \kappa_2) \subset \mathbb{Z}^{n_2}$ be digital images. If $f : (X, \kappa_1) \to (Y, \kappa_2)$ is a digitally continuous map, $c^p \in H^{p,\kappa}(X, G_1)$ and $c^q \in H^{q,\kappa}(X, G_2)$ are digital cocycles, then*

$$f^*(c^p \smile c^q) = f^*(c^p) \smile f^*(c^q).$$

Proof. We have

$$\begin{aligned} < f^*(c^p \smile c^q), [v_0, ..., v_{p+q}] > &=< c^p \smile c^q, [f(v_0), ..., f(v_{p+q})] > \\ &=< c^p, [f(v_0), ..., f(v_p)] > . < c^q, [f(v_p), ..., f(v_{p+q})] > \\ &=< f^*(c^p), [v_0, ..., v_p] > . < f^*(c^q), [v_p, ..., v_{p+q}] > \\ &=< f^*(c^p) \smile f^*(c^q), [v_0, ..., v_{p+q}] >. \end{aligned}$$ □

Definition 4.7 ([30])**.** Let (X, κ) be a digital simplicial complex. $H^{*,\kappa}(X; G) = \oplus H^{i,\kappa}(X; G)$ is the ring with the cup product. This is called the digital simplicial cohomology ring of X.

Example 4.8. Consider $MSS_{18} \sharp MSS_{18}$.

$$H^{q,18}(MSS_{18} \sharp MSS_{18}) = \begin{cases} \mathbb{Z} & , \quad q = 0 \\ \mathbb{Z}^7 & , \quad q = 1 \\ 0 & , \quad q \geq 2. \end{cases}$$

By example 3.1, we obtain 1-cocycles of simplicial complex:

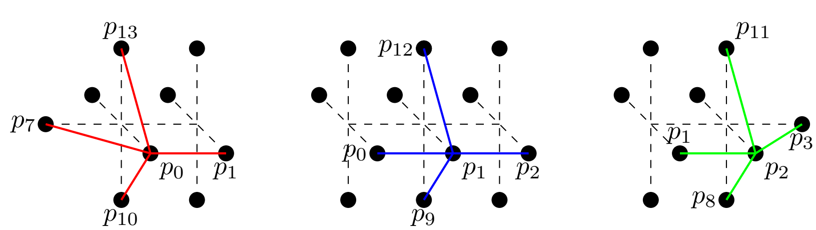

FIGURE 8. Cocycle x, cocycle y and cocycle z

We compute the cup product of 1-cocycles a, b, c, d, f, g, h, k, l, m, n, p, q, r and s, where the cup product of two 1-cocycles is equal to standart generator.

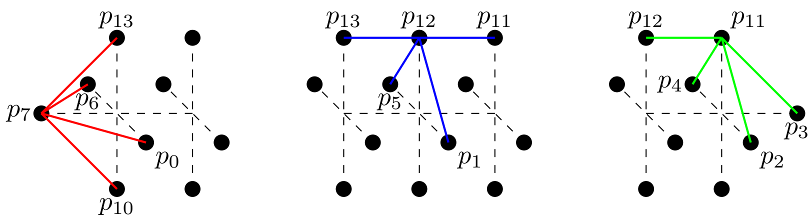

FIGURE 9. Cocycle b, cocycle h and cocycle g

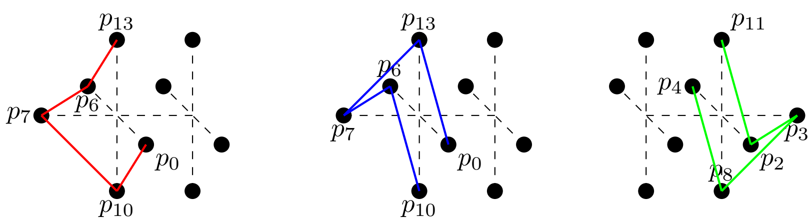

FIGURE 10. Cocycle q, cocycle r and cocycle l

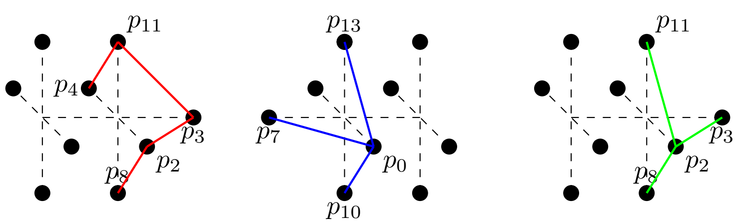

FIGURE 11. Cocycle m, cocycle θ and cocycle α

5. Simplicial cohomology algebra of digital images

Definition 5.1. If M_i is module, then $M = \oplus M_i$ is a graded module for all $i \in I$. If $\Phi : M \otimes M \to M$ is a homomorphism for the graded module M, then M is a graded algebra.

Theorem 5.2. *Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex. Then $H^{*,\kappa}(X, G)$ is a graded G-algebra with the cup product.*

Proof. Let us show that $H^{*,\kappa}(X, G)$ is the graded G-module. Since $H^{*,\kappa}(X, G)$ $= \oplus H^{q,\kappa}(X, G)$, we must show that $H^{q,\kappa}(X, G)$ is a G-module. G is a commutative ring and $H^{q,\kappa}(X, G)$ is a ring. Also, the following statements hold for scalar multiplication

$$G \times H^{q,\kappa}(X,G) \to H^{q,\kappa}(X,G), \quad (g,\alpha) \to g.\alpha$$

- $g.(\alpha_1 + \alpha_2) = g.\alpha_1 + g.\alpha_2$
- $(g_1 + g_2).\alpha = g_1.\alpha + g_2.\alpha$
- $(g_1.g_2).\alpha = g_1.(g_2.\alpha)$
- $1.\alpha = \alpha$

where $\alpha, \alpha_1, \alpha_2 \in H^{q,\kappa}(X,G)$ and $g, g_1, g_2, 1 \in G$. Hence $H^{q,\kappa}(X,G)$ is a G-module. So $H^{*,\kappa}(X,G)$ is a graded G-module. Then $H^{*,\kappa}(X,G)$ is a graded G-algebra with cup product

$$\smile: H^{q,\kappa}(X,G) \times H^{p,\kappa}(X,G) \to H^{q+p,\kappa}(X,G). \qquad \square$$

Theorem 5.3. *There is no continuous map $g : S^2 \to S^1$ with $g(-x) = -g(x)$ for all $x \in S^2$.*

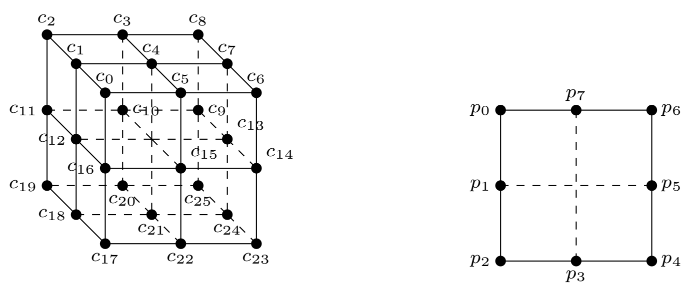

FIGURE 12. S^2 ve S^1

Proof. $S^1 = \{p_0 = (-1,1), p_1 = (-1,0), p_2 = (-1,-1), p_3 = (0,-1), p_4 = (1,-1), p_5 = (1,0), p_6 = (1,1), p_7 = (0,1)\}$ is digital 1-sphere with 4-adjacency in $\mathbb{Z}^2$. For points of S^1,

$$p_0 = -p_4, \quad p_1 = -p_5, \quad p_2 = -p_6, \quad p_3 = -p_7.$$

$S^2 = [-1,1]^3_{\mathbb{Z}} / \{(0,0,0)\}$ is digital 2-sphere with 6-adjacency in $\mathbb{Z}^3$. For points of S^2,

$$\begin{array}{ll} c_0 = -c_{25}, & c_7 = -c_{18}, \\ c_1 = -c_{24}, & c_8 = -c_{17}, \\ c_2 = -c_{23}, & c_9 = -c_{16}, \\ c_3 = -c_{22}, & c_{10} = -c_{15}, \\ c_4 = -c_{21}, & c_{11} = -c_{14}, \\ c_5 = -c_{20}, & c_{12} = -c_{13}, \\ c_6 = -c_{19}, & \end{array}$$

a function $g : S^2 \to S^1$ is defined as

$$\begin{array}{lll} g(c_0) = p_0, & g(c_9) = p_5, & g(c_{18}) = p_2, \\ g(c_1) = p_0, & g(c_{10}) = p_5, & g(c_{19}) = p_2, \\ g(c_2) = p_0, & g(c_{11}) = p_1, & g(c_{20}) = p_3, \\ g(c_3) = p_7, & g(c_{12}) = p_1, & g(c_{21}) = p_3, \\ g(c_4) = p_7, & g(c_{13}) = p_5, & g(c_{22}) = p_3, \\ g(c_5) = p_7, & g(c_{14}) = p_5, & g(c_{23}) = p_4, \\ g(c_6) = p_6, & g(c_{15}) = p_1, & g(c_{24}) = p_4, \\ g(c_7) = p_6, & g(c_{16}) = p_1, & g(c_{25}) = p_4. \\ g(c_8) = p_6, & g(c_{17}) = p_2, & \end{array}$$

Then the function $g : S^2 \to S^1$ satisfies the condition $g(-x) = -g(x)$ for all $x \in S^2$. On the other hand the function g is not $(6, 4)$-continuous. $c_{10}, c_{11} \in S^2$ are 6-adjacent each other, $g(c_{10}) = p_5$ and $g(c_{11}) = p_1$ are not 4-adjacent each other. □

One of the most useful results from topology is the Borsuk-Ulam Theorem. It states that some pair of antipodal points has the same image. We state it in the following form.

Theorem 5.4. ([5]) *Suppose that $f : (S^n, \kappa) \to \mathbb{R}^n$ is a continuous map. Then there exists a point $x \in S^n \subseteq \mathbb{R}^{n+1}$ such that $f(x) = f(-x)$.*

Theorem 5.5. *(Digital Borsuk-Ulam) If $f : (S^n, \kappa) \to \mathbb{Z}^n$ is continuous for $n = 1, 2$, where $\kappa = 4$ for S^1 and $\kappa = 6$ for S^2, then there exists $x \in S^n$ with $f(x) = f(-x)$.*

Proof. If no such x exists, then the map $g : S^2 \to S^1$ given by

$$g(x) = \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}$$

is a well-defined continuous and $g(-x) = -g(x)$ for every $x \in S^n$, contradicting Theorem 5.3. So there exists $x \in S^n$ with $f(x) = f(-x)$.□ □

Example 5.6. $S^1 = \{p_0 = (-1,1), p_1 = (-1,0), p_2 = (-1,-1), p_3 = (0,-1), p_4 = (1,-1), p_5 = (1,0), p_6 = (1,1), p_7 = (0,1)\}$ is digital 1-sphere with 4-adjacency in $\mathbb{Z}^2$. It is clear that,

$$p_0 = -p_4, \quad p_1 = -p_5, \quad p_2 = -p_6, \quad p_3 = -p_7.$$

Let $f : S^1 \to \mathbb{Z}$ be a map defined by

$$\begin{array}{l} f(p_0) = f(p_4) = 0, \\ f(p_1) = f(p_5) = 1, \\ f(p_2) = f(p_6) = 1, \\ f(p_3) = f(p_7) = 0. \end{array}$$

This map is a $(4, 2)$-continuous map.

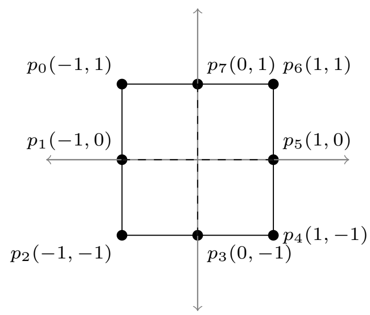

FIGURE 13. S^1

6. **Conclusion**

First, we compute cohomology groups of certain digital surface. Secondly, we present that ring and algebra structure that exists on the digital simplicial cohomology groups with the cup product. The main result is a digital version of the Borsuk-Ulam theorem.

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REFERENCES

[1] J.W. Alexander, On the connectivity ring of an abstract space, *Ann. of Math. (2)* **37** (1936), no. 3, 698–708.
[2] H. Arslan, I. Karaca, and A. Oztel, Homology groups of n-dimensional digital images XXI, Proc. Turkish National Mathematics Symposium, B. pp. 1-13, 2008.
[3] G. Bertrand, Simple points, topological numbers and geodesic neighborhoods in cubic grids, *Pattern Recognit. Lett.* **15** (1994) 1003–1011.
[4] G. Bertrand and R. Malgouyres, Some topological properties of discrete surfaces, *J. Math. Imaging Vision,* **11** (1999) 207–221.
[5] K. Borsuk, Drei sätze über die n-dimensionale euklidische sphäre, *Fund. Math.* **20** (1933) 177–190.
[6] L. Boxer, Digitally continuous functions, *Pattern Recognit. Lett.* **15** (1994) 833–839.
[7] L. Boxer, A classical construction for the digital fundamental group, *J. Math. Imaging Vision* **10** (1999), no. 1, 51–62.
[8] L. Boxer, Homotopy properties of sphere-like digital images, *J. Math. Imaging Vision* **24** (2006), no. 2, 167–175.
[9] L. Boxer, Digital products, wedges, and covering spaces, *J. Math. Imaging Vision* **25** (2006), no. 2, 159–171.
[10] L. Boxer, Continuous maps on digital simple closed curves, *Appl. Math.* **1** (2010) 377–386.

[11] L. Boxer, I. Karaca and A. Oztel, Topological invariants in digital images, *J. Math. Sci. Adv. Appl.* **11** (2011), no. 2, 109–140.
[12] E. Čech, Multiplication on a complex, *Ann. of Math. (2)* **37** (1936), no. 3, 681–697.
[13] L. Chen, Gradually varied surfaces and its optimal uniform approximation, *Proc. SPIE* 2182 (1994) 300–307.
[14] L.M. Chen, Digital and Discrete Geometry Theory and Algorithms, Springer, Berlin, 2014.
[15] M.C. Crabb and J. Jaworowski, Aspects of the Borsuk-Ulam theorem, *J. Fixed Point Theory Appl.* **13** (2013) 459–488.
[16] H. Edelsbrunner and J.L. Harer, Computational Topology An Introduction, American Mathematical Society, Providence, RI, 2010.
[17] O. Ege and I. Karaca, Fundamental properties of digital simplicial homology groups, *Amer. J. Comput. Technol. Appl.* **1** (2013), no. 2, 25–42.
[18] O. Ege and I. Karaca, Cohomology theory for digital images, *Rom. J. Inform. Sci. Technol.* **16** (2013), no. 1, 10–28.
[19] S.E. Han, Non-product property of the digital fundamental group, *Inform. Sci.* **171** (2005), no. 1-3, 73–91.
[20] S.E. Han, Connected sum of digital closed surfaces, *Inform. Sci.* **176** (2006), no. 3, 332–348.
[21] S.E. Han, Minimal simple closed 18-surfaces and a topological preservation of 3D surfaces, *Inform. Sci.* **176** (2006), no. 2, 120–134.
[22] S.E. Han, Digital fundamental group and Euler characteristic of a connected sum of digital closed surfaces, *Inform. Sci.* **177** (2007), no. 16, 3314–3326.
[23] A. Hatcher, Algebraic Topology, Cambridge Univ. Press, 2002.
[24] G.T. Herman, Oriented surfaces in digital spaces, *CVGIP: Graphical Models and Image Processing* **55** (1993) 381–396.
[25] I. Karaca and G. Burak, Simplicial relative cohomology rings of digital images, *Appl. Math. Inf. Sci.* **8** (2014), no. 5, 2375–2387.
[26] A. Kolmogoroff, Homologiering des komplexes und des lokal bikompakten raumes, *Matem. Sb.* **1** (1936) 701–705.
[27] T.Y. Kong, A digital fundamental group, *Computers & Graphics* **13** (1989), no. 2, 159–166.
[28] R. Kopperman, R. Meyer and R.G. Wilson, A Jordan surface theorem for three-dimensional digital spaces, *Discrete Comput. Geom.* **6** (1991), no. 2, 155–161.
[29] R. Malgouyres and G. Bertrand, A new local property of strong n-surfaces, *Pattern Recognit. Lett.* **20** (1999) 417–428.
[30] J.R. Munkres, Elements of Algebraic Topology, Addison-Wesley, 1984.
[31] V.V. Prasolov, Elements of Homology Theory, American Mathematical Society, Providence, RI, 2007.
[32] A. Rosenfeld, Digital topology, *Amer. Math. Monthly* **86** (1979), no. 8, 621–630.
[33] A. Rosenfeld, Continuous functions on digital images, *Pattern Recognit. Lett.* **4** (1986) 177–184.
[34] J.J. Rotman, An Introduction to Algebraic Topology, Springer-Verlag, New York, 1998.
[35] S. Roy and W. Steiger, Some combinatorial and algorithmic applications of the Borsuk-Ulam Theorem, *Graphs Combin.* **23** (2007) 331–341.
[36] E. Spanier, Algebraic Topology, McGraw-Hill, New York, 1966.
[37] E. Ünver Demir and I. Karaca, Simplicial homology groups of certain digital surfaces, *Hacet. J. Math. Stat.* **44** (2015), no. 5, 1011–1022.
[38] H. Whitney, On products in a complex, *Ann. of Math. (2)* **39** (1938), no. 2, 397–432.

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